Advances in the Theory of Nonlinear Analysis and its Applications **5** (2021) No. 4, 580–599.
https://doi.org/10.31197/atnaa.799236
Available online at www.atnaa.org
Research Article

Advances in the Theory of Nonlinear Analysis and its Applications

ISSN: 2587-2648 Peer-Reviewed Scientific Journal

Regional Reconstruction of Semilinear Caputo Type Time-Fractional Systems Using the Analytical Approach

Khalid Zguaid[a], Fatima-Zahrae El Alaoui[a], Ali Boutoulout[a]

[a] *Department of Mathematics, Moulay Ismail University, Meknes, Morocco.*

Abstract

The aim of this paper is to investigate the concept of regional observability, precisely regional reconstruction of the initial state, for a semilinear Caputo type time-fractional diffusion system. The approaches attempted in this work are both based on fixed point techniques that leads to a successful algorithm which is tested by numerical examples.

Keywords: Regional Obervability Control Theory Fractional Calculus Semilinear Systems.
2010 MSC: 93B07 93B28 93B99 26A33.

1. Introduction

"An apparent paradox from which one day useful consequences will be drawn", the answer that Leibniz gave when l'Hospital had asked him about the meaning of $\dfrac{d^n}{dx^n}f(x)$ when $n = \dfrac{1}{2}$, and then fractional calculus was born.

Even-though fractional calculus is approximately 300 years of age, it only caught so much attention in the last 25 years or so, the main reason being, its capability of a better describing of real world phenomena. In fact a lot of works show that non-integer order ordinary and partial differential equations present, most times, better modeling systems than integer order ones, for instance in [6] the authors presented a model of beam heating process, and with experimental setup (thermo-electrical module), theoretical results were

Email addresses: `zguaid.khalid@gmail.com` (Khalid Zguaid), `fzelalaoui2011@yahoo.fr` (Fatima-Zahrae El Alaoui), (Ali Boutoulout)

Received :September 23, 2020; Accepted: July 28, 2021; Online: July 30, 2021

verified and a high degree of accuracy was obtained in the experimental ones. Also the modeling of the ultra-capacitor is given in [8], and in [7] both examples of the heat beam process and the ultra-capacitor are given. More information on fractional calculus can be found in [1, 12, 13, 14, 22, 24, 32, 35, 36, 37, 39, 40]

Control theory is an important and very active branch of mathematics which serves as a link between theoretical mathematics and its applications in the real world where most processes are modeled by nonlinear distributed parameter systems, which explains the big interest of researchers in the study (Controllability, Observability, Stability...) of nonlinear and semilinear systems. As for the study of linear systems, there exists a very wide literature for integer order system, see [9] - [34] and the references therein, whereas for fractional order systems there is a much less literature, see [17, 26, 29, 30, 38] and the references therein.

Most works in control theory precisely in observability deal with state estimation in the whole evolution domain, namely Ω, of the considered system (global observability) [34], but in the 90's the regional observability concept was born by professor El Jai et al. [31], and was after that developed by others, its purpose is to estimate the initial state of a given system only in a subregion $\omega \subset \Omega$, with positive Lebesgue measure, several works have treated this notion for various kind of systems [3, 4, 5, 18, 20, 23]. The main reason for introducing such a concept is its applicability to non observable systems in the whole domain.

Here we give an extension of the results of regional observability of semilinear systems to time-fractional semilinear systems. We make two approaches in order to reconstruct the initial state in ω, the first one consists of reconstructing the trajectory of the system in ω, "$y(t)|_\omega$", than substituting t with 0, we get the wanted result (the direct approach), while the second one, where we make some assumptions so that the dynamic of the system generates an analytical semigroup on the state space, gives directly the initial state in ω as a fixed point of a function to be defined later.

This manuscript is organized as follows. In the second section we give some introductory notions and useful tools for a better comprehension of the manuscript, as for the third section we introduce the considered system, the definition of its mild solution and we also talk about regional observability. The fourth section deals with the direct approach whereas in the fifth one we present the analytical approach , and just before the conclusion, we give, in section six an algorithm for the regional reconstruction of the initial state also as two numerical results.

2. Preliminary Notes

We layout, in this section, some definitions and properties, which will be used all over this manuscript. We will start with the definition of the Caputo left fractional derivative.

Definition 2.1. *[2] The left sided fractional derivative of order $\alpha \in]0,1[$ in Caputo's sense of $y(x,t)$ with respect to t, is given by the following expression*

$$^{C}D^{\alpha}_{0^+}y(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{\partial}{\partial s}y(x,s)ds.$$

For two Banach spaces E and F we denote the space of all linear bounded mappings defined from E to F by $\mathcal{L}(E,F)$ and $\mathcal{L}(E) := \mathcal{L}(E,E)$, also we mean by $L^p(0,T;E)$ (resp. $L^p_{loc}(0,T;E)$), the space of all vector-valued functions f going from the interval $[0,T]$ to E, which are measurable, such that $\|f(.)\|_E$ is in $L^p(0,T)$ (resp. $L^p_{loc}(0,T)$).

