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AN EXTENSION OF TRAPEZOID INEQUALITY TO THE COMPLEX INTEGRAL

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Abstract. In this paper we extend the trapezoid inequality to the complex integral by providing upper bounds for the quantity

$$\left|(v-u)\, f(u)+(w-v)\, f(w)-\int_{\gamma} f(z)\, dz\right|$$

under the assumptions that γ is a smooth path parametrized by $z(t)$, $t\in[a,b]$, $u=z(a)$, $v=z(x)$ with $x\in(a,b)$ and $w=z(b)$ while f is holomorphic in G, an open domain and $\gamma\subset G$. An application for circular paths is also given.

1. Introduction

Inequalities providing upper bounds for the quantity

$$\left|(t-a)\, f(a)+(b-t)\, f(b)-\int_a^b f(s)\, ds\right|, \qquad t\in[a,b] \tag{1}$$

are known in the literature as *generalized trapezoid inequalities* and it has been shown in [2] that

$$\left|(t-a)\, f(a)+(b-t)\, f(b)-\int_a^b f(s)\, ds\right| \tag{2}$$
$$\leq\left[\frac{1}{2}+\left|\frac{t-\frac{a+b}{2}}{b-a}\right|\right](b-a)\bigvee_a^b(f)$$

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for any $t\in[a,b]$, provided that f is of bounded variation on $[a,b]$. The constant $\frac{1}{2}$ is the best possible.

If f is *absolutely continuous* on $[a,b]$, then (see [1, p. 93])

$$\left|(t-a)f(a)+(b-t)f(b)-\int_a^b f(s)\,ds\right| \tag{3}$$

$$\leq\begin{cases}\left[\frac{1}{4}+\left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)^2\|f'\|_\infty & \text{if } f'\in L_\infty[a,b];\\[2ex] \frac{1}{(q+1)^{1/q}}\left[\left(\frac{t-a}{b-a}\right)^{q+1}+\left(\frac{b-t}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}(b-a)^{1+1/q}\|f'\|_p & \text{if } f'\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \left[\frac{1}{2}+\left|\frac{t-\frac{a+b}{2}}{b-a}\right|\right](b-a)\|f'\|_1 & \end{cases}$$

for any $t\in[a,b]$. The constants $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{(q+1)^{1/q}}$ are the best possible.

Finally, for *convex functions* $f:[a,b]\to\mathbb{R}$, we have [4]

$$\begin{aligned}\frac{1}{2}&\left[(b-t)^2 f'_+(t)-(t-a)^2 f'_-(t)\right]\\ &\leq (b-t)f(b)+(t-a)f(a)-\int_a^b f(s)\,ds\\ &\leq\frac{1}{2}\left[(b-t)^2 f'_-(b)-(t-a)^2 f'_-(a)\right]\end{aligned} \tag{4}$$

for any $t\in(a,b)$, provided that $f'_-(b)$ and $f'_+(a)$ are finite. As above, the second inequality also holds for $t=a$ and $t=b$ and the constant $\frac{1}{2}$ is the best possible on both sides of (4).

For other recent results on the trapezoid inequality, see [3], [7], [8], [9] and [11].

In order to extend this result for the complex integral, we need some preparations as follows.

Suppose γ is a *smooth path* parametrized by $z(t)$, $t\in[a,b]$ and f is a complex function which is continuous on γ. Put $z(a)=u$ and $z(b)=w$ with $u,w\in\mathbb{C}$. We define the integral of f on $\gamma_{u,w}=\gamma$ as

$$\int_\gamma f(z)\,dz=\int_{\gamma_{u,w}} f(z)\,dz:=\int_a^b f(z(t))\,z'(t)\,dt.$$

We observe that that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are *piecewise smooth*. Suppose γ is parametrized by $z(t)$, $t\in[a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\,dz:=\int_{\gamma_{u,v}} f(z)\,dz+\int_{\gamma_{v,w}} f(z)\,dz$$

where $v := z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)|dz| := \int_a^b f(z(t))|z'(t)|\,dt$$

and the length of the curve γ is then

$$\ell(\gamma) = \int_{\gamma_{u,w}} |dz| = \int_a^b |z'(t)|\,dt.$$

Let f and g be holomorphic in G, an open domain and suppose $\gamma \subset G$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then we have the *integration by parts formula*

$$\int_{\gamma_{u,w}} f(z)g'(z)\,dz = f(w)g(w) - f(u)g(u) - \int_{\gamma_{u,w}} f'(z)g(z)\,dz. \tag{5}$$

We recall also the *triangle inequality* for the complex integral, namely

$$\left|\int_\gamma f(z)\,dz\right| \le \int_\gamma |f(z)||dz| \le \|f\|_{\gamma,\infty}\,\ell(\gamma) \tag{6}$$

where $\|f\|_{\gamma,\infty} := \sup_{z\in\gamma}|f(z)|$.

We also define the p-norm with $p \ge 1$ by

$$\|f\|_{\gamma,p} := \left(\int_\gamma |f(z)|^p|dz|\right)^{1/p}.$$

For $p = 1$ we have

$$\|f\|_{\gamma,1} := \int_\gamma |f(z)||dz|.$$

If p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's inequality we have

$$\|f\|_{\gamma,1} \le [\ell(\gamma)]^{1/q}\,\|f\|_{\gamma,p}.$$

In this paper we extend the trapezoid inequality to the complex integral, by providing upper bounds for the quantity

$$\left|(v-u)f(u) + (w-v)f(w) - \int_\gamma f(z)\,dz\right|$$

under the assumptions that γ is a smooth path parametrized by $z(t)$, $t \in [a,b]$, $u = z(a)$, $v = z(x)$ with $x \in (a,b)$ and $w = z(b)$ while f is holomorphic in G, an open domain and $\gamma \subset G$. An application for circular paths is also given.

