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Research Article

Certain results on hybrid relatives of the Sheffer polynomials

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Abstract

The multi-variable special matrix polynomials have been identified significantly both in mathematical and applied frameworks. Due to its usefulness and various applications, a variety of its extensions and generalizations have been investigated and presented. The purpose of the paper is intended to study and emerge with a new generalization of Hermite matrix based Sheffer polynomials by involving integral transforms and some known operational rules. Their properties and quasi-monomial nature are also established. Further, these sequences are expressed in determinant forms by utilizing the relationship between the Sheffer sequences and Riordan arrays. An analogous study of these results is also carried out for certain members belonging to generalized Hermite matrix based Sheffer polynomials.

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1. Introduction and preliminaries

The Sheffer polynomials are one of the most important class of polynomial sequences and have been extensively studied [2, 10, 11, 13] not only due to the fact that they arise in numerous branches of mathematics but also because of their importance in applied sciences, such as physics and engineering. The Sheffer polynomials $S_n(p)$ for the pair $(g(t), f(t))$ are defined by the generating function [14, Pg. 18]:

$$\frac{1}{g(f^{-1}(t))} \exp\left(pf^{-1}(t)\right) = \sum_{n=0}^{\infty} S_n(p)\frac{t^n}{n!}, \tag{1.1}$$

where $g(t)$ is invertible series and $f(t)$ is delta series and is given by

$$g(t) = \sum_{n=0}^{\infty} g_n \frac{t^n}{n!} \quad (g_0 \neq 0) \tag{1.2}$$

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and

$$f(t) = \sum_{n=0}^{\infty} f_n \frac{t^n}{n!} \quad (f_0 = 0,\ f_1 \neq 0). \tag{1.3}$$

Sheffer class of polynomials contain two subclasses that are the corresponding associated sequences and the Appell sequences. The pair $(1, f(t))$ reduces the Sheffer sequences to the associated Sheffer sequence, and the pair $(g(t), t)$ reduces to the Appell sequence [14, Pg. 17]. The applications of Appell polynomials lie in a variety of mathematical fields, e.g., statistics, number theory, and probability theory. The generalized Appell polynomials play an essential role in approximating 3D-mappings in combination with Clifford analysis methods. These polynomials also appear in representation theory in the field of quantum physics.

The concept of the Riordan arrays was introduced by Shapiro et al. [15]. For the invertible series $g(t)$ and delta series $f(t)$ defined by (1.2) and (1.3), the generalized Riordan array $(g(t), f(t))$ with respect to the sequence $(c_n)_{n\in\mathbb{N}}$, defines an infinite, lower triangular array $(a_{n,k})_{0\leq k\leq n<\infty}$ according to the rule:

$$a_{n,k} = \left[\frac{t^n}{c_n}\right] g(t)\frac{(f(t))^k}{c_k}, \tag{1.4}$$

where the notation $\left[\frac{t^n}{c_n}\right]$ stands for the "coefficient of" operator and the functions $g(t)\frac{(f(t))^k}{c_k}$ are called the column generating functions of the Riordan array $(g(t), f(t))$ i.e.,

$$g(t)\frac{(f(t))^k}{c_k} = \sum_{n=k}^{\infty} a_{n,k}\frac{t^n}{c_n}. \tag{1.5}$$

Particularly, the classical Riordan arrays correspond to the case of $c_n = 1$, and the exponential Riordan arrays correspond to the case of $c_n = n!$.

The Sheffer polynomial sequences can also be represented via algebraic (determinant) form [18] and provide significant advantages in several numerical and computational viewpoint. If $(S_n(p))_{n\in\mathbb{N}_0}$ is a Sheffer sequence for the pair $(g(t), f(t))$ satisfying the following condition:

$$p^n = \sum_{k=0}^{n} a_{n,k}\ S_k(p), \tag{1.6}$$

then $S_n(p)$ can be expressed by the following determinant form :

$$S_0(p) = \frac{1}{a_{0,0}}, \tag{1.7}$$

$$S_n(p) = \frac{(-1)^n}{a_{0,0}\ a_{1,1}\cdots a_{n,n}} \begin{vmatrix} 1 & p & p^2 & \cdots & p^{n-1} & p^n \\ a_{0,0} & a_{1,0} & a_{2,0} & \cdots & a_{n-1,0} & a_{n,0} \\ 0 & a_{1,1} & a_{2,1} & \cdots & a_{n-1,1} & a_{n,1} \\ 0 & 0 & a_{2,2} & \cdots & a_{n-1,2} & a_{n,2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1,n-1} & a_{n,n-1} \end{vmatrix}, \tag{1.8}$$

where $a_{n,k}$ is the (n, k)th entry of the Riordan array $(g(t), f(t))$.

Conversely, if a polynomial sequence $(S_n(p))_{n\in\mathbb{N}_0}$ is defined by (1.7) and (1.8), where $a_{n,k}$ is the (n, k)th entry of the Riordan array $(g(t), f(t))$, then

$$S_n(p) = \sum_{k=0}^{n} b_{n,k}\ p^k, \tag{1.9}$$

where $b_{n,k}$ is the (n, k)th entry of the Riordan array $\left(\frac{1}{g(f^{-1}(t))}, f^{-1}(t)\right)$ and $(S_n(p))_{n\in\mathbb{N}_0}$ is Sheffer sequence for $(g(t), f(t))$ (see [17]).

An extension to the matrix framework of the classical special polynomials has been extensively studied and investigated in recent years [4, 7, 8, 19–21]. Matrix polynomials are important due to their applications in certain areas of statistics, physics, and engineering and is an emergent field. Moreover the introduction of multi-variable special functions serves as an analytical foundation for the majority of problems in mathematical physics that have been solved exactly and finds a wide range of practical applications. For some physical problems, the utilization of hybrid classes of special functions provided solutions hardly achievable with conventional numerical and analytical means. We review the definitions and concepts related to the 3-index 3-variable Hermite matrix based Sheffer polynomials.

Throughout the paper unless otherwise stated, we assume that $\mathfrak{B}$ is a positive stable matrix in $\mathbb{C}^{N\times N}$, that is, $\mathfrak{B}$ satisfies the following condition:

$$Re(\nu) > 0, \quad \text{for all } \nu \in \sigma(\mathfrak{B}), \tag{1.10}$$

where $\sigma(\mathfrak{B})$ denotes the set of all the eigenvalues of $\mathfrak{B}$.

If D_0 is the complex plane cut along the negative real axis and $\log(r)$ denotes the principal logarithm of r, then $r^{1/2}$ represents $\exp(\frac{1}{2}\log(r))$. If $\mathfrak{B}$ is a matrix in $\mathbb{C}^{N\times N}$ with $\sigma(\mathfrak{B}) \subset D_0$, then $\mathfrak{B}^{1/2} = \sqrt{\mathfrak{B}}$ denotes the image by $r^{1/2}$ of the matrix functional calculus [6] acting on the matrix $\mathfrak{B}$.

The 3-index 3-variable Hermite matrix based Sheffer polynomials (3I3VHMSP) denoted by ${}_HS_n^{(m,\eta)}(p,q,r;\mathfrak{B})$ are defined by the generating function [21]:

$$\frac{1}{g(f^{-1}(t))}\exp\left(pf^{-1}(t)\sqrt{m\mathfrak{B}} - q(f^{-1}(t))^m I + r(f^{-1}(t))^\eta I\right) = \sum_{n=0}^{\infty} {}_HS_n^{(m,\eta)}(p,q,r;\mathfrak{B})\frac{t^n}{n!}, \tag{1.11}$$

where m and η are both positive integers.

