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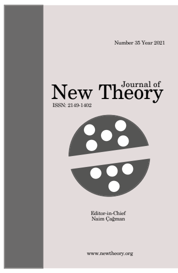

A New View of Homomorphic Properties of BCK-Algebra in Terms of Some Notions of Discrete Dynamical System

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Abstract − In the present manuscript, we introduce the concept of a discrete dynamical system (Ȥ, Ψ) in BCK-algebra where Ȥ is a BCK-algebra and Ψ is a homomorphism from Ȥ to Ȥ and establish some of their related properties. We prove that the set of all fixed points and the set of all periodic points in BCK-algebra Ȥ are the BCK-subalgebras. We show that when a subset of BCK-algebra Ȥ is invariant concerning Ψ. We prove that the set of all fixed points and the set of all periodic points in commutative BCK-algebra Ȥ with relative cancellation property are the ideals of Ȥ. We also prove that the set of all fixed points in Ȥ is an S-invariant subset of a BCK-algebra Ȥ.

Keywords − *BCK-Algebra, discrete dynamical system, periodic points, fixed points, invariant set, strongly invariant*

Mathematics Subject Classification (2020) − 37A15, 06F35

1. Introduction

The foundation of the concept of BCK-Algebra was laid down by the famous mathematicians Imai and Iseki in their pioneering paper [1]. Their theory about BCK-algebra and related ideas and properties are nowadays utilized extensively in different areas of science like artificial intelligence information sciences, cybernetics, and computer sciences. BCK-algebra has been inspired by two considerations; one based on classical and non-classical propositional calculi of Meredith and the other based on set theory [2]. The BCK-algebra can also be considered as the algebraic formulation of Meredith's BCK-implicational calculus [3]. The concept of the ideal theory of BCK-algebra playing a fundamental role in the evolution of BCK-Algebra was first introduced by Iseki in [4]. Besides, the notion of BCK-homomorphism was also first defined by Iseki in [5]. During the past four decades, several researchers have extensively investigated this field and have produced much literature about the theory of BCK-algebra [6].

The foundation of the dynamical system was laid down by the eminent mathematician Henri Poincare in 1899 in his famous paper celestial mechanics [7]. His theories about dynamical systems and connected ideas and properties are nowadays widely used in various fields of science like physics, biology, meteorology, astronomy, economics, and many others. The main idea of the study of the dynamical system indicates the mathematical techniques for describing the eventual or asymptotic behaviour of an iterative process in different specified scientific disciplines. In 1917, Julia and Fatau diverted the concept of the dynamical system theory in the relationship of complex analysis, established a new notion, and provided the name as complex analytic dynamical system [8]. Latterly, Birkhoff enthusiastically adopted Poincare's viewpoint, realized the

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significance of the concept of mappings, and introduced a discrete dynamical system [9]. A discrete dynamical system is an interesting and active area of applied and pure mathematics that involves tools and techniques from different fields such as Number Theory, Analysis, and Geometry. According to a fixed rule, discrete dynamical systems are those dynamical systems whose states evolve over a state space in discrete steps. Birkhoff's opinion based on discrete steps was highly captivating and attracted the attention of mathematicians to use in various fields of mathematics. Many researchers applied the concept of a discrete dynamical system in their related areas. This becomes the extension of the theory of discrete dynamical systems in many branches of mathematical sciences. In 1927 Birkhoff infused the notion of discrete dynamical systems in topology and laid down the foundation of another field known as topological Dynamics [10]. Topological dynamics then flourished and further generated algebraic topology and differential topology [11]. Differential Topological techniques enabled Peixoto and Smale to understand the chaotic behaviour of a large class of dynamical systems and introduced a new area of the dynamical system known as hyperbolic dynamical system [12]. Von Neumann, Birkhoff, and Koopman introduced a discrete dynamical system in measure theory, due to which a new field emerged by the name of Ergodic theory [13]. Dikranjan and Bruno embedded the discrete dynamical system in group theory and established a new algebraic structure known as discrete dynamical systems in group theory [14]. Dawood et al. have recently infused the discrete dynamical system in BCI-algebra and obtained specific interesting properties [15].

