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Research Article

Helix Surfaces in Euclidean 3-Space with Density

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Abstract

The differential geometry of helix curves and helix hypersurfaces in different spaces has important application areas in many disciplines. Also, the notion of weighted manifold is become to be a very popular topic for scientists in recent years. In this context, after defining the notions of weighted mean curvature (or φ-mean curvature) and weighted Gaussian curvature (or φ-Gaussian curvature) of an n-dimensional hypersurface on manifolds with density, lots of studies have been done by differential geometers in different spaces with different densities. So, in the present study, firstly we give the normal vector field, mean curvature and Gaussian curvature of a helix surface in three dimensional Euclidean space and after that, we obtain the weighted mean curvature and weighted Gaussian curvature of a helix surface generated by a unit speed planar curve in three dimensional Euclidean space with different three densities by stating the parametric equation of this surface. However, we know that a hypersurface is weighted minimal and weighted flat in Eucilidean 3-space with density if the weighted mean curvature and the weighted Gaussian curvature vanish, respectively. So, by using these definitions, we obtain the weighted minimal helix surfaces for these different densities and give some results for weighted flatness of the helix surfaces in Euclidean 3-space. We hope that, this study will bring a new viewpoint to differential geometers who are dealing with constant angle surfaces and in near future, one can handle these surfaces in different spaces with another densities.

Keywords: Helix hypersurface, Weighted mean curvature, Space with density.

Yoğunluklu Öklidyen 3-Uzayında Helis Yüzeyleri

Öz

Farklı uzaylarda helis eğrilerinin ve helis hiperyüzeylerinin diferensiyel geometrisi, birçok bilim dalında önemli uygulama alanlarına sahiptir. Ayrıca, ağırlıklı manifold kavramı son yıllarda bilim insanları için çok popular bir konu olmaya başlamıştır. Bu bağlamda, yoğunluklu manifoldlar üzerinde n-boyutlu bir hiperyüzeyin ağırlıklı ortalama eğriliği (veya φ-ortalama eğriliği) ve ağırlıklı Gaussian eğriliği (veya φ-Gaussian eğriliği) kavramları tanımlandıktan sonra, farklı yoğunluğa sahip değişik uzaylarda diferensiyel geometriciler tarafından pek çok çalışma yapılmaktadır. Bu sebeple, biz de bu çalışmada, ilk olarak Öklidyen 3-uzayında bir helis yüzeyinin normal vektör alanını, ortalama eğriliğini ve Gaussian eğriliğini verdik ve ardından üç farklı yoğunluğa sahip üç boyutlu Öklidyen uzayında birim hızlı bir düzlemsel eğri tarafından oluşturulan bir helis yüzeyinin, parametrik denklemini ifade ederek, ağırlıklı ortalama eğriliğini ve ağırlıklı Gaussian eğriliğini elde ettik. Bununla birlikte, biliyoruz ki yoğunluklu Öklidyen 3-uzayında bir hiperyüzey ağırlıklı minimal ve ağırlıklı flattır, eğer sırasıyla ağırlıklı ortalama eğriliği ve ağırlıklı Gaussian eğriliği sıfır oluyorsa. Dolayısıyla, bu tanımları kullanarak, Öklidyen 3-uzayında bu farklı yoğunluklar için ağırlıklı minimal helis yüzeyleri elde ettik ve bu yüzeylerin ağırlıklı flatlığı için bazı sonuçlar verdik. Umuyoruz ki, bu çalışma sabit açılı yüzeyler ile ilgilenen diferensiyel geometricilere yeni bir bakış açısı kazandıracak ve yakın gelecekte değişik yoğunluklara sahip farklı uzaylarda bu yüzeyler ele alınabilecektir.

Anahtar Kelimeler: Helis hiperyüzeyi, Ağırlıklı ortalama eğrilik, Yoğunluklu uzay.