The two following propositions show, respectively, the existence of the convolution between an operator and a vector-valued function, and the Young inequality for this convolution [41].

Proposition 2.2. *[41] Let E and F be two Banach spaces. Let's consider $v \in L^1_{loc}(0,T;E)$ and $\mathcal{T} : [0,T] \to \mathcal{L}(E,F)$ be strongly continuous. Then the convolution*

$$(\mathcal{T} * v)(t) := \int_0^t \mathcal{T}(t-s)v(s)ds,$$

exists (in the Bochner sense) and is a continuous function ($\mathcal{T} * v : [0,T] \to F$).

The previous convolution $(*)$ is called the Laplace convolution operator.

Proposition 2.3. *[41] With the same considerations as in the previous proposition. Let's consider* $p, q, r \geq 1$ *such that* $\frac{1}{q} + \frac{1}{p} = 1 + \frac{1}{r}$.
If $\mathcal{T} \in L^p(0,T;\mathcal{L}(E,F))$ *and* $v \in L^q(0,T;E)$, *then* $\mathcal{T} * v \in L^r(0,T;F)$ *and we have*

$$\|\mathcal{T} * v\|_{L^r(0,T;F)} \leq \|v\|_{L^q(0,T;E)} . \|\mathcal{T}\|_{L^p(0,T;\mathcal{L}(E,F))}.$$

For more information about vector-valued analysis see [21, 41].
For a linear operator A, the set $\sigma(A) := \left\{\lambda \in \mathbb{C} \,\middle|\, (A - \lambda Id)^{-1} \text{ exists}\right\}$ is called the resolvant set and $R(\lambda, A) := (A - \lambda Id)^{-1}$, for all λ in $\sigma(A)$, is called the resolvant of A.
An important type of C_0-semigroup is the analytic one, especially in nonlinear systems. Before we give its definition, we need to define the following sector $\Delta_\theta := \{z \in \mathbb{C} \mid |arg(z)| < \theta \ , \ z \neq 0\}$, where $\theta \in]0, \pi[$.

Definition 2.4. *[21] Let* $\{S(t)\}_{t\geq 0}$ *be a* C_0*-semigroup on a Banach space* E. *We say that* $\{S(t)\}_{t\geq 0}$ *is an analytic semigroup if there exists* θ *in* $]0, \pi[$ *such that* $\{S(t)\}_{t\geq 0}$ *can be extended to the sector* Δ_θ *and it satisfies the following properties :*

- *i-* $S(0) = Id_E$.
- *ii-* $S(w + z) = S(w)S(z), \quad \forall z, w \in \Delta_\theta \cup \{0\}$.
- *iii-* $\lim\limits_{\substack{z \to 0 \\ z \in \Delta_\theta}} S(z)x = 0, \quad \forall x \in E$.
- *iv-* $z \longrightarrow S(z)$ *is analytic from* Δ_θ *to* $\mathcal{L}(E)$.

Here is a useful characterization of an analytic semigroup.

Proposition 2.5. *[28] Let* A *be a densely defined, linear closed operator from* E *to itself. If we assume that* A *enjoys the following conditions :*

- *i- For some* $\theta \in]\frac{\pi}{2}, \pi[$, *we have that* $\Delta_\theta \subset \sigma(A)$.
- *ii-* $\exists M_0 > 0$ *such that the resolvant* $R(\lambda, A)$ *satisfies the estimate*

$$\|R(\lambda, A)\|_{\mathcal{L}(E,E)} \leq \frac{M_0}{|\lambda|}, \quad \forall \lambda \in \Delta_\theta. \tag{1}$$

Then, A *generates an analytic semigroup on* E.

We now introduce the notion of fractional powers for operators.

Definition 2.6. *[21] Let* $A : D(A) \subset E \longrightarrow E$ *be a linear, possibly unbounded, operator that generates a* C_0*-semigroup* $\{S(t)\}_{t\geq 0}$ *on* E. *Let's consider* $\alpha \in]0, 1[$, *The fractional power, of order* α, *of* A *is defined by,*

$$\begin{array}{rcl} (-A)^{-\alpha} & : & D\left((-A)^{-\alpha}\right) \subset E \longrightarrow E \\ & & y \longmapsto \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^{+\infty} t^{\alpha-1} S(t) dt, \end{array}$$

where $D\left((-A)^{-\alpha}\right) = \left\{x \in E \mid \displaystyle\int_0^{+\infty} t^{\alpha-1} S(t) dt < +\infty\right\}$ *is its domain of definition. By definition* $A^0 = Id_E$.

The operator $(-A)^{-\alpha}$ is in fact injective and eventually invertibly bounded, from X into $\mathcal{I}m((-A)^{-\alpha})$, for all α in $]0,1[$ see [28]. Hence $(-A)^{\alpha}$ can be defined as the inverse of $(-A)^{-\alpha}$, where $D((-A)^{\alpha}) := \mathcal{I}m((-A)^{-\alpha})$ see [33].