In the present paper, we emphasize the mathematical aspects of the theory of discrete dynamical system $(Ꝩ, \Psi)$. We define the concept of discrete dynamical system $(Ꝩ\, \Psi)$ in BCK-algebra where Ꝩ is a BCK-algebra and Ψ is a homomorphism from Ꝩ to Ꝩ and establish some properties of the set of periodic points and the set of fixed points of BCK-algebra Ꝩ. We prove that the sets of all fixed points and periodic points of BCK-algebra Ꝩ are the BCK-subalgebras of Ꝩ. We show that when a subset of a BCK-algebra Ꝩ is invariant concerning Ψ. We prove that the sets of all fixed and periodic points of a commutative BCK-algebra Ꝩ with relative cancellation property are the ideals of Ꝩ.we also prove that in the discrete dynamical system $(Ꝩ, \Psi)$ the set of all fixed points is S-Invariant subset of a BCK-algebra Ꝩ.

2. Preliminaries

This section consists of some preliminary definitions and basic facts about BCK-algebra, useful to prove our results. Throughout this research work, we consistently denote the BCK-algebra by Ꝩ without any specification.

Here we only mention those concepts of BCK-algebra which are necessary for our treatment. For further information regarding BCK-algebra, the readers are referred to the references [16-21].

Definition 2.1. [7] A BCK-algebra $(Ꝩ,*,0)$ is an algebra of the type $(2,0)$ satisfying the following five axioms for all $\mathfrak{a},\mathfrak{b},\mathfrak{c} \in Ꝩ$

(BCK-i) $((\mathfrak{a} * \mathfrak{b}) * (\mathfrak{a} * \mathfrak{c})) * (\mathfrak{c} * \mathfrak{b}) = 0$

(BCK-ii) $(\mathfrak{a} * (\mathfrak{a} * \mathfrak{b})) * \mathfrak{b} = 0$

(BCK-iii) $\mathfrak{a} * \mathfrak{a} = 0$

(BCK-iv) $0 * \mathfrak{a} = 0$

(BCK-v) $\mathfrak{a} * \mathfrak{b} = 0$ and $\mathfrak{b} * \mathfrak{a} = 0 \Longrightarrow \mathfrak{a} = \mathfrak{b}$

Moreover $\leq$ is a partial order on Ꝩ and is defined by

$$\mathfrak{a} \leq \mathfrak{b} \Leftrightarrow \mathfrak{a} * \mathfrak{b} = 0$$

Definition 2.2. [7] BCK-algebra $(Ꝩ,*,0)$ is said to be commutative if the following condition holds in Ꝩ for any $\mathfrak{a},\mathfrak{b} \in Ꝩ$

$$\mathfrak{a} * (\mathfrak{a} * \mathfrak{b}) = \mathfrak{b} * (\mathfrak{b} * \mathfrak{a})$$

Definition 2.3. [7] Let $Ɀ_s$ be a non-vacuous subset of a BCK-algebra $(Ɀ,*,0)$. Then, $Ɀ_s$ is said to be a BCK-subalgebra of Ɀ if it satisfies the (s-i) and (s-ii) conditions where

(s-i) $0 \epsilon Ɀ_s$

(s-ii) $\mathfrak{a} * \mathfrak{b} \epsilon Ɀ_s$ for any $\mathfrak{a}, \mathfrak{b} \in Ɀ_s$

Definition 2.4. [7] Let Ɀ be a BCK-algebra and I_d be a non-vacuous subset of Ɀ**.** Then, I_d is said to be an ideal of Ɀ if it satisfies (I_d-i) and (I_d-ii) conditions, where

(I_d-i) $0 \epsilon I_d$

(I_d-ii) $\mathfrak{a} * \mathfrak{b} \epsilon I_d$ and $\mathfrak{b} \in I_d \Longrightarrow \mathfrak{a} \in I_d$, $\forall \mathfrak{a}, \mathfrak{b} \in Ɀ$

Definition 2.5. [5,17] A mapping $\Psi: Ɀ_1 \longrightarrow Ɀ_2$ where $Ɀ_1$ and $Ɀ_2$ are two BCK-algebras is said to be a BCK-homomorphism if it satisfies the following condition

$$\Psi(\mathfrak{a} * \mathfrak{b}) = \Psi(\mathfrak{a}) * \Psi(\mathfrak{b}), \quad \forall \mathfrak{a}, \mathfrak{b} \in Ɀ_1$$

Definition 2.6. [18] A commutative BCK-algebra (Ɀ, $*$, 0) has the relative cancellation property if for $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in Ɀ$ and $\mathfrak{a} \geq \mathfrak{c}$, $\mathfrak{b} \geq \mathfrak{c}$ such that $\mathfrak{a} * \mathfrak{c} = \mathfrak{b} * \mathfrak{c}$, then $\mathfrak{a} = \mathfrak{b}$.