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1. Introduction

It is known that, if the tangent space of a submanifold of $\mathbb{R}^n$ makes a constant angle with a fixed direction ξ, then it is a helix. So, Ψ is called a helix hypersurface with respect to a unit vector field $\xi \neq 0$ in $\mathbb{R}^n$, if for each $q \in M$, the angle function $\phi \in [0, \pi)$ between ξ and TqM is constant [7,10]. According to this definition, lots of studies have been done by differential geometers about helix hypersurfaces. For instance in [10], the author has determined all helix surfaces with parallel mean curvature vector field which are not minimal or pseudo-umbilical in $M^n(c) \times \mathbb{R}$, where $M^n(c)$ is a simply connected n-manifold with constant sectional curvature c. In [20], some special curves on helix hypersurfaces have been studied and in [5], the authors have studied helix surfaces with parallel mean curvature vector in Euclidean spaces. Minimal helix submanifolds of any dimension and codimension immersed in Euclidean space has been investigated and the result of "A ruled minimal helix submanifold is a cylinder." has been proved in [8]. For further studies about helix hypersurfaces, we refer to [9,15,18,21] and etc.

Furthermore, the notion of weighted manifold is become to be a very popular topic for scientists in recent years. The weighted mean curvature (or φ-mean curvature) of an n-dimensional hypersurface on manifolds with density e^{φ} has been introduced by Gromov as

$$H_{\varphi} = H - \frac{1}{n-1}\frac{d\varphi}{d\mathcal{N}}, \tag{1}$$

where $\mathcal{N}$ is the normal vector field and H is the mean curvature of the hypersurface [11]. Also, the notion of weighted Gaussian curvature (or φ-Gaussian curvature) of an n-dimensional hypersurface on manifolds with density e^{φ} is introduced as

$$K_{\varphi} = K - \triangle \varphi, \tag{2}$$

where $\triangle$ is the Laplacian operator and K is the Gaussian curvature of the hypersurface [6].

It is said that, a hypersurface is weighted minimal and weighted flat if the weighted mean curvature and weighted Gaussian curvature vanish, respectively. After defining these notions, lots of studies have been done in different spaces with different densities by giving important characterizations for some types of curves and surfaces (for instance, [1-4,6,12-14,16,17,19] and etc).

In the present study, firstly we give the parametric expression of a helix surface generated by a unit speed planar curve in Euclidean 3-space and get the mean and Gaussian curvatures of it. After that, we find the weighted mean curvature and weighted Gaussian curvature of this helix surface in E^3 with densities e^x, e^{x+y} and $e^{x^2+y^2+z^2}$ seperately and give some characterizations for their minimality and flatness.

2. Helix Surfaces in E^3

In this section, we'll give the normal vector field, mean curvature and Gaussian curvature of a helix surface in Euclidean 3-space E^3.

We know that, if $\gamma: I \subset \mathbb{R} \longrightarrow E^3$, $s \longrightarrow \gamma(s)$ is a unit speed planar curve in E^3, then

$$\Psi: U \subset E^2 \longrightarrow E^3$$
$$(s,v) \longrightarrow \Psi(s,v) = \gamma(s) + v(sin\phi . N(s) + cos\phi . B) \tag{3}$$

is a helix surface with direction B. Here, ϕ is constant and the binormal vector B is a constant vector which is vertical to plane of the curve γ [21]. Also, for the unit speed planar curve $\gamma(s) = (\gamma_1(s), \gamma_2(s), 0)$, we have

$$(\gamma_1'(s))^2 + (\gamma_2'(s))^2 = 1 \tag{4}$$

and

$$T(s) = (\gamma_1'(s), \gamma_2'(s), 0),$$
$$N(s) = (-\gamma_2'(s), \gamma_1'(s), 0), \tag{5}$$
$$B(s) = (0,0,1),$$

where T, N and B are tangent vector, principal normal vector and binormal vector of γ, respectively.