2. Trapezoid Type Inequalities

We have the following result for functions of complex variable:

Theorem 1. *Let f be holomorphic in G, an open domain and suppose $\gamma \subset G$ is a smooth path from $z(a) = u$ to $z(b) = w$. If $v = z(x)$ with $x \in (a,b)$, then $\gamma_{u,w} = \gamma_{u,v} \cup \gamma_{v,w}$,*

$$\left|(v-u) f(u) + (w-v) f(w) - \int_\gamma f(z)\,dz\right| \le \|f'\|_{\gamma_{u,v};\infty} \int_{\gamma_{u,v}} |z-v|\,|dz| + \|f'\|_{\gamma_{v,w};\infty} \int_{\gamma_{v,w}} |z-v|\,|dz| \le \|f'\|_{\gamma_{u,w};\infty} \int_{\gamma_{u,w}} |z-v|\,|dz|, \quad (7)$$

and

$$\left|(v-u) f(u) + (w-v) f(w) - \int_\gamma f(z)\,dz\right| \le \|f'\|_{\gamma_{u,v};1} \max_{z\in\gamma_{u,v}} |z-v| + \|f'\|_{\gamma_{v,w};1} \max_{z\in\gamma_{v,w}} |z-v| \le \|f'\|_{\gamma_{u,w};1} \max_{z\in\gamma_{u,w}} |z-v|. \quad (8)$$

If $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\left|(v-u) f(u) + (w-v) f(w) - \int_\gamma f(z)\,dz\right| \le \|f'\|_{\gamma_{u,v};p} \left(\int_{\gamma_{u,v}} |z-v|^q\,|dz|\right)^{1/q} + \|f'\|_{\gamma_{v,w};p} \left(\int_{\gamma_{v,w}} |z-v|^q\,|dz|\right)^{1/q} \le \|f'\|_{\gamma_{u,w};p} \left(\int_{\gamma_{u,w}} |z-v|^q\,|dz|\right)^{1/q}. \quad (9)$$

Proof. Using the integration by parts formula (5) twice we have

$$\int_{\gamma_{u,v}} (z-v) f'(z)\,dz = (v-u) f(u) - \int_{\gamma_{u,v}} f(z)\,dz$$

and

$$\int_{\gamma_{v,w}} (z-v) f'(z)\,dz = (w-v) f(w) - \int_{\gamma_{v,w}} f(z)\,dz.$$

If we add these two equalities, we get the following equality of interest

$$(v-u) f(u) + (w-v) f(w) - \int_\gamma f(z)\,dz$$

$$= \int_{\gamma_{u,v}} (z-v) f'(z)\,dz + \int_{\gamma_{v,w}} (z-v) f'(z)\,dz = \int_{\gamma} (z-v) f'(z)\,dz \quad (10)$$

with the above assumptions for u, v and w on γ.

Using the properties of modulus and the triangle inequality for the complex integral we have

$$\begin{aligned}
&\left|(v-u) f(u) + (w-v) f(w) - \int_{\gamma} f(z)\,dz\right| \\
&\quad = \left|\int_{\gamma_{u,v}} (z-v) f'(z)\,dz + \int_{\gamma_{v,w}} (z-v) f'(z)\,dz\right| \\
&\quad \le \left|\int_{\gamma_{u,v}} (z-v) f'(z)\,dz\right| + \left|\int_{\gamma_{v,w}} (z-v) f'(z)\,dz\right| \\
&\quad \le \int_{\gamma_{u,v}} |z-v|\,|f'(z)|\,|dz| + \int_{\gamma_{v,w}} |z-v|\,|f'(z)|\,|dz| \\
&\le \|f'\|_{\gamma_{u,v};\infty} \int_{\gamma_{u,v}} |z-v|\,|dz| + \|f'\|_{\gamma_{v,w};\infty} \int_{\gamma_{v,w}} |z-v|\,|dz| \le \|f'\|_{\gamma_{u,w};\infty} \int_{\gamma_{u,w}} |z-v|\,|dz|,
\end{aligned}$$

which proves the inequality (7).

We also have

$$\begin{aligned}
&\int_{\gamma_{u,v}} |z-v|\,|f'(z)|\,|dz| + \int_{\gamma_{v,w}} |z-v|\,|f'(z)|\,|dz| \\
&\le \max_{z\in\gamma_{u,v}} |z-v| \int_{\gamma_{u,v}} |f'(z)|\,|dz| + \max_{z\in\gamma_{v,w}} |z-v| \int_{\gamma_{v,w}} |f'(z)|\,|dz| \\
&\le \max\left\{\max_{z\in\gamma_{u,v}} |z-v|,\, \max_{z\in\gamma_{v,w}} |z-v|\right\} \\
&\times \left(\int_{\gamma_{u,v}} |f'(z)|\,|dz| + \int_{\gamma_{v,w}} |f'(z)|\,|dz|\right) = \max_{z\in\gamma_{u,w}} |z-v| \int_{\gamma_{u,w}} |f'(z)|\,|dz|,
\end{aligned}$$

which proves the inequality (8).