3. Definitions of Some Notions of Discrete Dynamical System in BCK-Algebra

This section picks some terminologies of discrete dynamical systems and defines them in terms of BCK-algebra.

Definition 3.1. Let Ɀ be a BCK-algebra and $\Psi: Ɀ \longrightarrow Ɀ$ be a homomorphism. Then, $(Ɀ, \Psi)$ is called a discrete dynamical system in BCK-algebra

In the present paper, whenever we say a discrete dynamical system, it means we are taking an ordered pair $(Ɀ, \Psi)$ where Ɀ is a BCK-algebra and Ψ is a homomorphism from Ɀ to Ɀ.

Definition 3.2. In the discrete dynamical system $(Ɀ, \Psi)$ a point $\mathfrak{a} \epsilon Ɀ$ is a fixed point if $\Psi(\mathfrak{a}) = \mathfrak{a}$.

Definition 3.3. In the discrete dynamical system $(Ɀ, \Psi)$ a point $\mathfrak{a} \in Ɀ$ is a periodic point if $\Psi^m(\mathfrak{a}) = \mathfrak{a}$ for some positive integer 'm', the least value of 'm' is said to be the period of '$\mathfrak{a}$'.

Definition 3.4. In the discrete dynamical system $(Ɀ, \Psi)$ a subset A of Ɀ is an invariant subset of Ɀ if $\Psi(A) \subset A$.

Definition 3.5. In the discrete dynamical system $(Ɀ, \Psi)$ a subset A of Ɀ is a strongly invariant subset of Ψ if $\Psi(A) = A$.

The following are simple examples regarding the definitions given in the paper.

Example 3.1. Suppose that $Ɀ = \{0, p, q, r\}$ is a BCK-algebra.

$*$	0	p	q	r
0	0	0	0	0
p	p	0	p	p
q	q	q	0	q
r	r	r	r	0

And a mapping $\Psi: Ɀ \longrightarrow Ɀ$ defined by $\Psi(0) = 0, \Psi(p) = 0, \Psi(q) = q$, and $\Psi(r) = r$ is a homomorphism. Then, the points 0, q, and r are fixed points and the periodic points of period 1.

Example 3.2. Consider the BCK-algebra Ɀ of example 3.1. Where the subset $A = \{0, q, r\}$ is a strongly invariant subset of Ɀ because $\Psi(A) = A$ while the subset $B = \{0, p\}$ is invariant because $\Psi(B) \subset B$.

4. Basic Results

In this section, we prove some properties essential in proving the theorems in this paper.

Proposition 4.1. If a mapping $\Psi: \text{Ꝣ} \longrightarrow \text{Ꝣ}$ is a homomorphism from BCK-algebra Ꝣ to Ꝣ and $0 \in$ Ꝣ, then $\Psi\,(0) = 0$.

PROOF. $\Psi\,(0) = \Psi\,(0 * 0)$ ∵ (BCK-iii)

$= \Psi\,(0) * \Psi\,(0) = 0$ ∵ (BCK-iii)

Proposition 4.2. If $\Psi: \text{Ꝣ} \longrightarrow \text{Ꝣ}$ is a homomorphism, then $\Psi^{\,n}: \text{Ꝣ} \longrightarrow \text{Ꝣ}$ is a homomorphism. (Here Ψ^n means $\Psi \circ \Psi \circ \Psi \ \dots \circ \Psi$ (n time)).

PROOF. We prove this result by using the principle of mathematical induction. We have to show that for any two elements a, b in Ꝣ

$$\Psi^n(\mathfrak{a} * \mathfrak{b}) = \Psi^n(\mathfrak{a}) * \Psi^n(\mathfrak{b}) \tag{1}$$

where n is the positive integer.