Before we finish the current section, let's introduce the Following weighted Lebesgue space, for all $q \geq 1$ and $\alpha \leq 1$

$$L^q_{\alpha-1}[0,T] = \left\{ f : [0,T] \longrightarrow \mathbb{K} \text{ measurable } \quad \middle| \quad \int_0^T \left|t^{\alpha-1}f(t)\right|^q dt < +\infty \right\},$$

which is a Banach space endowed with the norm

$$\|f\|_{L^q_{\alpha-1}[0,T]} = \left[\int_0^T \left|t^{\alpha-1}f(t)\right|^q dt\right]^{\frac{1}{q}}.$$

Remark 2.7. *For all $q \geq 1$, we have the following inclusions,*

$$L^q_{\alpha-1}[0,T] \subset L^q[0,T] \quad \text{and} \quad L^p_{\alpha-1}[0,T] \subset L^q_{\alpha-1}[0,T], \quad \forall p \geq q.$$

3. Considered System

Let Ω be an open bounded subset on $\mathbb{R}^n$, with smooth boundary $\Gamma = \partial\Omega$ and $]0,T]$ a time interval. Let's denote $Q = \Omega \times]0,T]$ and $\Sigma = \Gamma \times]0,T]$. Consider the following fractional semi-linear evolution equation :

$$\begin{cases} {}^C D^{\alpha}_{0+} y(x,t) = Ay(x,t) + Ny(x,t) & in \ Q \ , \quad \alpha \in]0,1[\, , \\ y(\xi,t) = 0 & in \ \Sigma, \\ y(x,0) = y_0(x) \in L^2(\Omega) & in \ \Omega, \end{cases} \tag{2}$$

together with the output function :

$$z(t) = Cy(.,t). \tag{3}$$

Where :

∗ ${}^C D^{\alpha}_{0+}$ is the left sided fractional derivative in Caputo's sense.
∗ y_0 is the initial state.
∗ A is a linear, second order, differential operator which generates a C_0-semigroup $\{S(t)\}_{t\geq 0}$ on the state Space $L^2(\Omega)$.
∗ C is a linear operator (operator of observation) from $L^2(\Omega)$ into $\mathcal{O}$ (the observation space).
∗ N is a nonlinear operator, assumed to be defined to ensure the existence and uniqueness of a mild solution of the system (2) in $L^2(0,T;L^2(\Omega))$ see [25, 42, 43].

Without loss of generality we denote $y(.,t) = y(t)$.

We associate with the system (2), the following linear system

$$\begin{cases} {}^C D^{\alpha}_{0+} y(t) = Ay(t), & t \in]0,T] \, , \\ y(\xi,t) = 0, & in \ \Sigma, \\ y(0) = y_0 \in L^2(\Omega). \end{cases} \tag{4}$$

We call a mild solution of (2) any function $y \in C\left(0,T;L^2(\Omega)\right)$ which is written as follows [25, 42, 43]:

$$y(t) = S_{\alpha}(t)y_0 + \int_0^t (t-\tau)^{\alpha-1}\mathcal{H}_{\alpha}(t-\tau)Ny(\tau)d\tau, \quad in \quad [0,T] \, , \tag{5}$$

where $S_\alpha(t) = \int_0^\infty \mathcal{W}_\alpha(\theta)S(t^\alpha\theta)d\theta$, and $\mathcal{H}_\alpha(t) = \alpha\int_0^\infty \theta\mathcal{W}_\alpha(\theta)S(t^\alpha\theta)d\theta$.

The function $\mathcal{W}_\alpha$ is called "The Mairandi function", and it is written as follows :

$$\mathcal{W}_\alpha(\theta) = \sum_{n=1}^{\infty}\frac{(-\theta)^{n-1}}{\Gamma(n)\Gamma(1-\alpha n)} = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\varrho_\alpha(\theta^{-\frac{1}{\alpha}}), \quad \theta \geq 0, \tag{6}$$

where ϱ_α is a probability density function defined on $]0, +\infty[$ by :

$$\varrho_\alpha(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}sin(n\pi\alpha). \tag{7}$$

Remark 3.1. *The Mairandi function is an alternating series, hence its sign is the same as its first term, which is* $\frac{1}{\Gamma(1)\Gamma(1-\alpha)} > 0$*, thus* $\mathcal{W}_\alpha(\theta) > 0$*.*

The Mairandi function satisfies the following

Proposition 3.2. *[25, 43] For all* $v \geq -1$*, we have :*

$$\int_0^\infty \theta^v\mathcal{W}_\alpha(\theta)d\theta = \frac{\Gamma(1+v)}{\Gamma(1+\alpha v)}. \tag{8}$$

Let ω be a subregion of Ω with positive Lebesgue measure, we define the restriction operator χ_ω : $L^2(\Omega) \longrightarrow L^2(\omega)$ for all $y \in L^2(\Omega)$ by $\chi_\omega y = y_{|\omega}$ and we denote by χ_ω^* its adjoint. We decompose the initial state to two parts : $\tilde{y}_0 = \chi_\omega y_0$, the restriction of y_0 in ω or the initial state in ω (to be reconstructed), and $\overline{y}_0$, the residual (undesired) part of y_0, we then have $y_0 = \chi_\omega^*\tilde{y}_0 + \chi_{\Omega\backslash\omega}^*\overline{y}_0$.