If p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then by Hölder's weighted integral inequality we have

$$\begin{aligned}
&\int_{\gamma_{u,v}} |z-v|\,|f'(z)|\,|dz| + \int_{\gamma_{v,w}} |z-v|\,|f'(z)|\,|dz| \\
&\quad \le \left(\int_{\gamma_{u,v}} |z-v|^q\,|dz|\right)^{1/q} \left(\int_{\gamma_{u,v}} |f'(z)|^p\,|dz|\right)^{1/p} \\
&\qquad + \left(\int_{\gamma_{v,w}} |z-v|^q\,|dz|\right)^{1/q} \left(\int_{\gamma_{v,w}} |f'(z)|^p\,|dz|\right)^{1/p} =: B.
\end{aligned}$$

By the elementary inequality

$$ab + cd \le (a^p + c^p)^{1/p} (b^q + d^q)^{1/q},$$

where a, b, c, $d \ge 0$ and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we also have

$$\begin{aligned} B &\le \left(\int_{\gamma_{u,v}} |z - v|^q |dz| + \int_{\gamma_{v,w}} |z - v|^q |dz| \right)^{1/q} \\ &\times \left(\int_{\gamma_{u,v}} |f'(z)|^p |dz| + \int_{\gamma_{v,w}} |f'(z)|^p |dz| \right)^{1/p} \\ &= \left(\int_{\gamma_{u,w}} |z - v|^q |dz| \right)^{1/q} \left(\int_{\gamma_{u,w}} |f'(z)|^p |dz| \right)^{1/p}, \end{aligned}$$

which prove the desired result (9). □

If the path γ is a segment $[u, w] \subset G$ connecting two distinct points u and w in G then we write $\int_\gamma f(z)\, dz$ as $\int_u^w f(z)\, dz$.

Using the p-norms defined in the introduction for the segments, namely

$$\|h\|_{[u,w];\infty} = \sup_{z \in [u,w]} |h(z)|$$

and

$$\|h\|_{[u,w];p} = \left(\int_u^w |h(z)|^p |dz| \right)^{1/p} \text{ for } p \ge 1,$$

we can state the following particular case as well:

Corollary 1. *Let f be holomorphic in G, an open domain and suppose $[u, w] \subset G$ is a segment connecting two distinct points u and w in G and $v \in [u, w]$. Then for $v = (1 - s) u + sw$ with $s \in [0, 1]$, we have*

$$\begin{aligned} &\left| (v - u) f(u) + (w - v) f(w) - \int_u^w f(z)\, dz \right| \\ &\le \frac{1}{2} |w - u|^2 \left[s^2 \|f'\|_{\gamma_{u,v};\infty} + (1 - s)^2 \|f'\|_{\gamma_{v,w};\infty} \right] \\ &\le |w - u|^2 \left[\frac{1}{4} + \left(s - \frac{1}{2} \right)^2 \right] \|f'\|_{[u,w];\infty}, \quad (11) \end{aligned}$$

and

$$\begin{aligned} &\left| (v - u) f(u) + (w - v) f(w) - \int_u^w f(z)\, dz \right| \\ &\le |w - u| \left\{ s \|f'\|_{[u,v];1} + (1 - s) \|f'\|_{[v,w];1} \right\} \end{aligned}$$

$$\leq |w-u|\left(\frac{1}{2}+\left|s-\frac{1}{2}\right|\right)\|f'\|_{[u,w];1}. \quad (12)$$

If $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$\left|(v-u)f(u)+(w-v)f(w)-\int_\gamma f(z)\,dz\right|$$
$$\leq \frac{1}{(q+1)^{1/q}}|w-u|^{1+1/q}\left[s^{1+1/q}\|f'\|_{[u,v];p}+(1-s)^{1+1/q}\|f'\|_{[v,w];p}\right]$$
$$\leq \frac{1}{(q+1)^{1/q}}|w-u|^{1+1/q}\left[s^{q+1}+(1-s)^{q+1}\right]^{1/q}\|f'\|_{[u,w];p}. \quad (13)$$

Proof. Observe that if the segment $[u,w]$ is parametrized by $z(t)=(1-t)u+tw$, then $z'(t)=w-u$

$$\int_u^v |z-v|\,|dz| = |w-u|\int_0^s |(1-t)u+tw-(1-s)u-sw|\,dt$$
$$= |w-u|^2\int_0^s (s-t)\,dt = \frac{1}{2}|w-u|^2 s^2$$

and

$$\int_v^w |z-v|\,|dz| = |w-u|\int_s^1 |(1-t)u+tw-(1-s)u-sw|\,dt$$
$$= |w-u|^2\int_s^1 (t-s)\,dt = \frac{1}{2}|w-u|^2(1-s)^2.$$

Using the inequality (7) we get

$$\left|(v-u)f(u)+(w-v)f(w)-\int_\gamma f(z)\,dz\right|$$
$$\leq \frac{1}{2}|w-u|^2 s^2\|f'\|_{\gamma_{u,v};\infty}+\frac{1}{2}|w-u|^2(1-s)^2\|f'\|_{\gamma_{v,w};\infty}$$
$$\leq \frac{1}{2}|w-u|^2\left[s^2+(1-s)^2\right]\|f'\|_{\gamma_{u,w};\infty} = |w-u|^2\left[\frac{1}{4}+\left(s-\frac{1}{2}\right)^2\right]\|f'\|_{[u,w];\infty},$$

which proves (11).