It has been given that statement (1) is valid for $n = 1$, and we assume that it is valid for $n = k$ then we have

$$\Psi^k(\mathfrak{a} * \mathfrak{b}) = \Psi^k(\mathfrak{a}) * \Psi^k(\mathfrak{b})$$

$$\Psi(\Psi^k(\mathfrak{a} * \mathfrak{b})) = \Psi(\Psi^k(\mathfrak{a}) * \Psi^k(\mathfrak{b}))$$

$$\Psi^{k+1}(\mathfrak{a} * \mathfrak{b}) = \Psi(\Psi^k(\mathfrak{a})) * \Psi(\Psi^k(\mathfrak{b})), \quad \because \Psi \text{ is homomorphism}$$

$$\Psi^{k+1}(\mathfrak{a} * \mathfrak{b}) = \Psi^{k+1}(\mathfrak{a}) * \Psi^{k+1}(\mathfrak{b})$$

Thus, the validity of statement (1) at $n = k$ implies the validity of (1) at $n = k + 1$. Hence, (1) holds for all positive integers 'n'.

Proposition 4.3. If $\Psi: \text{Ꝣ} \longrightarrow \text{Ꝣ}$ is a homomorphism,$\Psi^n(\mathfrak{a}) = \mathfrak{a}$ and n, p are positive integers such that n divides p then $\Psi^p(\mathfrak{a}) = \mathfrak{a}$.

PROOF. Since $\Psi^n(\mathfrak{a}) = \mathfrak{a}$ (2)

Where, n is a positive integer which divides p then there exists an integer q such that $p = nq$, then we have

$$\begin{aligned}
\Psi^p(\mathfrak{a}) &= \Psi^{nq}(\mathfrak{a}) = \Psi^{n(q-1)}(\Psi^n(\mathfrak{a})) \\
&= \Psi^{n(q-1)}(\mathfrak{a}) && \because \text{using (2)} \\
&= \Psi^{n(q-2)}(\Psi^n(\mathfrak{a})) \\
&= \Psi^{n(q-2)}(\mathfrak{a}) && \because \text{using (2)} \\
&\ \ \vdots \\
&= \Psi^{n(q-(q-1))}(\mathfrak{a}) = \Psi^{n(q-q+1)}(\mathfrak{a}) = \Psi^n(\mathfrak{a}) = \mathfrak{a} && \because \text{using (2)}
\end{aligned}$$

Hence $\Psi^p(\mathfrak{a}) = \mathfrak{a}$.

Proposition 4.4. In the discrete dynamical system (Ꝣ, Ψ) if '$\mathfrak{b}$' is a fixed point in Ꝣ and $\mathfrak{a} \geq \mathfrak{b}$ for any $\mathfrak{a} \in$ Ꝣ then $\Psi\,(\mathfrak{a}) \geq \mathfrak{b}$.

PROOF. Since $\mathfrak{a} \geq \mathfrak{b}$ can also be written as $\mathfrak{b} \leq \mathfrak{a} \Longrightarrow \mathfrak{b} * \mathfrak{a} = 0 \Longrightarrow \Psi\,(\mathfrak{b} * \mathfrak{a}) = \Psi\,(0)$.

$$\Longrightarrow \Psi\,(\mathfrak{b}) * \Psi\,(\mathfrak{a}) = 0 \tag{3}$$

Where '$\mathfrak{b}$' is a fixed point. Therefore, (3) becomes $\mathfrak{b} * \Psi(\mathfrak{a}) = 0 \Longrightarrow \mathfrak{b} \leq \Psi(\mathfrak{a})$ or $\Psi(\mathfrak{a}) \geq \mathfrak{b}$.

Proposition 4.5. In the discrete dynamical system $(\mathfrak{Z}, \Psi)$, if '$\mathfrak{b}$' is a fixed point in $\mathfrak{Z}$ and $\mathfrak{a} \geq \mathfrak{b}$ for any $\mathfrak{a} \in \mathfrak{Z}$, then $\Psi^{n}(\mathfrak{a}) \geq \mathfrak{b}$. Where '$n$' is a positive integer.

Proof. We prove this result by using the principle of mathematical induction, so by Proposition 4.4, the statement $\Psi^n(\mathfrak{a}) \geq \mathfrak{b}$ is true for $n = 1$. Next, we assume that the statement $\Psi^n(\mathfrak{a}) \geq \mathfrak{b}$ is true for $n = k$ such that

$$\Psi^k(\mathfrak{a}) \geq \mathfrak{b} \tag{4}$$

from equation (4), we get $\mathfrak{b} * \Psi^k(\mathfrak{a}) = 0 \Longrightarrow \Psi(\mathfrak{b} * \Psi^k(\mathfrak{a})) = \Psi(0)$

$$\Longrightarrow \Psi(\mathfrak{b}) * \Psi^{k+1}(\mathfrak{a}) = 0 \tag{5}$$

where '$\mathfrak{b}$' is a fixed point.