We introduce the operator $L_\alpha(.) : L^2(0,T;L^2(\Omega)) \longrightarrow L^2(0,T;L^2(\Omega))$, defined as follows :
$\forall x \in L^2(0,T;L^2(\Omega)), \quad \forall t \in [0,T], \quad L_\alpha(t)x = \int_0^t (t-s)^{\alpha-1}\mathcal{H}_\alpha(t-s)x(s)ds.$
The solution of the system (2) can be written

$$y(t) = S_\alpha(t)\chi_\omega^*\tilde{y}_0 + S_\alpha(t)\chi_{\Omega\backslash\omega}^*\overline{y}_0 + L_\alpha(t)Ny. \tag{9}$$

The observability operator can be given by

$$\begin{aligned} K_\alpha : L^2(\Omega) &\longrightarrow L^2(0,T;\mathcal{O}) \\ x &\longmapsto K_\alpha x, \end{aligned}$$

for all $t \in [0,T]$, $(K_\alpha x)(t) = CS_\alpha(t)x$, and let $K_\alpha^\omega = K_\alpha\chi_\omega^*$.
If C is unbounded, we assume that it is an admissible observation operator for S_α, that is :

$$\exists N > 0, \text{ satisfying } \int_0^T \|(K_\alpha v)(t)\|_{\mathcal{O}}^2 dt \leq N\|v\|_{L^2(\Omega)}^2, \quad \forall v \in L^2(\Omega).$$

Remark 3.3. *If* C *is bounded, then it is an admissible observation operator.*

The condition of admissibility on C gives us the wright to extend the operator $CS_\alpha(t)$ to a bounded linear operator from $L^2(\Omega)$ to $\mathcal{O}$, see [11, 16].
In both cases (bounded or not) the adjoint operator of K_α, can be written

$$\begin{aligned} (K_\alpha)^* \ : \ \mathcal{D}((K_\alpha)^*) \subset L^2(0,T;\mathcal{O}) &\longrightarrow X \\ z^* &\longmapsto \int_0^T S_\alpha^*(s)C^*z^*(s)ds, \end{aligned}$$

Definition 3.4. *[19] The system (4)-(3) is said to be approximately observable in ω (or approximately ω-observable) if and only if*

$$\overline{Im(\chi_\omega K_\alpha^*)} = L^2(\omega).$$

This definition is equivalent to $\mathcal{K}er(K_\omega^\alpha) = \{0\}$.
The operator $\left(K_\alpha^\omega\right)^\dagger := \left[\left(K_\alpha^\omega\right)^* \left(K_\alpha^\omega\right)\right]^{-1} \left(K_\alpha^\omega\right)^*$, called the pseudo inverse of K_α^ω, is well defined if the system (4)-(3) is approximately ω-observable [10].

Definition 3.5. *We say that the system (2), augmented with the measurements (3), is initially continuously observable in ω, if it is possible to reconstruct $y(0)|_\omega$, depending on z in a continuous way.*

Problem : Considering any system (2), with the output (3), in $[0,T]$, can we reconstruct the initial state $\tilde{y}_0$ in ω ?

4. Direct Approach

For $\omega \subset \Omega$, we assume that (4)-(3) is approximately ω-observable. We define the following mapping

$$\begin{array}{rcccl} \Phi & : & L^2(0,T;L^2(\Omega)) & \longrightarrow & L^2(0,T;L^2(\Omega)) \\ & & y & \longmapsto & \Phi(y), \end{array}$$

$$\begin{aligned} \Phi(y)(t) \;=\; & S_\alpha(t)\chi_\omega^* \left(K_\alpha^\omega\right)^\dagger \left(z(.) - CS_\alpha(.)\chi_{\Omega\backslash\omega}^* \overline{y} - CL_\alpha(.)Ny\right) \\ & + S_\alpha(t)\chi_{\Omega\backslash\omega}^* \overline{y} + L_\alpha(t)Ny, \end{aligned} \tag{10}$$

for all t in $[0,T]$, where $\overline{y}$ is in $L^2(\Omega \setminus \omega)$, such that $\chi_\omega \overline{y} = 0$.

Proposition 4.1. *The initial state of (2) in ω is the restriction in ω of a fixed point of Φ at $t=0$.*

Proof. We have

$$y(.) = S_\alpha(.)\chi_\omega^* \tilde{y}_0 + S_\alpha(.)\chi_{\Omega\backslash\omega}^* \overline{y}_0 + L_\alpha(.)Ny. \tag{11}$$

By applying the operator C, we get

$$z(.) = CS_\alpha(.)\chi_\omega^* \tilde{y}_0 + CS_\alpha(.)\chi_{\Omega\backslash\omega}^* \overline{y}_0 + CL_\alpha(.)Ny.$$