Also

$$\max_{z\in\gamma_{u,v}}|z-v| = \max_{t\in[0,s]}|(1-t)u+tw-(1-s)u-sw| = |w-u|\,s$$

and

$$\max_{z\in\gamma_{v,w}}|z-v| = \max_{t\in[s,1]}\{|w-u|(1-t)\} = |w-u|(1-s),$$

then by (8)

$$\left|(v-u) f(u)+(w-v) f(w)-\int_{\gamma} f(z) dz\right|$$
$$\leq|w-u|\left\{s\|f'\|_{[u, v] ; 1}+(1-s)\|f'\|_{[v, w] ; 1}\right\}$$
$$\leq|w-u| \max \{s, 1-s\}\|f'\|_{[u, w] ; 1}=|w-u|\left(\frac{1}{2}+\left|s-\frac{1}{2}\right|\right)\|f'\|_{[u, w] ; 1},$$

which proves (12).

Finally, since

$$\int_{u}^{v}|z-v|^{q}|dz|=|w-u| \int_{0}^{s}|(1-t) u+t w-(1-s) u-s w|^{q} dt$$
$$=|w-u|^{q+1} \int_{0}^{s}(s-t)^{q} dt=\frac{1}{q+1} s^{q+1}|w-u|^{q+1}$$

and

$$\int_{v}^{w}|z-v|^{q}|dz|=|w-u| \int_{s}^{1}|(1-t) u+t w-(1-s) u-s w|^{q} dt$$
$$=|w-u|^{q+1} \int_{s}^{1}(t-s)^{q} dt=\frac{1}{q+1}(1-s)^{q+1}|w-u|^{q+1},$$

hence by (9) we get (13). □

Remark 1. *Let f be holomorphic in G, an open domain and suppose $[u, w] \subset G$ is a segment connecting two distinct points u and w in G. Then*

$$\left|\frac{f(u)+f(w)}{2}(w-u)-\int_{u}^{w} f(z) dz\right|$$
$$\leq \frac{1}{8}|w-u|^{2}\left[\|f'\|_{\gamma_{u, \frac{u+w}{2}} ; \infty}+\|f'\|_{\gamma_{\frac{u+w}{2}, w} ; \infty}\right] \leq \frac{1}{4}|w-u|^{2}\|f'\|_{[u, w] ; \infty}, \tag{14}$$

and

$$\left|\frac{f(u)+f(w)}{2}(w-u)-\int_{u}^{w} f(z) dz\right| \leq \frac{1}{2}|w-u|\|f'\|_{[u, w] ; 1}. \tag{15}$$

If $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then

$$\left|\frac{f(u)+f(w)}{2}(w-u)-\int_{u}^{w} f(z) dz\right|$$
$$\leq \frac{1}{2^{1+1/q}(q+1)^{1/q}}|w-u|^{1+1/q}\left[\|f'\|_{\left[u, \frac{u+w}{2}\right] ; p}+\|f'\|_{\left[\frac{u+w}{2}, w\right] ; p}\right]$$
$$\leq \frac{1}{2(q+1)^{1/q}}|w-u|^{1+1/q}\|f'\|_{[u, w] ; p}. \tag{16}$$

Suppose that $\gamma \subset G$ is a smooth path from $z(a)=u$ to $z(b)=w$. If $v=z(x)$ with $x \in(a, b)$, then $\gamma_{u, w}=\gamma_{u, v} \cup \gamma_{v, w}$.

If we consider $f(z) = \exp(z)$ with $z \in \mathbb{C}$, then

$$\int_{\gamma_{u,w}} \exp(z)\, dz = \exp(w) - \exp(u),$$

$$|\exp(z)| = |\exp(\operatorname{Re}(z) + i \operatorname{Im}(z))| = \exp(\operatorname{Re}(z))$$

and by Theorem 1 we have

$$\begin{aligned} &|(v-u)\exp u + (w-v)\exp w - \exp(w) + \exp(u)| \\ &\quad \leq \|\exp(\operatorname{Re}(\cdot))\|_{\gamma_{u,v};\infty} \int_{\gamma_{u,v}} |z-v|\,|dz| \\ &\quad + \|\exp(\operatorname{Re}(\cdot))\|_{\gamma_{v,w};\infty} \int_{\gamma_{v,w}} |z-v|\,|dz| \\ &\quad \leq \|\exp(\operatorname{Re}(\cdot))\|_{\gamma_{u,w};\infty} \int_{\gamma_{u,w}} |z-v|\,|dz|, \end{aligned} \tag{17}$$

and

$$\begin{aligned} &|(v-u)\exp u + (w-v)\exp w - \exp(w) + \exp(u)| \\ &\quad \leq \|\exp(\operatorname{Re}(\cdot))\|_{\gamma_{u,v};1} \max_{z\in\gamma_{u,v}} |z-v| + \|\exp(\operatorname{Re}(\cdot))\|_{\gamma_{v,w};1} \max_{z\in\gamma_{v,w}} |z-v| \\ &\quad \leq \|\exp(\operatorname{Re}(\cdot))\|_{\gamma_{u,w};1} \max_{z\in\gamma_{u,w}} |z-v|. \end{aligned} \tag{18}$$

If $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\begin{aligned} &|(v-u)\exp u + (w-v)\exp w - \exp(w) + \exp(u)| \\ &\quad \leq \|\exp(\operatorname{Re}(\cdot))\|_{\gamma_{u,v};p} \left(\int_{\gamma_{u,v}} |z-v|^q\,|dz| \right)^{1/q} \\ &\quad + \|\exp(\operatorname{Re}(\cdot))\|_{\gamma_{v,w};p} \left(\int_{\gamma_{v,w}} |z-v|^q\,|dz| \right)^{1/q} \\ &\quad \leq \|\exp(\operatorname{Re}(\cdot))\|_{\gamma_{u,w};p} \left(\int_{\gamma_{u,w}} |z-v|^q\,|dz| \right)^{1/q}. \end{aligned} \tag{19}$$