Thus, using (5) in (3), the helix surface generated by unit speed planar curve $\gamma(s) = (\gamma_1(s), \gamma_2(s), 0)$ can be written as

$$\Psi(s,v) = (\gamma_1(s) - v\gamma_2'(s)sin\phi, \gamma_2(s) + v\gamma_1'(s)sin\phi, vcos\phi). \tag{6}$$

For this surface, from (4) and (6), we have

$$\Psi_s \times \Psi_v = ((\gamma_2'(s) + v\gamma_1''(s)sin\phi)cos\phi,$$
$$(v\gamma_2''(s)sin\phi - \gamma_1'(s))cos\phi,$$
$$sin\phi + vsin^2\phi(\gamma_1''(s)\gamma_2'(s) - \gamma_1'(s)\gamma_2''(s))) \tag{7}$$

and

$$\|\Psi_s \times \Psi_v\| = \frac{\gamma_1'(s) - vsin\phi\gamma_2''(s)}{\gamma_1'(s)}. \tag{8}$$

So, from (7) and (8), we obtain the normal vector field of the helix surface (6) as

$$\mathcal{N} = \frac{\Psi_s \times \Psi_v}{\|\Psi_s \times \Psi_v\|} = (\gamma_2'(s)cos\phi, -\gamma_1'(s)cos\phi, sin\phi). \tag{9}$$

Also, from (4) and (6), we have the coefficients of first fundamental form as

$$E = \langle\Psi_s, \Psi_s\rangle = \left(1 - vsin\phi\frac{\gamma_2''(s)}{\gamma_1'(s)}\right)^2, \ F = \langle\Psi_s, \Psi_v\rangle = 0, \ G = \langle\Psi_v, \Psi_v\rangle = 1 \tag{10}$$

and the coefficients of second fundamental form as

$$L = \langle\Psi_{ss}, \mathcal{N}\rangle = \frac{-\gamma_1'(s)\gamma_2''(s)cos\phi + vsin\phi cos\phi(\gamma_2''(s))^2}{(\gamma_1'(s))^2}, \ M = \langle\Psi_{sv}, \mathcal{N}\rangle = 0,$$
$$N = \langle\Psi_{vv}, \mathcal{N}\rangle = 0. \tag{11}$$

Hence, from (10) and (11), we obtain the mean curvature and Gaussian curvature of the helix surface generated by unit speed planar curve $\gamma(s) = (\gamma_1(s), \gamma_2(s), 0)$ as

$$H = \frac{GL - 2FM + EN}{2(EG - F^2)} = -\frac{\gamma_2''(s)cos\phi}{2(\gamma_1'(s) - vsin\phi\gamma_2''(s))} \tag{12}$$

and

$$K = \frac{LN - M^2}{EG - F^2} = 0, \tag{13}$$

respectively.

Remark 1. *We note that, the author has found the normal vector field, mean curvature and Gaussian curvature of helix surface (3) as*

$$\mathcal{N} = -cos\phi N + sin\phi B, \ H = \frac{1}{2}\left(\frac{k_1 cos\phi}{1 - k_1 vsin\phi}\right), \ K = 0,$$

respectively [21]. Obtaining the first curvature of unit speed planar curve $\gamma(s) = (\gamma_1(s), \gamma_2(s), 0)$, one can see that these equations are equivalent to (9), (12) and (13) which have been obtained by us for the parametric expression of a helix surface stated by (6).

3. Main Results

In this section, we'll give some characterizations for helix surface (6) in Euclidean 3-space with different densities. Throughout this section, we suppose that, $c_i \in \mathbb{R}, (i = 1,2,3, \dots)$.

3.1. Helix Surfaces in E^3 with Different Densities

Firstly, let we assume that the density is e^x.

In this case, from (1), (9) and (12), the weighted mean curvature of helix surface (6) generated by unit speed planar curve $\gamma(s) = (\gamma_1(s), \gamma_2(s), 0)$ is obtained as

$$H_\varphi = H - \frac{1}{2}\langle \nabla\varphi, \mathcal{N}\rangle$$

$$= -\frac{\cos\phi}{2(\gamma_1'(s) - v\gamma_2''(s)\sin\phi)}\{\gamma_2''(s) + \gamma_2'(s)(\gamma_1'(s) - v\gamma_2''(s)\sin\phi)\}. \quad (14)$$

So, we have

Theorem 1. *If*

i) $\phi = \frac{\pi}{2}$ *or*

ii) $\gamma_1(s) = \pm s + c_1$ *and* $\gamma_2(s) = c_2$ *or*

iii) $\phi = \arcsin\left(\frac{\gamma_1'(s)\gamma_2'(s) + \gamma_2''(s)}{v\gamma_2'(s)\gamma_2''(s)}\right)$*(=constant),*

then the helix surface (6) generated by unit speed planar curve $\gamma(s) = (\gamma_1(s), \gamma_2(s), 0)$ *is weighted minimal in* E^3 *with density* e^x.