Which gives

$$K_\alpha^\omega \tilde{y}_0 = z(.) - CS_\alpha(.)\chi_{\Omega\backslash\omega}^* \overline{y}_0 - CL_\alpha(.)Ny,$$

and, since (4)-(3) is approximately ω-observable, by applying the pseudo inverse of K_α^ω, we have

$$\tilde{y}_0 = \left(K_\alpha^\omega\right)^\dagger \left(z(.) - CS_\alpha(.)\chi_{\Omega\backslash\omega} \overline{y}_0 - CL_\alpha(.)Ny\right). \tag{12}$$

Substituting (12) in (11), we get

$$\begin{aligned} \forall t \in [0,T], \quad y(t) \;=\; & S_\alpha(t)\chi_\omega^* \left(K_\alpha^\omega\right)^\dagger \left(z(.) - CS_\alpha(.)\chi_{\Omega\backslash\omega}^* \overline{y}_0 - CL_\alpha(.)Ny\right) \\ & + S_\alpha(t)\chi_{\Omega\backslash\omega}^* \overline{y}_0 + L_\alpha(t)Ny \\ =\; & \Phi(y)(t). \end{aligned} \tag{13}$$

Thus $y(.)$ is a fixed point of Φ, and

$$y(0)|_\omega = \left(K_\alpha^\omega\right)^\dagger \left(z(.) - CS_\alpha(.)\chi_{\Omega\backslash\omega}^* \overline{y}_0 - CL_\alpha(.)Ny\right).$$

□

For the next result we suppose that Φ has a unique fixed point $y^*(.)$, for example if Φ is a strict contraction.

Proposition 4.2. *If the following condition is satisfied,*

$$\left(z(.) - CS_\alpha(.)\chi^*_{\Omega\backslash\omega}\overline{y} - CL_\alpha(.)Ny^*\right) \in \mathcal{I}m(K^\omega_\alpha). \tag{14}$$

Then $y^*(0)|_\omega$ is the estimated initial state of (2) in ω.

Proof. We have

$$y^*(t) = \Phi(y^*)(t), \quad \forall t \in [0,T],$$

thus

$$\begin{aligned} Cy^*(t) \;\; &= \;\; \left(K^\omega_\alpha\,(K^\omega_\alpha)^\dagger\left(z(.) - CS_\alpha(.)\chi^*_{\Omega\backslash\omega}\overline{y} - CL_\alpha(.)Ny^*\right)\right)(t) \\ &\quad +CS_\alpha(t)\chi^*_{\Omega\backslash\omega}\overline{y} + CL_\alpha(t)Ny^*\;, \quad \forall t \in [0,T], \end{aligned}$$

by virtue of (14), we have

$$\exists v \in L^2(\omega) \text{ such that } z(.) - CS_\alpha(.)\chi^*_{\Omega\backslash\omega}\overline{y} - CL_\alpha(.)Ny^* = K^\omega_\alpha v,$$

then by applying $\left(K^\omega_\alpha\right)^\dagger$ we get

$$\left(K^\omega_\alpha\right)^\dagger\left(z(.) - CS_\alpha(.)\chi^*_{\Omega\backslash\omega}\overline{y} - CL_\alpha(.)Ny^*\right) = \left(K^\omega_\alpha\right)^\dagger K^\omega_\alpha v = v,$$

thus

$$\begin{aligned} K^\omega_\alpha\left(K^\omega_\alpha\right)^\dagger\left(z(.) - CS_\alpha(.)\chi^*_{\Omega\backslash\omega}\overline{y} - CL_\alpha(.)Ny^*\right) &= K^\omega_\alpha v \\ &= z(.) - CS_\alpha(.)\chi^*_{\Omega\backslash\omega}\overline{y} - CL_\alpha(.)Ny^*, \end{aligned}$$

which gives, for all t

$$Cy^*(t) = z(t) - CS_\alpha(t)\chi^*_{\Omega\backslash\omega}\overline{y} - CL_\alpha(t)Ny^* + CS_\alpha(t)\chi^*_{\Omega\backslash\omega}\overline{y} + CL_\alpha(t)Ny^* = z(t),$$

then

$$Cy^*(.) = z(.),$$

and we have

$$y^*(0)|_\omega = (K^\omega_\alpha)^\dagger\left(z(.) - CS_\alpha(.)\chi^*_{\Omega\backslash\omega}\overline{y} - CL_\alpha(.)Ny^*\right).$$

□

In all the previous results we worked with the residual part being any function in $L^2(\Omega\setminus\omega)$, so we can take $\overline{y}_0 = 0$ for the rest of this work.

5. Analytical Approach

In this section, we shall use another approach where we make some assumptions that will allow our dynamic, A, to generate an analytic semigroup and $-A$ to have a fractional power of order $\alpha \in]0,1[$. Both of these consequences play an important role in the resolution of semilinear evolution systems. In fact the benefit of working with a dynamic that generates an analytic semigroup is that it provides good information one has on the behavior of the solution at time $t \longrightarrow 0^+$, whereas, fractional powers of $-A$ allow us to define interpolation spaces ,between $D(A)$ and $L^2(\Omega)$, in which might lie the solution of our system, since, for semilinear systems, the solution might not live in the evolution space, in our case $L^2(\Omega)$.