It has been shown in [21], that 3I3VHMSP ${}_HS_n^{(m,\eta)}(p,q,r;\mathfrak{B})$ are quasi-monomial under the action of the following multiplicative and derivative operators:

$$\Phi_{HS}^{+} := \left(p\sqrt{m\mathfrak{B}} - mq(\sqrt{m\mathfrak{B}})^{-(m-1)}\frac{\partial^{m-1}}{\partial p^{m-1}} + \eta r(\sqrt{m\mathfrak{B}})^{-(\eta-1)}\frac{\partial^{\eta-1}}{\partial p^{\eta-1}} - \frac{g'(D_P)}{g(D_P)}\right)\frac{1}{f'(D_P)} \tag{1.12}$$

and

$$\Phi_{HS}^{-} := f(D_P) \tag{1.13}$$

respectively, where

$$\frac{D_p}{\sqrt{m\mathfrak{B}}} = D_P.$$

The operational rule connecting the 3I3VHMSP ${}_HS_n^{(m,\eta)}(p,q,r;\mathfrak{B})$ with the Sheffer polynomials $S_n(p)$ is given by

$${}_HS_n^{(m,\eta)}(p,q,r;\mathfrak{B}) = \exp\left(r(\sqrt{m\mathfrak{B}})^{-\eta}\frac{\partial^\eta}{\partial p^\eta} - q(\sqrt{m\mathfrak{B}})^{-m}\frac{\partial^m}{\partial p^m}\right)\left\{S_n(p\sqrt{m\mathfrak{B}})\right\}. \tag{1.14}$$

In view of (1.9) for the pair $(g(t), f(t))$, explicit series representation for 3I3VHMSP ${}_HS_n^{(m,\eta)}(p,q,r;\mathfrak{B})$ is given by

$${}_HS_n^{(m,\eta)}(p,q,r;\mathfrak{B}) = \sum_{k=0}^{n} b_{n,k}H_k^{(m,\eta)}(p,q,r;\mathfrak{B}), \tag{1.15}$$

where $b_{n,k}$ is the (n,k)th entry of the Riordan array $(\frac{1}{g(f^{-1}(t))}, f^{-1}(t))$ and $H_n^{(m,\eta)}(p,q,r;\mathfrak{B})$ are the 3-index 3-variable Hermite matrix polynomials given by the following operational rule [7]:

$$H_n^{(m,\eta)}(p,q,r;\mathfrak{B}) = \exp\left(r(\sqrt{m\mathfrak{B}})^{-\eta}\frac{\partial^\eta}{\partial p^\eta} - q(\sqrt{m\mathfrak{B}})^{-m}\frac{\partial^m}{\partial p^m}\right)\left\{(p\sqrt{m\mathfrak{B}})^n\right\}. \tag{1.16}$$

Operational methods can be exploited to simplify the derivation of properties associated with ordinary and generalized special matrix functions and to define new families of hybrid special matrix polynomials. In addition, operational methods, developed within the context of the fractional derivative formalism [12] have opened new possibilities in the application of calculus. The combined use of integral transforms and operational methods provides a powerful computational tool to allow further progress and reveal new avenues for the study of fractional derivatives. We recall that the Euler integral [16, Pg. 218] is given by

$$a^{-\mu} = \frac{1}{\Gamma(\mu)}\int_0^\infty e^{-at}t^{\mu-1}dt, \quad \min\{Re(\mu),\ Re(a)\} \geq 0. \tag{1.17}$$

The Euler integral form the foundation of new generalizations of special matrix polynomials. The fusion of the properties of exponential operators with suitable integral representations for the special polynomials leads to a new and efficient method of treating fractional operators, for example, see [5].

This article is an attempt to generalize 3I3VHMSP ${}_HS_n^{(m,\eta)}(p,q,r;\mathfrak{B})$ in view of fractional calculus which looks very promising for developing a new perspective on the theory of special matrix polynomials. In Section 2, generalized three index three variable Hermite matrix based Sheffer polynomials are introduced. These sequences are studied within the framework of the monomiality principle, fractional calculus, and Riordan array. Its subclasses are discussed in Section 3. Further, in order to show some applications of the main results, several illustrative examples are also constructed in Section 4. In the last section, summation formula and identities corresponding to some well known identities are derived using operational formalism.

2. Generalized form of three index three variable Hermite matrix based Sheffer polynomials

In order to introduce the generalized three index three variable Hermite matrix based Sheffer polynomials, denoted by ${}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$, first we prove the following theorem:

Theorem 2.1. *For the generalized three index three variable Hermite matrix based Sheffer polynomials* ${}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$*, the following operational rule holds true:*

$$\left(\alpha + q(\sqrt{m\mathfrak{B}})^{-m}\frac{\partial^m}{\partial p^m} - r(\sqrt{m\mathfrak{B}})^{-\eta}\frac{\partial^\eta}{\partial p^\eta}\right)^{-\mu}\left\{S_n(p\sqrt{m\mathfrak{B}})\right\} = {}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha). \tag{2.1}$$

Proof. Replacing a in Euler integral (1.17) by $\left(\alpha - \left(r(\sqrt{m\mathfrak{B}})^{-\eta}\frac{\partial^\eta}{\partial p^\eta} - q(\sqrt{m\mathfrak{B}})^{-m}\frac{\partial^m}{\partial p^m}\right)\right)$ and then operating the resultant expression on Sheffer polynomials $S_n(p\sqrt{m\mathfrak{B}})$, we obtain

$$\left(\alpha + q(\sqrt{m\mathfrak{B}})^{-m}\frac{\partial^m}{\partial p^m} - r(\sqrt{m\mathfrak{B}})^{-\eta}\frac{\partial^\eta}{\partial p^\eta}\right)^{-\mu}\left\{S_n(p\sqrt{m\mathfrak{B}})\right\} = \frac{1}{\Gamma(\mu)}\int_0^\infty t^{\mu-1}$$
$$\exp\left(-\left(\alpha + q(\sqrt{m\mathfrak{B}})^{-m}\frac{\partial^m}{\partial x^m} - r(\sqrt{m\mathfrak{B}})^{-\eta}\frac{\partial^\eta}{\partial x^\eta}\right)t\right)\left\{S_n(p\sqrt{m\mathfrak{B}})\right\}dt, \tag{2.2}$$

which on using (1.14) gives

$$\left(\alpha + q(\sqrt{m\mathfrak{B}})^{-m}\frac{\partial^m}{\partial p^m} - r(\sqrt{m\mathfrak{B}})^{-\eta}\frac{\partial^\eta}{\partial p^\eta}\right)^{-\mu}\left\{S_n(p\sqrt{m\mathfrak{B}})\right\} = \frac{1}{\Gamma(\mu)}\int_0^\infty e^{-\alpha t}t^{\mu-1}{}_HS_n^{(m,\eta)}(p,qt,rt;\mathfrak{B})dt. \quad (2.3)$$

The integral transform on the right hand side of (2.3) defines a hybrid class of polynomials. Denoting this hybrid class of polynomials by ${}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ and naming it as the generalized three index three variable Hermite matrix based Sheffer polynomials (G3I3VHMSP), we have

$${}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) = \frac{1}{\Gamma(\mu)}\int_0^\infty e^{-\alpha t}t^{\mu-1}{}_HS_n^{(m,\eta)}(p,qt,rt;\mathfrak{B})dt. \quad (2.4)$$

In view of (2.3) and (2.4), (2.1) follows. □

Next, we derive the generating function of the G3I3VHMSP ${}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ by proving the following result.