Proof. When the above assumptions (i), (ii) and (iii) satisfy, we get $H_\varphi = 0$ in the equation (14). Since a surface is weighted minimal if weighted mean curvature vanishes, the proof completes. ■

Hence, using these assumptions in the parametric expression of the helix surface (6), we can state the following corollary:

Corollary 1. *Weighted minimal helix surfaces generated by unit speed planar curve* $\gamma(s) = (\gamma_1(s), \gamma_2(s), 0)$ *in Euclidean 3-space with density* e^x *can be parametrized by*

$$\Psi(s,v) = (\gamma_1(s) - v\gamma_2'(s), \gamma_2(s) + v\gamma_1'(s), 0), \ \text{or} \quad (15)$$

$$\Psi(s,v) = (\pm s + c_1, \pm v\sin\phi + c_2, v\cos\phi), \ \phi \neq \frac{\pi}{2} \ \text{or} \quad (16)$$

$$\Psi(s,v) = \Bigg(\gamma_1(s) - \frac{\gamma_1'(s)\gamma_2'(s) + \gamma_2''(s)}{\gamma_2''(s)}, \gamma_2(s) + \frac{(\gamma_1'(s))^2\gamma_2'(s) + \gamma_1'(s)\gamma_2''(s)}{\gamma_2'(s)\gamma_2''(s)}, v\sqrt{1 - \left(\frac{\gamma_1'(s)\gamma_2'(s) + \gamma_2''(s)}{v\gamma_2'(s)\gamma_2''(s)}\right)^2}\Bigg). \quad (17)$$

Also, from (2) and (13), the weighted Gaussian curvature of helix surface (6) generated by unit speed planar curve $\gamma(s) = (\gamma_1(s), \gamma_2(s), 0)$ in Euclidean 3-space with density e^x is

$$K_\varphi = K - \triangle\varphi = K = 0. \quad (18)$$

Since a surface is weighted flat if weighted Gaussian curvature vanishes, it is obvious that,

Corollary 2. *The helix surface (6) generated by unit speed planar curve* $\gamma(s) = (\gamma_1(s), \gamma_2(s), 0)$ *is weighted flat in Euclidean 3-space with density* e^x.

The following figures show the weighted minimal helix surfaces (15), (16) and (17), respectively, for $\gamma_1(s) = s$, $\gamma_2(s) = s^2$, $c_1 = 3$, $c_2 = 1$ and $\phi = \frac{\pi}{6}$.

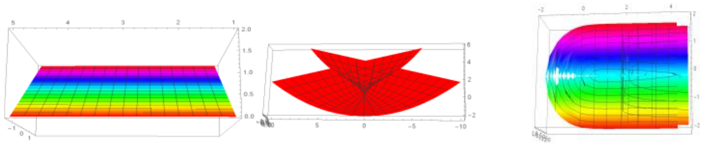

Figure 1

Now, let we assume that the density is e^{x+y}.

In this case, from (1), (9) and (12), the weighted mean curvature of helix surface (6) generated by unit speed planar curve $\gamma(s) = (\gamma_1(s), \gamma_2(s), 0)$ is obtained as

$$H_\varphi = -\frac{\cos\phi}{2(\gamma_1'(s) - v\gamma_2''(s)\sin\phi)}\{\gamma_2''(s) + (\gamma_2'(s) - \gamma_1'(s))(\gamma_1'(s) - v\gamma_2''(s)\sin\phi)\}. \quad (19)$$