Therefore, equation (5) becomes

$$\mathfrak{b} * \Psi^{k+1}(\mathfrak{a}) = 0 \Longrightarrow \mathfrak{b} \leq \Psi^{k+1}(\mathfrak{a}) \text{ or } \Psi^{k+1}(\mathfrak{a}) \geq \mathfrak{b}$$

Thus, the truth of the statement $\Psi^n(\mathfrak{a}) \geq \mathfrak{b}$ at $n = k$ implies the truth of $\Psi^n(\mathfrak{a}) \geq \mathfrak{b}$ at $n = k + 1$ hence the statement $\Psi^n(\mathfrak{a}) \geq \mathfrak{b}$ is true for any positive integer 'n'.

5. Main Theorems

Theorem 5.1. In the discrete dynamical system $(\mathfrak{Z}, \Psi)$, the set of all fixed points in $\mathfrak{Z}$ is a BCK-subalgebra of $\mathfrak{Z}$.

Proof. Let $\mathfrak{Z}$ be a BCK-algebra and a mapping $\Psi: \mathfrak{Z} \longrightarrow \mathfrak{Z}$ is a homomorphism. Suppose that $\mathfrak{Z}_f$ be the set of all fixed points in $\mathfrak{Z}$. We show that $\mathfrak{Z}_f$ is a BCK-subalgebra for this $\mathfrak{Z}_f$ has to satisfy the conditions of BCK-subalgebra. Since Ψ is a homomorphism therefore by Proposition 4.1, we have $\Psi(0) = 0$, which implies that 0 is a fixed point $\Longrightarrow 0 \in \mathfrak{Z}_f \Longrightarrow \mathfrak{Z}_f$ is a non-vacuous set.

Thus, condition (s-i) of BCK-sub algebra holds in $\mathfrak{Z}_f$. Now assume that $a, \mathfrak{b} \in \mathfrak{Z}_f$

Then,

$$\Psi(\mathfrak{a}) = \mathfrak{a} \tag{6}$$

and

$$\Psi(\mathfrak{b}) = \mathfrak{b} \tag{7}$$

Since Ψ is a homomorphism, therefore we get

$$\Psi(\mathfrak{a} * \mathfrak{b}) = \Psi(\mathfrak{a}) * \Psi(\mathfrak{b}) \tag{8}$$

Using (6) and (7) in (8) we get

$$\Psi(\mathfrak{a} * \mathfrak{b}) = \mathfrak{a} * \mathfrak{b} \Longrightarrow \mathfrak{a} * \mathfrak{b} \text{ is a fixed point} \Longrightarrow \mathfrak{a} * \mathfrak{b} \in \mathfrak{Z}_f$$

Thus, for any $\mathfrak{a}, \mathfrak{b} \in \mathfrak{Z}_f$ we have $\mathfrak{a} * \mathfrak{b} \in \mathfrak{Z}_f$. Hence $\mathfrak{Z}_f$ is a BCK-subalgebra.

Theorem 5.2. In the discrete dynamical system $(\mathfrak{Z}, \Psi)$, the set of all periodic points in $\mathfrak{Z}$ is a BCK-subalgebra of $\mathfrak{Z}$.

Proof. Let $\mathfrak{Z}$ be a BCK-algebra and a mapping $\Psi: \mathfrak{Z} \longrightarrow \mathfrak{Z}$ is a homomorphism Suppose that $\mathfrak{Z}_p$ be the set of all periodic points in BCK-algebra. We show that $\mathfrak{Z}_p$ is a BCK-subalgebra for this $\mathfrak{Z}_p$ has to satisfy the conditions of Definition 2.3. Since Ψ is a homomorphism therefore by Proposition 4.1, we have $\Psi(0) = 0 \Longrightarrow$

0 is a periodic point of a period 1 $\Longrightarrow$ $0 \in \mathfrak{Z}_p \Longrightarrow \mathfrak{Z}_p \neq \{\ \}$. Thus, condition (s-i) of Definition 2.3 holds in $\mathfrak{Z}_p$.

Let $\mathfrak{a}, \mathfrak{b} \in \mathfrak{Z}_p$ and suppose that the periods of 'a' and '$\mathfrak{b}$' are m and n respectively.