Theorem 2.2. *For the G3I3VHMSP ${}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$, the following generating function holds true:*

$$\frac{exp\left(pf^{-1}(u)\sqrt{m\mathfrak{B}}\right)}{g(f^{-1}(u))\left(\alpha + q(f^{-1}(u))^m I - r(f^{-1}(u))^\eta I\right)^\mu} = \sum_{n=0}^\infty {}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)\frac{u^n}{n!}. \quad (2.5)$$

Proof. Multiplying both sides of the integral representation (2.4) of G3I3VHMSP ${}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ by $\frac{u^n}{n!}$ and summing the resultant expression over n, we obtain

$$\sum_{n=0}^\infty {}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)\frac{u^n}{n!} = \sum_{n=0}^\infty \frac{1}{\Gamma(\mu)}\int_0^\infty e^{-\alpha t}t^{\mu-1}{}_HS_n^{(m,\eta)}(p,qt,rt;\mathfrak{B})\frac{u^n}{n!}dt. \quad (2.6)$$

Using (1.11) in the right hand side of (2.6), it follows that

$$\sum_{n=0}^\infty {}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)\frac{u^n}{n!} = \frac{\exp(pf^{-1}(u)\sqrt{m\mathfrak{B}})}{g(f^{-1}(u))}\frac{1}{\Gamma(\mu)}\int_0^\infty t^{\mu-1} \exp\left(-\left(\alpha + q(f^{-1}(u))^m I - r(f^{-1}(u))^\eta I\right)t\right)dt. \quad (2.7)$$

which in view of Euler integral (1.17), leads to (2.5). □

Remark 2.3. Using similar argument as in the proof of Theorem 2.1 on the operational rule (1.16), we can form a new class of the generalized 3-index 3-variable Hermite matrix polynomials (G3I3VHMP), denoted by $H_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$, defined by the following integral transform

$$H_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) = \frac{1}{\Gamma(\mu)}\int_0^\infty e^{-\alpha t}t^{\mu-1}H_n^{(m,\eta)}(p,qt,rt;\mathfrak{B})dt \quad (2.8)$$

and its operational representation is given by

$$H_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) = \left(\alpha + q(\sqrt{m\mathfrak{B}})^{-m}\frac{\partial^m}{\partial p^m} - r(\sqrt{m\mathfrak{B}})^{-\eta}\frac{\partial^\eta}{\partial p^\eta}\right)^{-\mu}\left\{(p\sqrt{m\mathfrak{B}})^n\right\}. \quad (2.9)$$

Theorem 2.4. *For the pair $(g(t), f(t))$, explicit series representation for G3I3VHMSP ${}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ is given by*

$${}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) = \sum_{k=0}^n b_{n,k}H_{k,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha), \quad (2.10)$$

where $b_{n,k}$ *is the* $(n,k)th$ *entry of the Riordan array* $\left(\frac{1}{g(f^{-1}(t))}, f^{-1}(t)\right)$ *and* $H_{k,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ *are the G3I3VHMP given by (2.8).*

Proof. In (1.15), replacing q by qt, r by rt and then multiplying both sides by $\frac{1}{\Gamma(\mu)}e^{-\alpha t}t^{\mu-1}$ and thereafter integrating with respect to t from $t=0$ to $t=\infty$, we obtain

$$\frac{1}{\Gamma(\mu)}\int_0^\infty e^{-\alpha t}t^{\mu-1}{}_HS_n^{(m,\eta)}(p,qt,rt;\mathfrak{B})dt = \frac{1}{\Gamma(\mu)}\int_0^\infty e^{-\alpha t}t^{\mu-1}\sum_{k=0}^{n} b_{n,k}H_k^{(m,\eta)}(p,qt,rt;\mathfrak{B})dt, \quad (2.11)$$

which on using integral transforms (2.4) and (2.8), leads to (2.10). □

Differentiating generating function (2.5) w.r.t α, the following result is obtained:

Lemma 2.5. *For the G3I3VHMSP* ${}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$*, the following recurrence relation holds true:*

$$\frac{\partial}{\partial\alpha}\left({}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)\right) = -\mu\ {}_HS_{n,\mu+1}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha). \quad (2.12)$$

To frame the G3I3VHMSP ${}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ within the context of monomiality principle, we prove the following result:

Theorem 2.6. *The G3I3VHMSP* ${}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ *are quasi-monomial with respect to the following multiplicative and derivative operators:*

$$\Phi_{GHS}^{+} := \left(p\sqrt{m\mathfrak{B}} + mq(\sqrt{m\mathfrak{B}})^{-(m-1)}\frac{\partial^m}{\partial\alpha\partial p^{m-1}} - \eta r(\sqrt{m\mathfrak{B}})^{-(\eta-1)}\frac{\partial^\eta}{\partial\alpha\partial p^{\eta-1}} - \frac{g'(D_P)}{g(D_P)}\right)\frac{1}{f'(D_P)} \quad (2.13)$$

and

$$\Phi_{GHS}^{-} := f(D_P) \quad (2.14)$$

respectively, where

$$D_P = \frac{D_p}{\sqrt{m\mathfrak{B}}}.$$

Proof. In view of monomiality principle, expression (1.12) and (1.13) can be written as:

$${}_HS_{n+1}^{(m,\eta)}(p,q,r;\mathfrak{B}) = \left(p\sqrt{m\mathfrak{B}} - mq(\sqrt{m\mathfrak{B}})^{-(m-1)}\frac{\partial^{m-1}}{\partial p^{m-1}} + \eta r(\sqrt{m\mathfrak{B}})^{-(\eta-1)}\frac{\partial^{\eta-1}}{\partial p^{\eta-1}} - \frac{g'(D_P)}{g(D_P)}\right)\frac{1}{f'(D_P)}{}_HS_n^{(m,\eta)}(p,q,r;\mathfrak{B}) \quad (2.15)$$

and

$$n\ {}_HS_{n-1}^{(m,\eta)}(p,q,r;\mathfrak{B})\Phi_{HS}^{-} := f(D_P){}_HS_n^{(m,\eta)}(p,q,r;\mathfrak{B}), \quad (2.16)$$

where

$$D_P = \frac{D_p}{\sqrt{m\mathfrak{B}}}.$$

Now in (2.15) and (2.16), replacing q by qt, r by rt and then multiplying both sides by $\frac{1}{\Gamma(\mu)}e^{-\alpha t}t^{\mu-1}$ and thereafter integrating with respect to t from $t=0$ to $t=\infty$, we obtain

$$\frac{1}{\Gamma(\mu)}\int_0^\infty e^{-\alpha t}t^{\mu-1}{}_HS_{n+1}^{(m,\eta)}(p,qt,rt;\mathfrak{B})dt = \frac{1}{\Gamma(\mu)}\int_0^\infty e^{-\alpha t}t^{\mu-1}\Big(p\sqrt{m\mathfrak{B}} - mqt(\sqrt{m\mathfrak{B}})^{-(m-1)}\frac{\partial^{m-1}}{\partial p^{m-1}} + \eta rt(\sqrt{m\mathfrak{B}})^{-(\eta-1)}\frac{\partial^{\eta-1}}{\partial p^{\eta-1}} - \frac{g'(D_P)}{g(D_P)}\Big)\frac{1}{f'(D_P)}{}_HS_n^{(m,\eta)}(p,qt,rt;\mathfrak{B})dt \quad (2.17)$$

and

$$\frac{n}{\Gamma(\mu)}\int_0^\infty e^{-\alpha t}t^{\mu-1}{}_HS_{n-1}^{(m,\eta)}(p,qt,rt;\mathfrak{B})dt = \frac{1}{\Gamma(\mu)}\int_0^\infty e^{-\alpha t}t^{\mu-1}f(D_P){}_HS_n^{(m,\eta)}(p,qt,rt;\mathfrak{B})dt, \quad (2.18)$$

which on using integral transform (2.4) and recurrence relation (2.12), leads to (2.13) and (2.14). □

In view of the monomiality principle, following consequences of the above result are obtained:

Corollary 2.7. *The G3I3VHMSP ${}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ satisfy the following differential recurrence relations:*

$$\Big(p\sqrt{m\mathfrak{B}} + mq(\sqrt{m\mathfrak{B}})^{-(m-1)}\frac{\partial^m}{\partial\alpha\partial p^{m-1}} - \eta r(\sqrt{m\mathfrak{B}})^{-(\eta-1)}\frac{\partial^\eta}{\partial\alpha\partial p^{\eta-1}} - \frac{g'(D_P)}{g(D_P)}\Big) \times \frac{1}{f'(D_P)}{}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) = {}_HS_{n+1,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) \quad (2.19)$$

and

$$f(D_P){}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) = n\ {}_HS_{n-1,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha). \quad (2.20)$$

Corollary 2.8. *The G3I3VHMSP ${}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ satisfy the following differential equation:*

$$\Big(\Big\{p\sqrt{m\mathfrak{B}} + mq(\sqrt{m\mathfrak{B}})^{-(m-1)}\frac{\partial^m}{\partial\alpha\partial p^{m-1}} - \eta r(\sqrt{m\mathfrak{B}})^{-(\eta-1)}\frac{\partial^\eta}{\partial\alpha\partial p^{\eta-1}} - \frac{g'(D_P)}{g(D_P)}\Big\}\frac{f(D_P)}{f'(D_P)} - n\Big){}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) = 0. \quad (2.21)$$

Corollary 2.9. *The G3I3VHMSP ${}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ satisfy the following explicit representation:*

$${}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) = \Big(\Big\{p\sqrt{m\mathfrak{B}} + mq(\sqrt{m\mathfrak{B}})^{-(m-1)} \times \frac{\partial^m}{\partial\alpha\partial p^{m-1}} - \eta r(\sqrt{m\mathfrak{B}})^{-(\eta-1)}\frac{\partial^\eta}{\partial\alpha\partial p^{\eta-1}} - \frac{g'(D_P)}{g(D_P)}\Big\}\frac{1}{f'(D_P)}\Big)^n\{1\}. \quad (2.22)$$

The determinant approach is equivalent to the corresponding approach based on operational methods. The simplicity of this approach allows non-specialists to use its applications and is beneficial in detecting the solution of general linear interpolation problems and also suitable for computation. Inspired by the novel work on determinant approaches of the Sheffer sequences proposed by W. Wang [17] in 2014, the determinant definition of the G3I3VHMSP ${}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ is established by proving the following result.

Theorem 2.10. *The G3I3VHMSP* ${}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ *of degree* n *are defined by the following determinant form:*

$$ {}_HS_{0,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) = \frac{1}{a_{0,0}} H_{0,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha), \tag{2.23}$$

$$ {}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) = \frac{(-1)^n}{a_{0,0}\ a_{1,1}\cdots a_{n,n}} \begin{vmatrix} \mathcal{H}_0 & \mathcal{H}_1 & \mathcal{H}_2 & \cdots & \mathcal{H}_{n-1} & \mathcal{H}_n \\ a_{0,0} & a_{1,0} & a_{2,0} & \cdots & a_{n-1,0} & a_{n,0} \\ 0 & a_{1,1} & a_{2,1} & \cdots & a_{n-1,1} & a_{n,1} \\ 0 & 0 & a_{2,2} & \cdots & a_{n-1,2} & a_{n,2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1,n-1} & a_{n,n-1} \end{vmatrix}, \tag{2.24}$$

where $a_{n,k}$ *is the* (n,k)*th entry of the Riordan array* $(g(t),f(t))$ *and* $\mathcal{H}_n = H_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ *are the G3I3VHMP defined by (2.9).*

Proof. In (1.6), replacing p by $p\sqrt{m\mathfrak{B}}$ and then operating $\left(\alpha + q(\sqrt{m\mathfrak{B}})^{-m}\frac{\partial^m}{\partial p^m} - r(\sqrt{m\mathfrak{B}})^{-\eta}\frac{\partial^\eta}{\partial p^\eta}\right)^{-\mu}$ on both sides. Afterwards, in the resultant expression using operational rules (2.1) and (2.9) in r.h.s and l.h.s. respectively, we find

$$ H_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) = \sum_{k=0}^{n} a_{n,k}\ {}_HS_{k,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha). \tag{2.25}$$

The above equality leads to the following system of infinite equations in the unknowns ${}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$, $n = 0,1,2,\cdots$,

$$\begin{cases} a_{0,0}\ {}_HS_{0,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) = H_{0,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha), \\ a_{1,0}\ {}_HS_{0,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) + a_{1,1}\ {}_HS_{1,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) = H_{1,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha), \\ a_{2,0}\ {}_HS_{0,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) + a_{2,1}\ {}_HS_{1,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) + a_{2,2}\ {}_HS_{2,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) \\ \qquad = H_{2,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha), \\ \vdots \\ a_{n,0}\ {}_HS_{0,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) + a_{n,1}\ {}_HS_{1,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) + a_{n,2}\ {}_HS_{2,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) + \cdots \\ \qquad + a_{n,n}\ {}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) = H_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha), \\ \vdots \end{cases} \tag{2.26}$$

From first equation of system (2.26), (2.23) follows. Applying Cramer's rule to the first $n+1$ equations, it follows that

$$ {}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) = \frac{1}{a_{0,0}\ a_{1,1}\cdots a_{n,n}} \times \begin{vmatrix} a_{0,0} & 0 & 0 & \cdots & 0 & H_{0,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) \\ a_{1,0} & a_{1,1} & 0 & \cdots & 0 & H_{1,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) \\ a_{2,0} & a_{2,1} & a_{2,2} & \cdots & 0 & H_{2,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,0} & a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} & H_{n-1,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) \\ a_{n,0} & a_{n,1} & a_{n,2} & \cdots & a_{n,n-1} & H_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) \end{vmatrix}. \tag{2.27}$$

Now, bringing the $(n+1)$th column to the first place by n transpositions of adjacent columns and noting that the determinant of a square matrix is the same as that of its transpose, (2.24) follows. □

3. Subclasses

The two particular subclasses of the Sheffer sequences are the sequences of the Appell polynomials and the associated Sheffer polynomials, which are discussed in Section 1. To study the subclasses related to the G3I3VHMSP ${}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$, we consider the following cases:

3.1. The subclass related to the associated Sheffer polynomials

As mentioned in Section 1, Sheffer polynomials $S_n(p)$ for the pair $(1,f(t))$ become the associated Sheffer polynomials $\mathfrak{s}_n(p)$. Therefore, by taking $g(t)=1$, so that

$$\frac{1}{g(f^{-1}(t))}=1 \quad \text{and} \quad g'(t)=0 \tag{3.1}$$

in the G3I3VHMSP ${}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$, we find that the generalized three index three variable Hermite matrix based associated Sheffer polynomials (G3I3VHMASP), denoted by ${}_H\mathfrak{s}_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$, related with the pair $(1,f(t))$ is defined by the following generating function:

$$\frac{exp\left(pf^{-1}(u)\sqrt{m\mathfrak{B}}\right)}{(\alpha+q(f^{-1}(u))^mI-r(f^{-1}(u))^\eta I)^\mu}=\sum_{n=0}^{\infty}{}_H\mathfrak{s}_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)\frac{u^n}{n!}. \tag{3.2}$$

The other results for the G3I3VHMASP ${}_H\mathfrak{s}_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ are given in Table 1, where $D_P=\frac{D_p}{\sqrt{m\mathfrak{B}}}$ and $b_{n,k}$ is the (n,k)th entry of the Riordan array $\left(1,f^{-1}(t)\right)$.