We make the following assumptions on the operator A :

i- $\exists\theta \in]\frac{\pi}{2},\pi[$, $\exists b > 0$, such that

$$\Delta_\theta - b := \left\{z \in \mathbb{C} \,\middle|\, |arg(z+b)| < \theta\right\} \subset \sigma(A).$$

ii- $\exists M_1 > 0$, such that $R(\lambda, A)$ satisfies

$$\|R(\lambda, A)\|_{\mathcal{L}(X,X)} \leq \frac{M_1}{1+|\lambda|}, \qquad \forall \lambda \in \Delta_\theta - b.$$

The conditions (i) and (ii) provide us with some useful consequences, see [28]. The first consequence is that the fractional power of the operator $-A$, of order $\alpha \in]0,1[$, is well defined and $D((-A)^\alpha)$ is dense in $L^2(\Omega)$. The second one is that, using proposition (2.5), A generates an analytic semigroup, denoted again by $\{S(t)\}_{t\geq 0}$, on the state space $L^2(\Omega)$, in fact one can see that if (i) is satisfies then $\Delta_\theta \subset \sigma(A)$ and if (ii) is verified, (1) is also true.
For any α in $]0,1[$ we denote, $X_\alpha := D((-A)^\alpha)$ in which we define the following norm $\|x\|_{X^\alpha} := \|(-A)^\alpha x\|$. Since $(-A)^\alpha$ is bounded then $\|.\|_{X^\alpha}$ is equivalent to the graph norm. In addition to the fact that $(-A)^\alpha$ is closed, we get that X_α, endowed with the norm $\|.\|_{X^\alpha}$, is a Banach space.

Remark 5.1. *The sequence $(D((-A)^\alpha))_{0\leq\alpha\leq 1}$ is a set of interpolation spaces between $D(A)$ and $L^2(\Omega)$ (i.e. $D((-A)^0) = L^2(\Omega)$, $D((-A)^1) = D(A)$ and for $0 < \alpha < \beta < 1$ we have $D((-A)^\beta) \subset D((-A)^\alpha)$).*

Remark 5.2. *For the fractional power of $-A$, it is possible to choose a different value from the order of derivation of the state in (2), one can reprove the upcoming results with a slight change in the expression of some constants, we kept the same order to simplify the calculations.*

For the operator N we need to make the following hypotheses :

iii- $\exists\, q, p, r \geq 1$ verifying $\dfrac{1}{q} + \dfrac{1}{p} = 1 + \dfrac{1}{r}$ such that :

- $*)$ $\exists g_1 \in L^{r}_{\alpha-1}[0,T], \forall\theta \geq 0, \forall t \in]0,T]$, $\quad ||S(t^\alpha\theta)||_{\mathcal{L}(X,X^\alpha)} \leq |g_1(t)|$.
- $**)$ The nonlinear operator $N : L^{r}(0,T;X^\alpha) \longrightarrow L^{p}(0,T;L^2(\Omega))$ is well defined and satisfies

$$\begin{cases} \bullet N(0) = 0. \\ \bullet ||Nx - Ny||_{L^p(0,T;L^2(\Omega))} \leq k(||x||,||y||)||x-y||_{L^r(0,T;X^\alpha)} \;, \\ \qquad \forall x,y \in L^{r}(0,T;X^\alpha). \\ \text{with} \\ k : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+ \text{ such that } k(\theta_1,\theta_2) \xrightarrow[\theta_1,\theta_2 \to 0]{} 0. \end{cases} \tag{15}$$

Even-though the condition (iii)-(**) on the operator N might seem harsh, yet, it can actually be achieved as shown in [27], in fact it was used to obtain the exact global observability of semilinear classical systems, also in [15] we find that this condition on N is valid for an important classes of systems, such as the Burgers' equation.

Proposition 5.3. *If (iii)-(*) is satisfied, we have*

$$\forall t \in [0,T],\ \forall\theta \geq 0 \quad \|S_\alpha(t)\|_{\mathcal{L}(X,X^\alpha)} \leq |g_1(t)| \quad \textit{and} \quad ||\mathcal{H}_\alpha(t)||_{\mathcal{L}(X,X^\alpha)} \leq \frac{|g_1(t)|}{\Gamma(\alpha)}.$$

Proof. Let's consider $t \in [0,T]$ and $\theta \geq 0$, then for the first inequality we have

$$\begin{aligned} ||S_\alpha(t)||_{\mathcal{L}(L^2(\Omega),X^\alpha)} &\leq \int_0^\infty |\mathcal{W}_\alpha(\theta)|\,||S(t^\alpha\theta)||_{\mathcal{L}(L^2(\Omega),X^\alpha)}\,d\theta \\ &\leq |g_1(t)| \int_0^\infty \mathcal{W}_\alpha(\theta)d\theta, \end{aligned}$$

remark (3.1) implies that $|\mathcal{W}_\alpha(\theta)| = \mathcal{W}_\alpha(\theta)$, and by using (8), we get