Thus, we have

Theorem 2. *If*

i) $\phi = \frac{\pi}{2}$ *or*

ii) $\gamma_1(s) = \pm\frac{s}{\sqrt{2}} + c_3$ *and* $\gamma_2(s) = \pm\frac{s}{\sqrt{2}} + c_4$ *or*

iii) $\phi = \arcsin\left(\frac{(\gamma_1'(s))^2 - \gamma_2''(s) - \gamma_1'(s)\gamma_2'(s)}{v\gamma_2''(s)(\gamma_1'(s) - \gamma_2'(s))}\right)$ *(=constant),*

then the helix surface (6) generated by unit speed planar curve $\gamma(s) = (\gamma_1(s), \gamma_2(s), 0)$ *is weighted minimal in* E^3 *with density* e^{x+y}.

Proof. If the above assumptions (i), (ii) and (iii) satisfy, we get $H_\varphi = 0$ in the equation (19). From the definition of weighted minimal surface, the proof completes. ■

Therefore, from Theorem 2, we get

Corollary 3. *Weighted minimal helix surfaces generated by unit speed planar curve* $\gamma(s) = (\gamma_1(s), \gamma_2(s), 0)$ *in Euclidean 3-space with density* e^{x+y} *can be parametrized by*

$$\Psi(s,v) = (\gamma_1(s) - v\gamma_2'(s), \gamma_2(s) + v\gamma_1'(s), 0) \ \text{or} \quad (20)$$

$$\Psi(s,v) = \left(\pm\frac{s}{\sqrt{2}} \mp \frac{v}{\sqrt{2}}\sin\phi + c_3, \pm\frac{s}{\sqrt{2}} \pm \frac{v}{\sqrt{2}}\sin\phi + c_4, v\cos\phi\right), \ \phi \neq \frac{\pi}{2} \ \text{or} \quad (21)$$

$$\Psi(s,v) = \Bigg(\gamma_1(s) - \frac{(\gamma_1'(s))^2\gamma_2'(s) - \gamma_2'(s)\gamma_2''(s) - \gamma_1'(s)(\gamma_2'(s))^2}{\gamma_2''(s)(\gamma_1'(s) - \gamma_2'(s))}, \gamma_2(s) + \frac{(\gamma_1'(s))^3 - \gamma_1'(s)\gamma_2''(s) - (\gamma_1'(s))^2\gamma_2'(s)}{\gamma_2''(s)(\gamma_1'(s) - \gamma_2'(s))}, v\sqrt{1 - \left(\frac{(\gamma_1'(s))^2 - \gamma_2''(s) - \gamma_1'(s)\gamma_2'(s)}{v\gamma_2''(s)(\gamma_1'(s) - \gamma_2'(s))}\right)^2}\Bigg). \quad (22)$$

From (2) and (13), the weighted Gaussian curvature of helix surface (6) generated by unit speed planar curve $\gamma(s) = (\gamma_1(s), \gamma_2(s), 0)$ in Euclidean 3-space with density e^{x+y} is

$$K_\varphi = K = 0.$$

Thus,

Corollary 4. *The helix surface (6) generated by unit speed planar curve* $\gamma(s) = (\gamma_1(s), \gamma_2(s), 0)$ *is weighted flat in Euclidean 3-space with density* e^{x+y}.

Finally, let we assume that the density is $e^{x^2+y^2+z^2}$.

In this case, from (1), (9) and (12), the weighted mean curvature of helix surface (6) generated by unit speed planar curve $\gamma(s) = (\gamma_1(s), \gamma_2(s), 0)$ is obtained as

$$H_\varphi = -\frac{cos\phi}{2(\gamma_1'(s) - v\gamma_2''(s)sin\phi)}\{\gamma_2''(s) + 2(\gamma_1'(s) - v\gamma_2''(s)sin\phi)(\gamma_1(s)\gamma_2'(s) - \gamma_1'(s)\gamma_2(s))\}. \quad (23)$$