With the same assumption of the path γ and if we consider $f(z) = z^n$ with $n \geq 1$, then

$$\int_{\gamma} z^n dz = \frac{w^{n+1} - u^{n+1}}{n+1}$$

and by Theorem 1 we get, by denoting $\ell(z) = z$, $z \in \mathbb{C}$, that

$$\left| (v-u)\, u^n + (w-v)\, w^n - \frac{w^{n+1} - u^{n+1}}{n+1} \right|$$

$$\leq n\left[\left\|\ell^{n-1}\right\|_{\gamma_{u,v};\infty}\int_{\gamma_{u,v}}|z-v|\,|dz|+\left\|\ell^{n-1}\right\|_{\gamma_{v,w};\infty}\int_{\gamma_{v,w}}|z-v|\,|dz|\right]$$
$$\leq n\left\|\ell^{n-1}\right\|_{\gamma_{u,w};\infty}\int_{\gamma_{u,w}}|z-v|\,|dz|, \quad (20)$$

and

$$\left|(v-u)\,u^{n}+(w-v)\,w^{n}-\frac{w^{n+1}-u^{n+1}}{n+1}\right|$$
$$\leq n\left[\left\|\ell^{n-1}\right\|_{\gamma_{u,v};1}\max_{z\in\gamma_{u,v}}|z-v|+\left\|\ell^{n-1}\right\|_{\gamma_{v,w};1}\max_{z\in\gamma_{v,w}}|z-v|\right]$$
$$\leq n\left\|\ell^{n-1}\right\|_{\gamma_{u,w};1}\max_{z\in\gamma_{u,w}}|z-v|. \quad (21)$$

If $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then

$$\left|(v-u)\,u^{n}+(w-v)\,w^{n}-\frac{w^{n+1}-u^{n+1}}{n+1}\right|$$
$$\leq n\left[\left\|\ell^{n-1}\right\|_{\gamma_{u,v};p}\left(\int_{\gamma_{u,v}}|z-v|^{q}\,|dz|\right)^{1/q}+\left\|\ell^{n-1}\right\|_{\gamma_{v,w};p}\left(\int_{\gamma_{v,w}}|z-v|^{q}\,|dz|\right)^{1/q}\right]$$
$$\leq n\left\|\ell^{n-1}\right\|_{\gamma_{u,w};p}\left(\int_{\gamma_{u,w}}|z-v|^{q}\,|dz|\right)^{1/q}, \quad (22)$$

where $\gamma\subset G$ is a smooth path from $z(a)=u$ to $z(b)=w$ and $v=z(x)$ with $x\in(a,b)$.

3. Examples For Circular Paths

Let $[a,b]\subseteq[0,2\pi]$ and the circular path $\gamma_{[a,b],R}$ centered in 0 and with radius $R>0$

$$z(t)=R\exp(it)=R(\cos t+i\sin t),\ t\in[a,b].$$

If $[a,b]=[0,\pi]$ then we get a half circle while for $[a,b]=[0,2\pi]$ we get the full circle.

Since

$$\left|e^{is}-e^{it}\right|^{2}=\left|e^{is}\right|^{2}-2\operatorname{Re}\left(e^{i(s-t)}\right)+\left|e^{it}\right|^{2}$$
$$=2-2\cos(s-t)=4\sin^{2}\left(\frac{s-t}{2}\right)$$

for any $t,\ s\in\mathbb{R}$, then

$$\left|e^{is}-e^{it}\right|^{r}=2^{r}\left|\sin\left(\frac{s-t}{2}\right)\right|^{r} \quad (23)$$

for any t, $s \in \mathbb{R}$ and $r > 0$. In particular,

$$\left|e^{is} - e^{it}\right| = 2\left|\sin\left(\frac{s-t}{2}\right)\right|$$

for any t, $s \in \mathbb{R}$.

For t, $x \in [a,b] \subseteq [0,2\pi]$ we then have

$$\left|e^{ix} - e^{it}\right| = 2\left|\sin\left(\frac{x-t}{2}\right)\right|.$$

If $u = R\exp(ia)$, $v = R\exp(ix)$ and $w = R\exp(ib)$ then

$$\begin{aligned} v - u &= R\left[\exp(ix) - \exp(ia)\right] = R\left[\cos x + i\sin x - \cos a - i\sin a\right] \\ &= R\left[\cos x - \cos a + i\left(\sin x - \sin a\right)\right]. \end{aligned}$$

Since

$$\cos x - \cos a = -2\sin\left(\frac{a+x}{2}\right)\sin\left(\frac{x-a}{2}\right)$$

and

$$\sin x - \sin a = 2\sin\left(\frac{x-a}{2}\right)\cos\left(\frac{a+x}{2}\right),$$

hence

$$\begin{aligned} v - u &= R\left[-2\sin\left(\frac{a+x}{2}\right)\sin\left(\frac{x-a}{2}\right) + 2i\sin\left(\frac{x-a}{2}\right)\cos\left(\frac{a+x}{2}\right)\right] \\ &= 2R\sin\left(\frac{x-a}{2}\right)\left[-\sin\left(\frac{a+x}{2}\right) + i\cos\left(\frac{a+x}{2}\right)\right] \\ &= 2Ri\sin\left(\frac{x-a}{2}\right)\left[\cos\left(\frac{a+x}{2}\right) + i\sin\left(\frac{a+x}{2}\right)\right] \\ &= 2Ri\sin\left(\frac{x-a}{2}\right)\exp\left[\left(\frac{a+x}{2}\right)i\right]. \end{aligned}$$

Similarly,

$$w - v = 2Ri\sin\left(\frac{b-x}{2}\right)\exp\left[\left(\frac{x+b}{2}\right)i\right]$$

for $a \le x \le b$.