$$||S_\alpha(t)||_{\mathcal{L}(L^2(\Omega),X^\alpha)} \leq |g_1(t)|.$$

As for the second

$$\begin{aligned} ||\mathcal{H}_\alpha(t)||_{\mathcal{L}(L^2(\Omega),X^\alpha)} &\leq \alpha \int_0^\infty \theta |\mathcal{W}_\alpha(\theta)| \, ||S(t^\alpha \theta)||_{\mathcal{L}(L^2(\Omega),X^\alpha)} \, d\theta \\ &\leq |g_1(t)| \alpha \int_0^\infty \theta \mathcal{W}_\alpha(\theta) d\theta, \end{aligned}$$

again by (8), we deduce that $||\mathcal{H}_\alpha(t)||_{\mathcal{L}(L^2(\Omega),X^\alpha)} \leq \dfrac{|g_1(t)|}{\Gamma(\alpha)}$.

□

The goal here is to study the regional reconstruction problem for (2)-(3) in $V = \mathcal{I}m(\chi_\omega K^*_\alpha)$, which is a Banach space endowed with the norm $||.||_V = ||K^\omega_\alpha(.)||_{L^2(0,T;\mathcal{O})}$.

Let's put $f(t) = t^{\alpha-1}\mathcal{H}_\alpha(t)$, which gives $L_\alpha(t)x = (f * x)(t)$,
We have the following propositions.

Proposition 5.4. *The following inequality is satisfied*

$$||f(.)||_{L^q(0,T;\mathcal{L}(L^2(\Omega),X^\alpha))} \leq \frac{||g_1(.)||_{L^q_{\alpha-1}[0,T]}}{\Gamma(\alpha)}. \tag{16}$$

Proof. We have

$$||f(.)||_{L^q(0,T;\mathcal{L}(L^2(\Omega),X^\alpha))} = \left[\int_0^T \|t^{\alpha-1}\mathcal{H}_\alpha(t)\|^q_{\mathcal{L}(X,X^\alpha)} dt\right]^{\frac{1}{q}},$$

then

$$||f(.)||_{L^q(0,T;\mathcal{L}(L^2(\Omega),X^\alpha))} = \left[\int_0^T t^{q(\alpha-1)}\|\mathcal{H}_\alpha(t)\|^q_{\mathcal{L}(X,X^\alpha)} dt\right]^{\frac{1}{q}},$$

using proposition 5.3, we get

$$||f(.)||_{L^q(0,T;\mathcal{L}(L^2(\Omega),X^\alpha))} \leq \frac{1}{\Gamma(\alpha)}\left[\int_0^T |t^{\alpha-1}g_1(t)|^q_{L^q[0,T]} dt\right]^{\frac{1}{q}},$$

finally

$$||f(.)||_{L^q(0,T;\mathcal{L}(L^2(\Omega),X^\alpha))} \leq \frac{1}{\Gamma(\alpha)}||g_1(.)||_{L^q_{\alpha-1}[0,T]}.$$

□

Proposition 5.5. *If the system (4) augmented by (3) is approximately observable, then the embedding $V \hookrightarrow X(\omega) := L^2(\omega)$, is continuous.*
That is

$$\exists \kappa > 0, \ \textit{\textbf{such that}} \ \|y_0\|_{X(\omega)} \leq \kappa \|y_0\|_V, \quad \forall y_0 \in V.$$

Proof. Let's consider $y_0 \in V$, then

$$\|y_0\|_{X(\omega)} = \left|\left|(K^\omega_\alpha)^\dagger K^\omega_\alpha y_0\right|\right|_{X(\omega)} \leq \left|\left|(K^\omega_\alpha)^\dagger\right|\right|_{\mathcal{L}(L^2(0,T;\mathcal{O}),X(\omega))} \|K^\omega_\alpha y_0\|_{L^2(0,T;\mathcal{O})},$$

and the fact that the operator $(K^\omega_\alpha)^\dagger$ is bounded comes from the admissibility condition of C, then

$$\exists \kappa > 0, \ \forall y_0 \in V, \quad \|y_0\|_{X(\omega)} \leq \kappa \|y_0\|_V.$$

□

Using (9) we set

$$\phi_{\tilde{y}_0}(y(.)) = S_\alpha(.)\chi^*_\omega \tilde{y}_0 + L_\alpha(.)Ny. \tag{17}$$

We shall now, in the following proposition, show the existence of a set (ball) of admissible initial states in ω, in the sense that they give a unique solution of (2) in a ball of $L^r(0,T;X^\alpha)$.

Theorem 5.6. *Assume that (iii)-(*) and (15) hold, the following results are satisfied.*

1 . *There exists positive numbers $a, m = m(a)$, sucht that :*
$\forall \tilde{y}_0 \in \overline{B(0,m)} \subset V$, the function $\phi_{\tilde{y}_0}$ has a unique fixed point, solution of (2), in $\overline{B(0,a)} \subset L^r(0,T;X^\alpha)$.

2 . *The mapping*

$$\begin{array}{rcccc} h & : & \overline{B(0,m)} & \longrightarrow & \overline{B(0,a)} \\ & & \tilde{y}_0 & \longmapsto & y(.), \end{array} \tag{18}$$

"which for every initial state $\tilde{y}_0$ in ω gives us the corresponding unique solution of (2)", satisfies the Lipschitz condition.