S. No.	Results	Expressions
I	Operational Representation	${}_H\mathfrak{s}_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) = \left(\alpha+q(\sqrt{m\mathfrak{B}})^{-m}\frac{\partial^m}{\partial p^m}-r(\sqrt{m\mathfrak{B}})^{-\eta}\frac{\partial^\eta}{\partial p^\eta}\right)^{-\mu}\left\{\mathfrak{s}_n(p\sqrt{m\mathfrak{B}})\right\}$
II	Series Representation	${}_H\mathfrak{s}_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)=\sum_{k=0}^{n}b_{n,k}H_{k,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$
III	Multiplicative Operator	$\Phi_{GHAS}^{+}:=\left(p\sqrt{m\mathfrak{B}}+mq(\sqrt{m\mathfrak{B}})^{-(m-1)}\frac{\partial^m}{\partial\alpha\partial p^{m-1}} -\eta r(\sqrt{m\mathfrak{B}})^{-(\eta-1)}\frac{\partial^\eta}{\partial\alpha\partial p^{\eta-1}}\right)\frac{1}{f'(D_P)}$
IV	Derivative Operator	$\Phi_{GHAS}^{-}:=f(D_P)$
V	Differential Equation	$\left(\left\{p\sqrt{m\mathfrak{B}}+mq(\sqrt{m\mathfrak{B}})^{-(m-1)}\frac{\partial^m}{\partial\alpha\partial p^{m-1}} -\eta r(\sqrt{m\mathfrak{B}})^{-(\eta-1)}\frac{\partial^\eta}{\partial\alpha\partial p^{\eta-1}}\right\}\frac{f(D_P)}{f'(D_P)}-n\right){}_H\mathfrak{s}_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)=0$

Table 1. Results for G3I3VHMASP ${}_H\mathfrak{s}_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$.

In view of (1.4) and considering the fact that $c_n=n!$ for associated Sheffer polynomials $\mathfrak{s}_n(p)$, the determinant form of Sheffer polynomials $S_n(p)$ reduces to that of associated Sheffer polynomials $\mathfrak{s}_n(p)$ for:

$$a_{n,0}=\left[\frac{t^n}{n!}\right]\frac{(f(t))^0}{0!}=n!\;[t^n]\;1=\delta_{n,0}. \tag{3.3}$$

Consequently by making the same substitution in the determinant form (2.23) and (2.24), we find that the G3I3VHMASP ${}_H\mathfrak{s}_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ are defined by means of the following determinant:

$${}_H\mathfrak{s}_{0,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)=H_{0,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha), \tag{3.4}$$

$$ {}_H\mathfrak{s}_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) = \frac{(-1)^{n+1}}{a_{1,1}\cdots a_{n,n}} \begin{vmatrix} \mathcal{H}_1 & \mathcal{H}_2 & \cdots & \mathcal{H}_{n-1} & \mathcal{H}_n \\ a_{1,1} & a_{2,1} & \cdots & a_{n-1,1} & a_{n,1} \\ 0 & a_{2,2} & \cdots & a_{n-1,2} & a_{n,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1,n-1} & a_{n,n-1} \end{vmatrix}, \tag{3.5} $$

where $a_{n,k}$ is the (n,k)th entry of the Riordan array associated with $(1, f(t))$ and $\mathcal{H}_n = H_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ are the G3I3VHMP defined by (2.9).

3.2. The subclass related to the Appell polynomials

We know that the Sheffer polynomials $S_n(p)$ for the pair $(g(t), t)$ become the Appell polynomials $A_n(p)$. Therefore, by taking $f(t) = t$, so that

$$ f^{-1}(t) = t, \quad \frac{1}{g(f^{-1}(t))} = \frac{1}{g(t)} \quad \text{and} \quad f'(t) = 1 \tag{3.6} $$

in the G3I3VHMSP ${}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$, we find that the generalized three index three variable Hermite matrix based Appell polynomials (G3I3VHMAP), denoted by ${}_HA_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$, related with the pair $(g(t), t)$ is defined by the following generating function:

$$ \frac{exp\left(pu\sqrt{m\mathfrak{B}}\right)}{g(u)\,(\alpha + qu^mI - ru^\eta I)^\mu} = \sum_{n=0}^{\infty} {}_HA_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)\frac{u^n}{n!}. \tag{3.7} $$

The other results for the G3I3VHMAP ${}_HA_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ are given in Table 2, where $D_P = \frac{D_p}{\sqrt{m\mathfrak{B}}}$ and $b_{n,k}$ is the (n,k)th entry of the Riordan array $\left(\frac{1}{g(t)}, t\right)$.

S. No.	Results	Expressions
I	Operational Representation	${}_HA_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) = \left(\alpha + q(\sqrt{m\mathfrak{B}})^{-m}\frac{\partial^m}{\partial p^m} - r(\sqrt{m\mathfrak{B}})^{-\eta}\frac{\partial^\eta}{\partial p^\eta}\right)^{-\mu}\left\{A_n(p\sqrt{m\mathfrak{B}})\right\}$
II	Series Representation	${}_HA_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) = \sum_{k=0}^{n} b_{n,k}H_{k,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$
III	Multiplicative Operator	$\Phi_{GHA}^{+} := p\sqrt{m\mathfrak{B}} + mq(\sqrt{m\mathfrak{B}})^{-(m-1)}\frac{\partial^m}{\partial\alpha\partial p^{m-1}} -\eta r(\sqrt{m\mathfrak{B}})^{-(\eta-1)}\frac{\partial^\eta}{\partial\alpha\partial p^{\eta-1}} - \frac{g'(D_P)}{g(D_P)}$
IV	Derivative Operator	$\Phi_{GHA}^{-} := D_P$
V	Differential Equation	$\left(\left\{p\sqrt{m\mathfrak{B}} + mq(\sqrt{m\mathfrak{B}})^{-(m-1)}\frac{\partial^m}{\partial\alpha\partial p^{m-1}} - \eta r(\sqrt{m\mathfrak{B}})^{-(\eta-1)}\frac{\partial^\eta}{\partial\alpha\partial p^{\eta-1}} -\frac{g'(D_P)}{g(D_P)}\right\}D_P - n\right){}_HA_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) = 0$

Table 2. Results for G3I3VHMAP ${}_HA_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$.

In view of (1.4) and considering the fact that $c_n = n!$ for Appell polynomials $A_n(p)$, the determinant form of Sheffer polynomials $S_n(p)$ reduces to that of Appell polynomials $A_n(p)$ for:

$$ a_{n,k} = \left[\frac{t^n}{n!}\right] g(t)\frac{t^k}{k!} = \frac{n!}{k!}[t^{n-k}]g(t) = \binom{n}{k} g_{n-k}. \tag{3.8} $$

Consequently by making the same substitution in the determinant form (2.23) and (2.24), we find that the G3I3VHMAP ${}_HA_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ are defined by means of the following determinant:

$$ {}_HA_{0,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) = \frac{1}{g_0}H_{0,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha), \tag{3.9} $$

$$
{}_HA_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)=\frac{(-1)^n}{{g_0}^{n+1}}\begin{vmatrix}\mathcal{H}_0 & \mathcal{H}_1 & \mathcal{H}_2 & \cdots & \mathcal{H}_{n-1} & \mathcal{H}_n\\ g_0 & g_1 & g_2 & \cdots & g_{n-1} & g_n\\ 0 & g_0 & \binom{2}{1}g_1 & \cdots & \binom{n-1}{1}g_{n-2} & \binom{n}{1}g_{n-1}\\ 0 & 0 & g_0 & \cdots & \binom{n-1}{2}g_{n-3} & \binom{n}{2}g_{n-2}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & g_0 & \binom{n}{n-1}g_1\end{vmatrix}, \tag{3.10}
$$

where $\mathcal{H}_n = H_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ are the G3I3VHMP defined by (2.9).

4. Illustrative examples

In order to give applications of the results derived above, we consider illustrative examples of certain members belonging to the class of G3I3VHMSP ${}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$.