Such that

$$\Psi^m(\mathfrak{a}) = \mathfrak{a} \tag{9}$$

and

$$\Psi^n(\mathfrak{b}) = \mathfrak{b} \tag{10}$$

Here we take r =LCM$[m, n]$, then by Proposition 4.3, equations (9) and (10) become

$$\Psi^r(\mathfrak{a}) = \mathfrak{a} \tag{11}$$

and

$$\Psi^r(\mathfrak{b}) = \mathfrak{b} \tag{12}$$

Next, by Proposition 2.2, we get

$$\Psi^r(\mathfrak{a} * \mathfrak{b}) = \Psi^r(\mathfrak{a}) * \Psi^r(\mathfrak{b}) \tag{13}$$

Using the values of (11) and (12) on the right-hand side of (13), we get

$$\Psi^r(\mathfrak{a} * \mathfrak{b}) = \mathfrak{a} * \mathfrak{b} \Longrightarrow \mathfrak{a} * \mathfrak{b} \text{ is periodic of period '}r\text{'} \Longrightarrow \mathfrak{a} * \mathfrak{b} \in \mathfrak{Z}_p$$

Thus, for any $\mathfrak{a}, \mathfrak{b} \in \mathfrak{Z}_p$ we have $\mathfrak{a} * \mathfrak{b} \in \mathfrak{Z}_p$. Hence $\mathfrak{Z}_p$ is a BCK-subalgebra.

Theorem 5.3. Let $(\mathfrak{Z}, \Psi)$ is a discrete dynamical system, then the set of all fixed points in $\mathfrak{Z}$ is a strongly invariant (S-invariant) subset of BCK-algebra $\mathfrak{Z}$.

PROOF. Let $\mathfrak{Z}_f$ be the set of all fixed points in $\mathfrak{Z}$.

Let $\Psi(\mathfrak{a}) \in \Psi(\mathfrak{Z}_f)$. Where '$\mathfrak{a}$' is any element of $\mathfrak{Z}$.

$\Longrightarrow \Psi(\mathfrak{a}) = \mathfrak{a}$, $\quad \because \Psi(\mathfrak{Z}_f)$ is a set of the images of all the fixed points

$\Longrightarrow \mathfrak{a} \in \mathfrak{Z}_f$

$\Longrightarrow \Psi(\mathfrak{a}) \in \mathfrak{Z}_f$, $\quad \because \Psi(\mathfrak{a}) = \mathfrak{a}$

Thus, $\Psi(\mathfrak{a}) \in \Psi(\mathfrak{Z}_f) \Longrightarrow \Psi(\mathfrak{a}) \in \mathfrak{Z}_f$

Therefore, we have

$$\Psi(\mathfrak{Z}_f) \subseteq \mathfrak{Z}_f \tag{14}$$

Now, we suppose that

$$\mathfrak{a} \in \mathfrak{Z}_f \Longrightarrow \Psi(\mathfrak{a}) = \mathfrak{a} \tag{15}$$

where

$$\Psi(\mathfrak{a}) \in \Psi(\mathfrak{Z}_f) \Longrightarrow \mathfrak{a} \in \Psi(\mathfrak{Z}_f), \qquad \because \text{using (15)}$$

Thus, $\mathfrak{a} \in \mathfrak{Z}_f \Longrightarrow \mathfrak{a} \in \Psi(\mathfrak{Z}_f)$. Therefore, we have

$$\mathfrak{Z}_f \subseteq \Psi(\mathfrak{Z}_f) \tag{16}$$

From (14) and (16), we have $\Psi(\mathfrak{Z}_f) = \mathfrak{Z}_f$. Hence, $\mathfrak{Z}_f$ in $\mathfrak{Z}$ is S-invariant or strongly invariant.

Theorem 5.4. In the discrete dynamical system $(Ȥ, \Psi)$ if '$\mathfrak{a}$' is a fixed point and '$\mathfrak{b}$' is a periodic point in Ȥ, then $\mathfrak{a} * \mathfrak{b}$ is a periodic point in Ȥ.