Moreover,

$$z - v = 2Ri\sin\left(\frac{t-x}{2}\right)\exp\left[\left(\frac{t+b}{2}\right)i\right]$$

and

$$|z - v| = \left|2Ri\sin\left(\frac{t-x}{2}\right)\exp\left[\left(\frac{t+b}{2}\right)i\right]\right| = 2R\left|\sin\left(\frac{t-x}{2}\right)\right|$$

for $a \le x$, $t \le b$.

We also have

$$z'(t) = Ri\exp(it) \text{ and } |z'(t)| = R$$

for $t\in[a,b]$.

Proposition 1. *Let f be holomorphic in G, on open domain and suppose $\gamma_{[a,b],R}\subset G$ with $[a,b]\subseteq[0,2\pi]$ and $R>0$. If $x\in[a,b]$, then*

$$\begin{aligned}&\left|\sin\left(\frac{x-a}{2}\right)\exp\left[\left(\frac{a+x}{2}\right)i\right]f\left(R\exp\left(ia\right)\right)\right.\\&\qquad+\sin\left(\frac{b-x}{2}\right)\exp\left[\left(\frac{x+b}{2}\right)i\right]f\left(R\exp\left(ib\right)\right)\\&\qquad\left.-\frac{1}{2}\int_a^b f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\right|\\&\leq 4R\left[\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,x],\infty}\sin^2\left(\frac{x-a}{4}\right)\right.\\&\qquad\left.+\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[x,b],\infty}\sin^2\left(\frac{b-x}{4}\right)\right]\\&\leq 4R\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,b],\infty}\left[\sin^2\left(\frac{x-a}{4}\right)+\sin^2\left(\frac{b-x}{4}\right)\right].\end{aligned}\tag{24}$$

Proof. We write the inequality (7) for $\gamma_{[a,b],R}$ and $x\in[a,b]$ to get

$$\begin{aligned}&\left|2Ri\sin\left(\frac{x-a}{2}\right)\exp\left[\left(\frac{a+x}{2}\right)i\right]f\left(R\exp\left(ia\right)\right)\right.\\&\qquad+2Ri\sin\left(\frac{b-x}{2}\right)\exp\left[\left(\frac{x+b}{2}\right)i\right]f\left(R\exp\left(ib\right)\right)\\&\qquad\left.-Ri\int_a^b f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\right|\\&\leq 2R^2\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,x],\infty}\int_a^b\left|\sin\left(\frac{t-x}{2}\right)\right|dt\\&\qquad+2R^2\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[x,b],\infty}\int_x^x\left|\sin\left(\frac{t-x}{2}\right)\right|dt.\end{aligned}$$

This is equivalent to

$$\begin{aligned}&\left|\sin\left(\frac{x-a}{2}\right)\exp\left[\left(\frac{a+x}{2}\right)i\right]f\left(R\exp\left(ia\right)\right)\right.\\&\qquad+\sin\left(\frac{b-x}{2}\right)\exp\left[\left(\frac{x+b}{2}\right)i\right]f\left(R\exp\left(ib\right)\right)\\&\qquad\left.-\frac{1}{2}\int_a^b f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\right|\end{aligned}$$

$$\leq R\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,x],\infty}\int_a^x\left|\sin\left(\frac{t-x}{2}\right)\right|dt + R\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[x,b],\infty}\int_x^b\left|\sin\left(\frac{t-x}{2}\right)\right|dt \quad (25)$$

for $x\in[a,b]$.

Observe that

$$\int_a^x\left|\sin\left(\frac{t-x}{2}\right)\right|dt=\int_a^x\sin\left(\frac{x-t}{2}\right)dt=2-2\cos\left(\frac{x-a}{2}\right) = 4\sin^2\left(\frac{x-a}{4}\right)$$

and

$$\int_x^b\left|\sin\left(\frac{t-x}{2}\right)\right|dt=\int_x^b\sin\left(\frac{t-x}{2}\right)dt=2-2\cos\left(\frac{b-t}{2}\right) = 4\sin^2\left(\frac{b-x}{4}\right),$$

which by (25) produce the desired result (24). □

Corollary 2. *With the assumptions of Proposition 1 we have*

$$\left|\sin\left(\frac{b-a}{4}\right)\exp\left[\left(\frac{3a+b}{4}\right)i\right]f\left(R\exp\left(ia\right)\right) + \sin\left(\frac{b-a}{4}\right)\exp\left[\left(\frac{a+3b}{4}\right)i\right]f\left(R\exp\left(ib\right)\right) -\frac{1}{2}\int_a^b f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\right|$$
$$\leq 4R\left[\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,x],\infty}+\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[x,b],\infty}\right]\sin^2\left(\frac{b-a}{8}\right)$$
$$\leq 8R\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,b],\infty}\sin^2\left(\frac{b-a}{8}\right). \quad (26)$$