Example 4.1. The Sheffer polynomials $S_n(p)$ for the pair $(g(t),f(t))$ given by

$$
g(t)=\frac{2}{1+\sqrt{1-t^2}} \quad \text{and} \quad f(t)=-\frac{t}{1+\sqrt{1-t^2}}, \tag{4.1}
$$

reduce to the Chebyshev polynomials $U_n(p)$ of the second kind [1, Pg. 778]. Consequently taking these values of $(g(t),f(t))$ and

$$
f^{-1}(t)=-\frac{2t}{1+t^2} \quad \text{and} \quad g(f^{-1}(t))=1+t^2 \tag{4.2}
$$

in the generating function (2.5) of G3I3VHMSP ${}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$, reduce it to the generalized three index three variable Hermite matrix based Chebyshev polynomials (G3I3VHMCP) ${}_HU_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ of the second kind which are defined by the following generating function:

$$
\frac{exp\left(-p\frac{2u}{1+u^2}\sqrt{m\mathfrak{B}}\right)}{(1+u^2)\left(\alpha+q(-\frac{2u}{1+u^2})^mI-r(-\frac{2u}{1+u^2})^\eta I\right)^\mu}=\sum_{n=0}^{\infty}{}_HU_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)\frac{u^n}{n!}. \tag{4.3}
$$

The explicit series representation (2.10) of G3I3VHMSP ${}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ reduces to that of G3I3VHMCP ${}_HU_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ for $b_{n,k}$, where $b_{n,k}$ is the (n,k)th entry of the Riordan array $\left(\frac{1}{1+t^2},\frac{-2t}{1+t^2}\right)$ (see (4.2)). So, in view of (1.4) and considering the fact that $c_n=(-1)^n$ for this case, $b_{n,k}$ is given by

$$
b_{n,k}=\begin{cases}0, & n-k \text{ odd},\\ (-1)^{n-k}(-2)^k\binom{-k-1}{\frac{n-k}{2}}, & n-k \text{ even}.\end{cases} \tag{4.4}
$$

Therefore we have

$$
{}_HU_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)=\sum_{k=0}^{[\frac{n}{2}]}\binom{-n+2k-1}{k}(-2)^{n-2k}H_{n-2k,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha). \tag{4.5}
$$

The other results for the G3I3VHMCP ${}_HU_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ are given in Table 3, where $D_P=\frac{D_p}{\sqrt{m\mathfrak{B}}}$.

S. No.	Results	Expressions
I	Operational Representation	${}_HU^{(m,\eta)}_{n,\mu}(p,q,r;\mathfrak{B};\alpha) = \left(\alpha + q(\sqrt{m\mathfrak{B}})^{-m}\frac{\partial^m}{\partial p^m} - r(\sqrt{m\mathfrak{B}})^{-\eta}\frac{\partial^\eta}{\partial p^\eta}\right)^{-\mu}\left\{U_n(p\sqrt{m\mathfrak{B}})\right\}$
II	Multiplicative Operator	$\Phi^+_{GHC} := \left(p\sqrt{m\mathfrak{B}} + mq(\sqrt{m\mathfrak{B}})^{-(m-1)}\frac{\partial^m}{\partial\alpha\partial p^{m-1}} - \eta r(\sqrt{m\mathfrak{B}})^{-(\eta-1)}\frac{\partial^\eta}{\partial\alpha\partial p^{\eta-1}} - \frac{D_P}{1-(D_P)^2+\sqrt{1-(D_P)^2}}\right)\left(-1+(D_P)^2-\sqrt{1-(D_P)^2}\right)$
III	Derivative Operator	$\Phi^-_{GHC} := -\frac{D_P}{1+\sqrt{1-D_P{}^2}}$
IV	Differential Equation	$\left(\left\{p\sqrt{m\mathfrak{B}} + mq(\sqrt{m\mathfrak{B}})^{-(m-1)}\frac{\partial^m}{\partial\alpha\partial p^{m-1}} - \eta r(\sqrt{m\mathfrak{B}})^{-(\eta-1)}\frac{\partial^\eta}{\partial\alpha\partial p^{\eta-1}} - \frac{D_P}{1-(D_P)^2+\sqrt{1-(D_P)^2}}\right\}\left(D_P\sqrt{1-(D_P)^2}\right) - n\right){}_HU^{(m,\eta)}_{n,\mu}(p,q,r;\mathfrak{B};\alpha) = 0$

Table 3. Results for G3I3VHMCP ${}_HU^{(m,\eta)}_{n,\mu}(p,q,r;\mathfrak{B};\alpha)$.

The determinant form (2.23) and (2.24) of G3I3VHMSP ${}_HS^{(m,\eta)}_{n,\mu}(p,q,r;\mathfrak{B};\alpha)$ reduces to that of G3I3VHMCP ${}_HU^{(m,\eta)}_{n,\mu}(p,q,r;\mathfrak{B};\alpha)$ for $a_{n,k}$, where $a_{n,k}$ is the (n,k)th entry of the Riordan array $\left(\frac{2}{1+\sqrt{1-t^2}}, -\frac{t}{1+\sqrt{1-t^2}}\right)$. In view of (1.4), $a_{n,k}$ is given by

$$a_{n,k} = \begin{cases} 0, & n-k \text{ odd}, \\ (-1)^{n-k}\frac{(-1)^k}{2^n}\frac{1+k}{1+n}\binom{1+n}{\frac{n-k}{2}}, & n-k \text{ even}. \end{cases} \tag{4.6}$$

In particular, $a_{n,n} = \left(-\frac{1}{2}\right)^n$. Therefore, for an even n, we have

$${}_HC^{(m,\eta)}_{0,\mu}(p,q,r;\mathfrak{B};\alpha) = H^{(m,\eta)}_{0,\mu}(p,q,r;\mathfrak{B};\alpha), \tag{4.7}$$

$${}_HC^{(m,\eta)}_{n,\mu}(p,q,r;\mathfrak{B};\alpha) = (-1)^{\frac{n(n+3)}{2}}2^{\frac{n(n+1)}{2}}\begin{vmatrix} \mathcal{H}_0 & \mathcal{H}_1 & \mathcal{H}_2 & \cdots & \mathcal{H}_{n-1} & \mathcal{H}_n \\ 1 & 0 & \frac{1}{4} & \cdots & 0 & \frac{1}{2^n}\frac{1}{1+n}\binom{1+n}{\frac{n}{2}} \\ 0 & -\frac{1}{2} & 0 & \cdots & \frac{-1}{2^{n-1}}\frac{2}{n}\binom{n}{\frac{n-2}{2}} & 0 \\ 0 & 0 & \frac{1}{4} & \cdots & 0 & \frac{1}{2^n}\frac{3}{1+n}\binom{1+n}{\frac{n-2}{2}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\left(\frac{1}{2}\right)^{n-1} & 0 \end{vmatrix}, \tag{4.8}$$

where $\mathcal{H}_n = H^{(m,\eta)}_{n,\mu}(p,q,r;\mathfrak{B};\alpha)$ are the G3I3VHMP defined by (2.9).