Proof. Since '$\mathfrak{a}$' is a fixed point, therefore we have

$$\Psi(\mathfrak{a}) = \mathfrak{a} \tag{17}$$

Let the period of the point 'b' is 'k' such that

$$\Psi^k(\mathfrak{b}) = \mathfrak{b} \tag{18}$$

As 'a' is a fixed point of period 1 and 1 divides 'k'; therefore, by Proposition 4.3, we have

$$\Psi^k(\mathfrak{a}) = \mathfrak{a} \tag{19}$$

Now by using the homomorphism property of Ψ^k we can get

$$\Psi^k(\mathfrak{a} * \mathfrak{b}) = \Psi^k(\mathfrak{a}) * \Psi^k(\mathfrak{b}) \tag{20}$$

Using (18) and (19) in (20), we get

$$\Psi^k(\mathfrak{a} * \mathfrak{b}) = \mathfrak{a} * \mathfrak{b} \tag{21}$$

Equation (5) implies that the period of $\mathfrak{a} * \mathfrak{b}$ is 'k' hence $\mathfrak{a} * \mathfrak{b}$ is a periodic point.

Theorem 5.5. In the discrete dynamical system $(Ȥ, \Psi)$ if A is a subset of Ȥ such that $\Psi(Ȥ) \subset A \subset Ȥ$, then A is invariant concerning Ψ.

Proof. Since

$$\Psi(Ȥ) \subset A \subset Ȥ \tag{22}$$

From equation (22), we have

$$A \subset Ȥ \tag{23}$$

and

$$\Psi(Ȥ) \subset A \tag{24}$$

From (23), we can get

$$\Psi(A) \subset \Psi(Ȥ) \tag{25}$$

From (24) and (25), we get

$$\Psi(A) \subset \Psi(Ȥ) \subset A \tag{26}$$

Equation (26) implies that $\Psi(A) \subset A$. Hence A is an invariant subset of BCK-algebra Ȥ.

Example 5.1. Consider the BCK-algebra $Ȥ = \{0, p, q, r\}$ and a mapping $\Psi: Ȥ \longrightarrow Ȥ$ where the mapping is a homomorphism defined by $\Psi(0) = 0, \Psi(p) = 0, \Psi(q) = 0$, and $\Psi(r) = r$. Let the subset of Ȥ is $A = \{0, p, r\}$ while $\Psi(Ȥ) = \{0, r\}$. Then, it is clear that $\Psi(Ȥ) \subset A \subset Ȥ$. Hence A is an invariant set of Ȥ.

Theorem 5.6. In the discrete dynamical system $(Ȥ, \Psi)$, if Ȥ is a commutative BCK-algebra and for any $\mathfrak{a} * \mathfrak{b}, \mathfrak{b} \in Ȥ_f$ where $Ȥ_f$ is the set of all fixed points in Ȥ and $\mathfrak{a} \geq \mathfrak{b}$ for all $\mathfrak{a}, \mathfrak{b} \in Ȥ$, then $Ȥ_f$ is an ideal of Ȥ.

Proof. Since Ȥ is a commutative BCK-algebra and a mapping $\Psi: Ȥ \longrightarrow Ȥ$ is a homomorphism. $Ȥ_f$ is the set of all fixed points in Ȥ. We show that $Ȥ_f$ is an ideal of Ȥ for this $Ȥ_f$ has to satisfy the conditions of Definition 2.4. Since Ψ is a homomorphism therefore by Proposition 4.1, we have

$$\Psi(0) = 0 \implies 0 \in Ȥ_f \implies Ȥ_f \neq \{\ \}$$

Thus, the first condition of ideal holds in $Ȥ_f$. Next, we have $\mathfrak{a} * \mathfrak{b} \in Ȥ_f$ and $\mathfrak{b} \in Ȥ_f$. Then,

$$\Psi\,(\mathfrak{a} * \mathfrak{b}) = \mathfrak{a} * \mathfrak{b} \tag{27}$$

and

$$\Psi(\mathfrak{b}) = \mathfrak{b} \tag{28}$$

We know that Ψ is a homomorphism, i.e., $\Psi\,(\mathfrak{a} * b) = \Psi(\mathfrak{a}) * \Psi(b)$ so using this in (27) we get

$$\Psi\,(\mathfrak{a}) * \Psi(\mathfrak{b}) = \mathfrak{a} * \mathfrak{b} \tag{29}$$

Using (28) in (29), we get

$$\Psi\,(\mathfrak{a}) * \mathfrak{b} = \mathfrak{a} * \mathfrak{b} \tag{30}$$

Since $\mathfrak{a} \geq \mathfrak{b}$ and by Proposition 4.4 $\Psi\,(\mathfrak{a}) \geq \mathfrak{b}$, then by Definition 2.6 from (30), we get

$$\Psi\,(\mathfrak{a}) = \mathfrak{a} \Longrightarrow \mathfrak{a} \in Ⱬ_f$$

Thus, $\mathfrak{a} * \mathfrak{b} \in Ⱬ_f$ and $\mathfrak{b} \in Ⱬ_f \Longrightarrow \mathfrak{a} \in Ⱬ_f$. Thus, the second condition of Definition 2.4 also holds in $Ⱬ_f$. Hence $Ⱬ_f$ is an ideal of commutative BCK-algebra Ⱬ.