Remark 2. *The case of semi-circle, namely $a=0$ and $b=\pi$ in (24) gives the inequality*

$$\left|\sin\left(\frac{x}{2}\right)\exp\left[\left(\frac{x}{2}\right)i\right]f\left(R\right)+i\cos\left(\frac{x}{2}\right)\exp\left[\left(\frac{x}{2}\right)i\right]f\left(-R\right) -\frac{1}{2}\int_0^\pi f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\right|$$
$$\leq 4R\left[\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[0,x],\infty}\sin^2\left(\frac{x}{4}\right)\right.$$

$$+\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[x,\pi],\infty}\sin^{2}\left(\frac{\pi-x}{4}\right)\Bigg]$$
$$\leq 4R\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[0,\pi],\infty}\left[\sin^{2}\left(\frac{x}{4}\right)+\sin^{2}\left(\frac{\pi-x}{4}\right)\right],\quad(27)$$

for $x\in[0,\pi]$.

Since
$$\sin^{2}\left(\frac{\pi}{8}\right)=\frac{1-\cos\left(\frac{\pi}{4}\right)}{2}=\frac{1-\frac{\sqrt{2}}{2}}{2}=\frac{2-\sqrt{2}}{4},$$
then by taking $x=\frac{\pi}{2}$ in (27), we get

$$\left|\frac{1+i}{2}f\left(R\right)+\frac{-1+i}{2}f\left(-R\right)-\frac{1}{2}\int_{0}^{\pi}f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\right|$$
$$\leq\left(2-\sqrt{2}\right)\left[\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[0,\frac{\pi}{2}],\infty}+\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[\frac{\pi}{2},\pi],\infty}\right]$$
$$\leq 2\left(2-\sqrt{2}\right)\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[0,\pi],\infty}.\quad(28)$$

Further, we have the following result as well:

Proposition 2. *With the assumptions of Proposition 1 we have*

$$\Bigg|\sin\left(\frac{x-a}{2}\right)\exp\left[\left(\frac{a+x}{2}\right)i\right]f\left(R\exp\left(ia\right)\right)$$
$$+\sin\left(\frac{b-x}{2}\right)\exp\left[\left(\frac{x+b}{2}\right)i\right]f\left(R\exp\left(ib\right)\right)$$
$$-\frac{1}{2}\int_{a}^{b}f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\Bigg|$$
$$\leq R\left[\max_{t\in[a,x]}\left|\sin\left(\frac{t-x}{2}\right)\right|\int_{a}^{x}\left|f'\left(R\exp\left(it\right)\right)\right|dt\right.$$
$$\left.+\max_{t\in[x,b]}\left|\sin\left(\frac{t-x}{2}\right)\right|\int_{x}^{b}\left|f'\left(R\exp\left(it\right)\right)\right|dt\right]$$
$$\leq R\max_{t\in[a,b]}\left|\sin\left(\frac{t-x}{2}\right)\right|\int_{a}^{b}\left|f'\left(R\exp\left(it\right)\right)\right|dt.\quad(29)$$

Proof. We write the inequality (8) for $\gamma_{[a,b],R}$ and $x\in[a,b]$ to get

$$\Bigg|2Ri\sin\left(\frac{x-a}{2}\right)\exp\left[\left(\frac{a+x}{2}\right)i\right]f\left(R\exp\left(ia\right)\right)$$
$$+2Ri\sin\left(\frac{b-x}{2}\right)\exp\left[\left(\frac{x+b}{2}\right)i\right]f\left(R\exp\left(ib\right)\right)$$

$$
\begin{aligned}
&-Ri\int_a^b f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\Bigg|\\
&\leq 2R^2\left[\max_{t\in[a,x]}\left|\sin\left(\frac{t-x}{2}\right)\right|\int_a^x\left|f'\left(R\exp\left(it\right)\right)\right|dt\right.\\
&\quad\left.+\max_{t\in[x,b]}\left|\sin\left(\frac{t-x}{2}\right)\right|\int_x^b\left|f'\left(R\exp\left(it\right)\right)\right|dt\right]\\
&\leq 2R^2\max_{t\in[a,b]}\left|\sin\left(\frac{t-x}{2}\right)\right|\int_a^b\left|f'\left(R\exp\left(it\right)\right)\right|dt,
\end{aligned}
$$

which is equivalent to (29). □

In particular, we have:

Corollary 3. *With the assumptions of Proposition 1 we have*

$$
\begin{aligned}
&\Bigg|\sin\left(\frac{b-a}{4}\right)\exp\left[\left(\frac{3a+b}{4}\right)i\right]f\left(R\exp\left(ia\right)\right)\\
&\quad+\sin\left(\frac{b-a}{4}\right)\exp\left[\left(\frac{a+3b}{4}\right)i\right]f\left(R\exp\left(ib\right)\right)\\
&-\frac{1}{2}\int_a^b f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\Bigg|\leq R\sin\left(\frac{b-a}{4}\right)\int_a^b\left|f'\left(R\exp\left(it\right)\right)\right|dt. \quad (30)
\end{aligned}
$$