Example 4.2. The Sheffer polynomials $S_n(p)$ for the pair $(g(t), f(t))$ given by

$$g(t) = 1 \quad \text{and} \quad f(t) = \log(1+t), \tag{4.9}$$

reduce to the exponential polynomials $\phi_n(p)$ [14, Pg. 63]. Consequently taking these values of $(g(t), f(t))$ and

$$f^{-1}(t) = e^t - 1 \quad \text{and} \quad g(f^{-1}(t)) = 1 \tag{4.10}$$

in the generating function (2.5) of G3I3VHMSP ${}_HS^{(m,\eta)}_{n,\mu}(p,q,r;\mathfrak{B};\alpha)$, reduce it to the generalized three index three variable Hermite matrix based exponential polynomials (G3I3VHMExP) ${}_H\phi^{(m,\eta)}_{n,\mu}(p,q,r;\mathfrak{B};\alpha)$ which are defined by the following generating function:

$$\frac{exp\left(p(e^u-1)\sqrt{m\mathfrak{B}}\right)}{(\alpha + q(e^u-1)^m I - r(e^u-1)^\eta I)^\mu} = \sum_{n=0}^{\infty} {}_H\phi^{(m,\eta)}_{n,\mu}(p,q,r;\mathfrak{B};\alpha)\frac{u^n}{n!}. \tag{4.11}$$

The explicit series representation (2.10) of G3I3VHMSP ${}_HS^{(m,\eta)}_{n,\mu}(p,q,r;\mathfrak{B};\alpha)$ reduces to that of G3I3VHMExP ${}_H\phi^{(m,\eta)}_{n,\mu}(p,q,r;\mathfrak{B};\alpha)$ for $b_{n,k}$, where $b_{n,k}$ is the (n,k)th entry of the Riordan array $(1, e^t - 1)$ (see (4.10)). So, in view of (1.4) and considering the fact that $c_n = n!$ for this case, $b_{n,k}$ is given by

$$b_{n,k} = \left[\frac{t^n}{n!}\right] \frac{(e^t-1)^k}{k!} = S(n,k), \tag{4.12}$$

where $S(n,k)$ are the the Stirling numbers of the second kind [3, Pg. 206]. Therefore we have

$$_H\phi_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) = \sum_{k=0}^{n} S(n,k) H_{k,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha). \tag{4.13}$$

The other results for the G3I3VHMExP $_H\phi_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ are given in Table 4, where $D_P = \frac{D_p}{\sqrt{m\mathfrak{B}}}$.

S. No.	Results	Expressions
I	Operational Representation	$_H\phi_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) = \left(\alpha + q(\sqrt{m\mathfrak{B}})^{-m}\frac{\partial^m}{\partial p^m} - r(\sqrt{m\mathfrak{B}})^{-\eta}\frac{\partial^\eta}{\partial p^\eta}\right)^{-\mu}\left\{\phi_n(p\sqrt{m\mathfrak{B}})\right\}$
II	Multiplicative Operator	$\Phi^+_{GHEX} := \left(p\sqrt{m\mathfrak{B}} + mq(\sqrt{m\mathfrak{B}})^{-(m-1)}\frac{\partial^m}{\partial\alpha\partial p^{m-1}} - \eta r(\sqrt{m\mathfrak{B}})^{-(\eta-1)}\frac{\partial^\eta}{\partial\alpha\partial p^{\eta-1}}\right)(1+D_P)$
III	Derivative Operator	$\Phi^-_{GHEX} := \log(1+D_P)$
IV	Differential Equation	$\left(\left\{p\sqrt{m\mathfrak{B}} + mq(\sqrt{m\mathfrak{B}})^{-(m-1)}\frac{\partial^m}{\partial\alpha\partial p^{m-1}} - \eta r(\sqrt{m\mathfrak{B}})^{-(\eta-1)}\frac{\partial^\eta}{\partial\alpha\partial p^{\eta-1}}\right\}(1+D_P)\log(1+D_P) - n\right){}_H\phi_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) = 0$

Table 4. Results for G3I3VHMExP $_H\phi_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$.

The determinant form (2.23) and (2.24) of G3I3VHMSP $_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ reduces to that of G3I3VHMExP $_H\phi_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ when $a_{n,k}$ is the (n,k)th entry of the Riordan array $(1, \log(1+t))$. In view of (1.4), $a_{n,k}$ is given by

$$a_{n,k} = \left[\frac{t^n}{n!}\right] \frac{(\log(1+t))^k}{k!} = s(n,k)\ , \tag{4.14}$$

where $s(n,k)$ are the the Stirling numbers of the first kind [3, Pg. 212].

In particular, $a_{n,n} = 1$ $(n = 0,1,2,\cdots)$ and $a_{n,0} = 0$ $(n = 1,2,3,\cdots)$. Therefore, we have

$$_H\phi_{0,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) = H_{0,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha), \tag{4.15}$$

$$_H\phi_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) = (-1)^{n+1}\begin{vmatrix} \mathcal{H}_1 & \mathcal{H}_2 & \cdots & \mathcal{H}_{n-1} & \mathcal{H}_n \\ 1 & s(2,1) & \cdots & s(n-1,1) & s(n,1) \\ 0 & 1 & \cdots & s(n-1,2) & s(n,2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & s(n,n-1) \end{vmatrix}, \tag{4.16}$$

where $\mathcal{H}_n = H_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ are the G3I3VHMP defined by (2.9).

Example 4.3. The Sheffer polynomials $S_n(p)$ for the pair $(g(t), f(t))$ given by

$$g(t) = \frac{e^t+1}{2} \quad \text{and} \quad f(t) = t, \tag{4.17}$$

reduce to the Euler polynomials $E_n(p)$ [14, Pg. 100]. Consequently taking these values of $(g(t), f(t))$ and

$$f^{-1}(t) = t \quad \text{and} \quad g(f^{-1}(t)) = \frac{e^t+1}{2} \tag{4.18}$$

in the generating function (2.5) of G3I3VHMSP ${}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$, reduce it to the generalized three index three variable Hermite matrix based Euler polynomials (G3I3VHMEP) ${}_HE_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ which are defined by the following generating function:

$$\frac{2\exp\left(pu\sqrt{m\mathfrak{B}}\right)}{(e^u+1)\left(\alpha+qu^mI-ru^\eta I\right)^\mu}=\sum_{n=0}^{\infty}{}_HE_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)\frac{u^n}{n!}. \tag{4.19}$$

The explicit series representation (2.10) of G3I3VHMSP ${}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ reduces to that of G3I3VHMEP ${}_HE_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ for $b_{n,k}$, where $b_{n,k}$ is the (n,k)th entry of the Riordan array $\left(\frac{2}{e^t+1},t\right)$ (see (4.18)). So, in view of (1.9), $b_{n,k}$ is given by

$$b_{n,k}=\binom{n}{k}E_{n-k}. \tag{4.20}$$

Therefore we have

$${}_HE_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)=\sum_{k=0}^{n}\binom{n}{k}E_{n-k}H_{k,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha). \tag{4.21}$$

The other results for the G3I3VHMEP ${}_HE_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ are given in Table 5, where $D_P=\frac{D_p}{\sqrt{m\mathfrak{B}}}$.

S. No.	**Results**	**Expressions**
I	Operational Representation	${}_HE_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ $=\left(\alpha+q(\sqrt{m\mathfrak{B}})^{-m}\frac{\partial^m}{\partial p^m}-r(\sqrt{m\mathfrak{B}})^{-\eta}\frac{\partial^\eta}{\partial p^\eta}\right)^{-\mu}\left\{E_n(p\sqrt{m\mathfrak{B}})\right\}$
II	Multiplicative Operator	$\Phi_{GHE}^{+}:=p\sqrt{m\mathfrak{B}}+mq(\sqrt{m\mathfrak{B}})^{-(m-1)}\frac{\partial^m}{\partial\alpha\partial p^{m-1}}-\eta r(\sqrt{m\mathfrak{B}})^{-(\eta-1)}\frac{\partial^\eta}{\partial\alpha\partial p^{\eta-1}}-\frac{\exp(D_P)}{1+\exp(D_P)}$
III	Derivative Operator	$\Phi_{GHE}^{-}:=D_P$
IV	Differential Equation	$\left(\left\{p\sqrt{m\mathfrak{B}}+mq(\sqrt{m\mathfrak{B}})^{-(m-1)}\frac{\partial^m}{\partial\alpha\partial p^{m-1}}-\eta r(\sqrt{m\mathfrak{B}})^{-(\eta-1)}\frac{\partial^\eta}{\partial\alpha\partial p^{\eta-1}}-\frac{\exp(D_P)}{1+\exp(D_P)}\right\}D_P-n\right){}_HE_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)=0$

Table 5. Results for G3I3VHMEP ${}_HE_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$.