Theorem 5.7. In the discrete dynamical system $(Ⱬ, \Psi)$ if Ⱬ is a commutative BCK-algebra and for any $\mathfrak{a} * \mathfrak{b}, \mathfrak{b} \in Ⱬ_p$ where $Ⱬ_p$ is a set of all periodic points in Ⱬ and $\mathfrak{a} \geq \mathfrak{b}$ for all $\mathfrak{a}, \mathfrak{b} \in Ⱬ$ then $Ⱬ_p$ is an ideal of BCK-algebra Ⱬ.

Proof. Since Ⱬ is a BCK-algebra and a mapping $\Psi: Ⱬ \longrightarrow Ⱬ$ is a homomorphism. $Ⱬ_p$ is a set of all periodic points in Ⱬ. We show that $Ⱬ_p$ is an ideal of Ⱬ for this $Ⱬ_p$ has to satisfy the conditions of Definition 2.4. Since Ψ is a homomorphism therefore by Proposition 4.1, we have $\Psi(0) = 0$ that implies that 0 is a periodic point of period 1 $\Longrightarrow 0 \in Ⱬ_p \Longrightarrow Ⱬ_p \neq \{\ \}$. Thus, the first condition of ideal holds in $Ⱬ_p$. Next, we have $\mathfrak{a} * \mathfrak{b} \in Ⱬ_p$ and $\mathfrak{b} \in Ⱬ_p$ and let their periods are 'm' and 'n', respectively, such that

$$\Psi^{m}(\mathfrak{a} * \mathfrak{b}) = \mathfrak{a} * \mathfrak{b} \tag{31}$$

and

$$\Psi^{n}(\mathfrak{b}) = \mathfrak{b} \tag{32}$$

Here, we take $r = LCM\,[m, n]$, then by Proposition 4.3 we can get

$$\Psi^{r}(\mathfrak{a} * \mathfrak{b}) = \mathfrak{a} * \mathfrak{b} \tag{33}$$

and

$$\Psi^{r}(\mathfrak{b}) = \mathfrak{b} \tag{34}$$

By Proposition 4.2 equation (33) becomes

$$\Psi^{r}(\mathfrak{a}) * \Psi^{r}(\mathfrak{b}) = \mathfrak{a} * \mathfrak{b}, \because \Psi^{r} \text{ is homomorphism} \tag{35}$$

Using equation (34) in (35), we get

$$\Psi^{r}(\mathfrak{a}) * \mathfrak{b} = \mathfrak{a} * \mathfrak{b} \tag{36}$$

Since $\mathfrak{a} \geq b$ and by Proposition 4.5 $\Psi^{r}(\mathfrak{a}) \geq \mathfrak{b}$, then by Definition 2.6 from (36), we get

$$\Psi^{r}(\mathfrak{a}) = \mathfrak{a} \Longrightarrow \mathfrak{a} \in Ⱬ_p$$

Thus, $\mathfrak{a} * \mathfrak{b} \in Ⱬ_p$ and $\mathfrak{b} \in Ⱬ_p \Longrightarrow \mathfrak{a} \in Ⱬ_p$. Thus, the second condition of Definition 2.4 also holds in $Ⱬ_p$. Hence $Ⱬ_p$ is an ideal of BCK-algebra Ⱬ.

6. Conclusion

We see that a discrete dynamical system with unique properties plays a central role in investigating the structure of an algebraic system.

We have no doubt that the research along this line can be kept up, and indeed, some results in this manuscript have already made up a foundation for further exploration concerning the further progression of a discrete dynamical system in BCK-algebra and their applications in other disciplines of algebra. The forthcoming study of a discrete dynamical system in BCK-algebras may be the following topics are worth to be taken into account.

(i) To describe other classes of BCK-algebra by using this concept.

(ii) To refer this concept to some other algebraic structures.

(iii) To consider the results of this concept to some possible applications in information systems and computer sciences.

Conflict of Interest

The authors declare no conflict of interest.

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