So, we have

Theorem 3. *If*

i) $\phi = \frac{\pi}{2}$ *or*

ii) $\gamma_1(s) = c_5 s + c_6$ *and* $\gamma_2(s) = c_7 s + c_8$,
where $(c_5)^2 + (c_7)^2 = 1$, $c_6 c_7 = c_5 c_8$ *or*

iii) $\phi = arcsin\left(\frac{\gamma_2''(s) + 2\gamma_1(s)\gamma_1'(s)\gamma_2'(s) - 2(\gamma_1'(s))^2\gamma_2(s)}{2v\gamma_2''(s)(\gamma_1(s)\gamma_2'(s) - \gamma_1'(s)\gamma_2(s))}\right)$*(=constant),*

then the helix surface (6) generated by unit speed planar curve $\gamma(s) = (\gamma_1(s), \gamma_2(s), 0)$ *is weighted minimal in* E^3 *with density* $e^{x^2+y^2+z^2}$.

Proof. If the assumptions (i), (ii) and (iii) satisfy, then we get $H_\varphi = 0$ in the equation (23). So, the helix surface is weighted minimal if these assumptions satisfy. ■

Hence, from Theorem 3, we get

Corollary 5. *Weighted minimal helix surfaces generated by unit speed planar curve* $\gamma(s) = (\gamma_1(s), \gamma_2(s), 0)$ *in Euclidean 3-space with density* $e^{x^2+y^2+z^2}$ *can be parametrized by*

$$\Psi(s,v) = (\gamma_1(s) - v\gamma_2'(s), \gamma_2(s) + v\gamma_1'(s), 0) \text{ or} \quad (24)$$

$$\Psi(s,v) = (c_5 s - c_7 v sin\phi + c_6, c_7 s + c_5 v sin\phi + c_8, v cos\phi),\ \phi \neq \frac{\pi}{2} \text{ or} \quad (25)$$

$$\Psi(s,v) = (\gamma_1(s) - \frac{\gamma_2'(s)\gamma_2''(s) + 2\gamma_1(s)\gamma_1'(s)(\gamma_2'(s))^2 - 2(\gamma_1'(s))^2\gamma_2(s)\gamma_2'(s)}{2\gamma_2''(s)(\gamma_1(s)\gamma_2'(s) - \gamma_1'(s)\gamma_2(s))}, \gamma_2(s) + \frac{\gamma_1'(s)\gamma_2''(s) + 2\gamma_1(s)(\gamma_1'(s))^2\gamma_2'(s) - 2(\gamma_1'(s))^3\gamma_2(s)}{2\gamma_2''(s)\left(\gamma_1(s)\gamma_2'(s) - \gamma_1'(s)\gamma_2(s)\right)}, v\sqrt{1 - \left(\frac{\gamma_2''(s) + 2\gamma_1(s)\gamma_1'(s)\gamma_2'(s) - 2(\gamma_1'(s))^2\gamma_2(s)}{2v\gamma_2''(s)\left(\gamma_1(s)\gamma_2'(s) - \gamma_1'(s)\gamma_2(s)\right)}\right)^2}), \quad (26)$$

where $(c_5)^2 + (c_7)^2 = 1$, $c_6 c_7 = c_5 c_8$.

From (2) and (13), the weighted Gaussian curvature of helix surface (6) generated by unit speed planar curve $\gamma(s) = (\gamma_1(s), \gamma_2(s), 0)$ in Euclidean 3-space with density $e^{x^2+y^2+z^2}$ is

$$K_\varphi = -6.$$

Thus,

Corollary 6. *The helix surface (6) generated by unit speed planar curve* $\gamma(s) = (\gamma_1(s), \gamma_2(s), 0)$ *has constant weighted Gaussian curvature* (-6) *in Euclidean 3-space with density* $e^{x^2+y^2+z^2}$.

We note that, the obtained above surfaces can be drawn with similar or different assumptions in Figure 1.

4. Conclusion and Future Work

In the present study, weighted minimal helix surfaces generated by unit speed planar curve have been given in Euclidean 3-space with different densities and some results have been stated for weighted flatness of these surfaces. We hope that, this study will bring a new viewpoint to differential geometers who are dealing with constant angle surfaces and in near future, one can handle these surfaces in different spaces with another densities.

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