The determinant form (2.23) and (2.24) of G3I3VHMSP ${}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ reduces to that of G3I3VHMEP ${}_HE_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ for $a_{n,k}$, where $a_{n,k}$ is the (n,k)th entry of the Riordan array $\left(\frac{e^t+1}{2},t\right)$. In view of (1.4) and considering the fact that $c_n=n!$ for this case, $a_{n,k}$ is given by

$$a_{n,k}=\frac{n!}{k!}\left[t^{n-k}\right]\frac{e^t+1}{2}=\begin{cases}1, & n=k,\\ \frac{1}{2}\binom{n}{k}, & n\neq k.\end{cases} \tag{4.22}$$

Therefore, we have

$${}_HE_{0,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)=H_{0,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha), \tag{4.23}$$

$${}_HE_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)=\left(-\frac{1}{2}\right)^n\begin{vmatrix}\mathcal{H}_0 & \mathcal{H}_1 & \mathcal{H}_2 & \cdots & \mathcal{H}_{n-1} & \mathcal{H}_n\\ 2 & 1 & 1 & \cdots & 1 & 1\\ 0 & 2 & \binom{2}{1} & \cdots & \binom{n-1}{1} & \binom{n}{1}\\ 0 & 0 & 2 & \cdots & \binom{n-1}{2} & \binom{n}{2}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 2 & \binom{n}{n-1}\end{vmatrix}, \tag{4.24}$$

where $\mathcal{H}_n=H_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ are the G3I3VHMP defined by (2.9).

5. Conclusion

In several cases, the solution of a given problem in engineering mathematics or physics requires the estimation of infinite sums involving special matrix polynomials. This leads to an increase in demand for solving problems by means of identities, functional equations, and formulas in research fields like classical and quantum optics. These identities, functional equations and formulas arise in combinatorial contexts and they lead systematically to well defined classes of functions. The summation formula of hybrid special matrix polynomials of several variables often appears in applications ranging from combinatorics to electromagnetic processes.

Theorem 5.1. *For the G3I3VHMSP ${}_HS^{(m,\eta)}_{n,\mu}(p,q,r;\mathfrak{B};\alpha)$, the following explicit summation formula in terms of the G3I3VHMASP ${}_H\mathfrak{s}^{(m,\eta)}_{n,\mu}(p,q,r;\mathfrak{B};\alpha)$ and Sheffer polynomials $S_n(p)$ holds true:*

$$ {}_HS^{(m,\eta)}_{n,\mu}(p,q,r;\mathfrak{B};\alpha)=\sum_{k=0}^{n}\binom{n}{k}{}_H\mathfrak{s}^{(m,\eta)}_{k,\mu}(0,q,r;\mathfrak{B};\alpha)S_{n-k}(p\sqrt{m\mathfrak{B}}). \tag{5.1}$$

Proof. Consider the product of generating functions (1.1) and (3.2) in the following form:

$$\frac{exp\left(pf^{-1}(t)\sqrt{m\mathfrak{B}}\right)}{g(f^{-1}(t))\,(\alpha+q(f^{-1}(t))^mI-r(f^{-1}(t))^\eta I)^\mu} =\sum_{k=0}^{\infty}{}_H\mathfrak{s}^{(m,\eta)}_{k,\mu}(0,q,r;\mathfrak{B};\alpha)\frac{t^k}{k!}\sum_{n=0}^{\infty}S_n(p\sqrt{m\mathfrak{B}})\frac{t^n}{n!}. \tag{5.2}$$

In view of generating function (2.5) on l.h.s. and Cauchy product rule on r.h.s. gives

$$\sum_{n=0}^{\infty}{}_HS^{(m,\eta)}_{n,\mu}(p,q,r;\mathfrak{B};\alpha)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}{}_H\mathfrak{s}^{(m,\eta)}_{k,\mu}(0,q,r;\mathfrak{B};\alpha)S_{n-k}(p\sqrt{m\mathfrak{B}})\frac{t^n}{n!}, \tag{5.3}$$

which on comparing coefficients of t gives (5.1). □

Several identities involving members of Sheffer polynomials are known. The operational formalism developed in the previous section can be used to obtain the corresponding identities involving the G3I3VHMSP ${}_HS^{(m,\eta)}_{n,\mu}(p,q,r;\mathfrak{B};\alpha)$. To achieve this, we perform the following operation:

Ψ: *Replacing p by $p\sqrt{m\mathfrak{B}}$ and operating* $\left(\alpha+q(\sqrt{m\mathfrak{B}})^{-m}\frac{\partial^m}{\partial p^m}-r(\sqrt{m\mathfrak{B}})^{-\eta}\frac{\partial^\eta}{\partial p^\eta}\right)^{-\mu}$.

Consider the following functional equations involving Bernoulli polynomials $B_n(p)$ [9, Pg. 26] and Euler polynomials $E_n(p)$ [9, Pg. 30]:

$$B_n(p+1)-B_n(p)=np^{n-1},\quad n=0,1,2,\cdots, \tag{5.4}$$

$$\sum_{m=0}^{n-1}\binom{n}{m}B_m(p)=np^{n-1},\quad n=2,3,4,\cdots, \tag{5.5}$$

$$E_n(p+1)+E_n(p)=2p^n, \tag{5.6}$$

$$E_n(mp)=m^n\sum_{k=0}^{m-1}(-1)^kE_n\left(p+\frac{k}{m}\right),\quad n=0,1,2,\cdots;\quad m \text{ odd}. \tag{5.7}$$

Performing operation Ψ on above equations and using operational rules (2.1) and (2.9) (corresponding to the Bernoulli and Euler polynomials), we obtain the following identities involving generalized three index three variable Hermite matrix based Bernoulli polynomials ${}_HB_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ and generalized three index three variable Hermite matrix based Euler polynomials ${}_HE_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$:

$$ {}_HB_{n,\mu}^{(m,\eta)}(p+1,q,r;\mathfrak{B};\alpha) - {}_HB_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) = nH_{n-1,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha), \quad n=0,1,2,\cdots, \tag{5.8} $$

$$ \sum_{m=0}^{n-1}\binom{n}{m}{}_HB_{m,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) = nH_{n-1,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha), \quad n=2,3,4,\cdots, \tag{5.9} $$

$$ {}_HE_{n,\mu}^{(m,\eta)}(p+1,q,r;\mathfrak{B};\alpha) + {}_HE_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha) = 2H_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha), \tag{5.10} $$

$$ {}_HE_{n,\mu}^{(m,\eta)}(mp,q,r;\mathfrak{B};\alpha) = m^n\sum_{k=0}^{m-1}(-1)^k{}_HE_{n,\mu}^{(m,\eta)}\left(p+\frac{k}{m},q,r;\mathfrak{B};\alpha\right), \quad n=0,1,2,\cdots; \quad m \text{ odd}. \tag{5.11} $$

The above examples illustrate that by using the operational correspondence between the Sheffer polynomials and G3I3VHMSP ${}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$, the corresponding results for the G3I3VHMSP ${}_HS_{n,\mu}^{(m,\eta)}(p,q,r;\mathfrak{B};\alpha)$ can be obtained. Operational methods is a widely exploited tool in analysis to simplify the derivation of the properties associated with ordinary and generalized special matrix functions and to define new families of functions.

The use of operational method combining with the fractional order operators are shown to be an effective means, providing a fairly unexhausted source of tool to strengthen the computational capabilities. The operational techniques can also be used for a more general insight into the theory of hybrid special matrix polynomials to represent its determinant form via Riordan array. The Riordan matrices naturally appear in a formulation of the umbral calculus. The Riordan group also appears in the new domain of combinatorial quantum physics, namely in the problem of the normal ordering of boson strings. The appropriate combination of methods relevant to generalized operational calculus and special matrix functions can be a very useful tool to treat a large body of problems both in physics and mathematics. Thus, we conclude that the formalism we have envisaged can be exploited to deal with the possibilities to frame different families of the hybrid special matrix polynomials and to establish their properties.

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