Proof. If we take in (29) $x=\frac{a+b}{2}$, then we get

$$
\begin{aligned}
&\Bigg|\sin\left(\frac{b-a}{4}\right)\exp\left[\left(\frac{3a+b}{4}\right)i\right]f\left(R\exp\left(ia\right)\right)\\
&+\sin\left(\frac{b-a}{4}\right)\exp\left[\left(\frac{a+3b}{4}\right)i\right]f\left(R\exp\left(ib\right)\right)-\frac{1}{2}\int_a^b f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\Bigg|\\
&\leq R\left[\max_{t\in\left[a,\frac{a+b}{2}\right]}\left|\sin\left(\frac{t-\frac{a+b}{2}}{2}\right)\right|\int_a^{\frac{a+b}{2}}\left|f'\left(R\exp\left(it\right)\right)\right|dt\right.\\
&\quad\left.+\max_{t\in\left[\frac{a+b}{2},b\right]}\left|\sin\left(\frac{t-\frac{a+b}{2}}{2}\right)\right|\int_{\frac{a+b}{2}}^b\left|f'\left(R\exp\left(it\right)\right)\right|dt\right]\\
&\leq R\max_{t\in[a,b]}\left|\sin\left(\frac{t-\frac{a+b}{2}}{2}\right)\right|\int_a^b\left|f'\left(R\exp\left(it\right)\right)\right|dt. \quad (31)
\end{aligned}
$$

Since the intervals $\left[a,\frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2},b\right]$ have a length less than π, then

$$
\max_{t\in\left[a,\frac{a+b}{2}\right]}\left|\sin\left(\frac{t-\frac{a+b}{2}}{2}\right)\right|=\max_{t\in\left[\frac{a+b}{2},b\right]}\left|\sin\left(\frac{t-\frac{a+b}{2}}{2}\right)\right|=\sin\left(\frac{b-a}{4}\right)
$$

and by (31) we get (30). □

The case of p-norms is as follows:

Proposition 3. *With the assumptions of Proposition 1 and* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *we have*

$$\begin{aligned} &\left| \sin\left(\frac{x-a}{2}\right) \exp\left[\left(\frac{a+x}{2}\right) i\right] f\left(R\exp\left(ia\right)\right) \right. \\ &\qquad + \sin\left(\frac{b-x}{2}\right) \exp\left[\left(\frac{x+b}{2}\right) i\right] f\left(R\exp\left(ib\right)\right) \\ &\qquad\qquad \left. -\frac{1}{2}\int_a^b f\left(R\exp\left(it\right)\right)\exp\left(it\right) dt \right| \end{aligned} \tag{32}$$

$$\begin{aligned} &\le R\left(\int_a^x \sin^q\left(\frac{x-t}{2}\right) dt\right)^{1/q} \left\| f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,x],p} \\ &\qquad + R\left(\int_x^b \sin^q\left(\frac{t-x}{2}\right) dt\right)^{1/q} \left\| f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[x,b],p} \\ &\le R\left[\int_a^x \sin^q\left(\frac{x-t}{2}\right) dt + \int_x^b \sin^q\left(\frac{t-x}{2}\right) dt\right]^{1/q} \left\| f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,b],p}. \end{aligned}$$

In particular, for $x = \frac{a+b}{2}$ *we get*

$$\begin{aligned} &\left| \sin\left(\frac{b-a}{4}\right) \exp\left[\left(\frac{3a+b}{4}\right) i\right] f\left(R\exp\left(ia\right)\right) \right. \\ &\qquad + \sin\left(\frac{b-a}{4}\right) \exp\left[\left(\frac{a+3b}{4}\right) i\right] f\left(R\exp\left(ib\right)\right) \\ &\qquad\qquad \left. -\frac{1}{2}\int_a^b f\left(R\exp\left(it\right)\right)\exp\left(it\right) dt \right| \end{aligned} \tag{33}$$

$$\begin{aligned} &\le R\left(\int_a^{\frac{a+b}{2}} \sin^q\left(\frac{\frac{a+b}{2}-t}{2}\right) dt\right)^{1/q} \left\| f'\left(R\exp\left(i\cdot\right)\right)\right\|_{\left[a,\frac{a+b}{2}\right],p} \\ &+ R\left(\int_{\frac{a+b}{2}}^b \sin^q\left(\frac{t-\frac{a+b}{2}}{2}\right) dt\right)^{1/q} \left\| f'\left(R\exp\left(i\cdot\right)\right)\right\|_{\left[\frac{a+b}{2},b\right],p} \\ &\le R\left[\int_a^b \sin^q\left(\left|\frac{t-\frac{a+b}{2}}{2}\right|\right) dt\right]^{1/q} \left\| f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,b],p}. \end{aligned}$$

Proof. By making use of the inequality (9) for $\gamma_{[a,b],R}$ and $x \in [a,b]$ we get

$$\Big|2Ri\sin\left(\frac{x-a}{2}\right)\exp\left[\left(\frac{a+x}{2}\right)i\right]f\left(R\exp\left(ia\right)\right)$$
$$+2Ri\sin\left(\frac{b-x}{2}\right)\exp\left[\left(\frac{x+b}{2}\right)i\right]f\left(R\exp\left(ib\right)\right)-Ri\int_a^b f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\Big|$$
$$\le 2R^2\left(\int_a^x \sin^q\left(\frac{x-t}{2}\right)dt\right)^{1/q}\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,x],p}$$
$$+2R^2\left(\int_x^b \sin^q\left(\frac{t-x}{2}\right)dt\right)^{1/q}\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[x,b],p}$$
$$\le 2R^2\left[\int_a^x \sin^q\left(\frac{x-t}{2}\right)dt+\int_x^b \sin^q\left(\frac{t-x}{2}\right)dt\right]^{1/q}\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,b],p},$$

which proves the desired result (32). □

The interested reader may consider for examples some fundamental complex functions such as $f(z) = z^n$ with n a natural number, $f(z) = \exp(z)$ or f a trigonometric or a hyperbolic complex function. The details are omitted.

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