Remark 5.7. *The constants a and $m(a)$ are not unique, in fact*
$\forall b \leq a, \exists\ m(b)$ such that the pair $(b, m(b))$ satisfy the last proposition.

Proof. :
1 . We have $k(\theta_1,\theta_2) \xrightarrow[\theta_1,\theta_2\to 0]{} 0$, then $\exists\ a > 0$, $\exists \nu > 0$ such that

$$k(\theta_1,\theta_2) < \nu < \frac{\Gamma(\alpha)}{||g_1(.)||_{L^q_{\alpha-1}[0,T]}} \qquad \forall \theta_1,\theta_2 \leq a,$$

which gives

$$\underset{\theta_i\leq a}{Sup}\ k(\theta_1,\theta_2) \leq \nu < \frac{\Gamma(\alpha)}{||g_1(.)||_{L^q_{\alpha-1}[0,T]}},$$

then

$$D_1 := \frac{||g_1(.)||_{L^q_{\alpha-1}[0,T]}}{\Gamma(\alpha)} \underset{\theta_i\leq a}{Sup}\ k(\theta_1,\theta_2) < 1.$$

$\forall x, y \in \overline{B(0,a)}$, we have :

$$\begin{aligned} ||\phi_{\tilde{y}_0}(x(.)) - \phi_{\tilde{y}_0}(y(.))||_{L^r(0,T;X^\alpha)} &= ||L_\alpha(.)(Nx - Ny)||_{L^r(0,T;X^\alpha)} \\ &= ||(f * (Nx - Ny))(.)||_{L^r(0,T;X^\alpha)} \\ &\leq ||f(.)||_{L^q(0,T;\mathcal{L}(L^2(\Omega),X^\alpha))}.||Nx - Ny||_{L^p(0,T;L^2(\Omega))}, \end{aligned}$$

using (16) and (15), we get

$$\begin{aligned} ||\phi_{\tilde{y}_0}(x(.)) - \phi_{\tilde{y}_0}(y(.))||_{L^r(0,T;X^\alpha)} &\leq \frac{||g_1(.)||_{L^q_{\alpha-1}[0,T]}}{\Gamma(\alpha)} k(||x||,||y||)||x-y||_{L^r(0,T;X^\alpha)} \\ &\leq \frac{||g_1(.)||_{L^q_{\alpha-1}[0,T]}}{\Gamma(\alpha)} \underset{\theta_i\leq a}{Sup}\ k(\theta_1,\theta_2)||x-y||_{L^r(0,T;X^\alpha)} \\ &\leq D_1||x-y||_{L^r(0,T;X^\alpha)}. \end{aligned}$$

Thus $\phi_{\tilde{y}_0}$ is a strict contraction.
Let's now show that $\phi_{\tilde{y}_0}\left(\overline{B(0,a)}\right) \subset \overline{B(0,a)}$.
Let y be an element in $\overline{B(0,a)}$, then

$$\begin{aligned}
||\phi_{\tilde{y}_0}(y(.))||_{L^r(0,T;X^\alpha)} &\leq ||S_\alpha(.)\chi^*_\omega\tilde{y}_0||_{L^r(0,T;X^\alpha)} + ||L_\alpha(.)Ny||_{L^r(0,T;X^\alpha)} \\
&\leq ||\tilde{y}_0||_{X(\omega)}.||g_1(.)||_{L^r[0,T]} \\
&\quad +||f(.)||_{L^q(0,T;\mathcal{L}(L^2(\Omega),X^\alpha))}.||Ny||_{L^p(0,T;L^2(\Omega))} \\
&\leq \kappa||\tilde{y}_0||_V.||g_1(.)||_{L^r[0,T]} \\
&\quad +\frac{||g_1(.)||_{L^{\frac{q}{\alpha-1}}[0,T]}}{\Gamma(\alpha)}\underset{\theta_1\leq a}{Sup}\ k(\theta_1,0).a,
\end{aligned}$$

then if $\quad m = \dfrac{a}{||g_1(.)||_{L^r[0,T]}\kappa}\left(1-\dfrac{||g_1(.)||_{L^{\frac{q}{\alpha-1}}[0,T]}}{\Gamma(\alpha)}\underset{\theta_1\leq a}{Sup}\ k(\theta_1,0)\right)$
we have

$$\tilde{y}_0 \in \overline{B(0,m)} \Longrightarrow \phi_{\tilde{y}_0}(y(.)) \in \overline{B(0,a)}.$$

Let's show that m is positive, in fact we have

$$\frac{||g_1(.)||_{L^{\frac{q}{\alpha-1}}[0,T]}}{\Gamma(\alpha)}\underset{\theta_1\leq a}{Sup}\ k(\theta_1,0) \leq D_1 = \frac{||g_1(.)||_{L^{\frac{q}{\alpha-1}}[0,T]}}{\Gamma(\alpha)}\underset{\theta_i\leq a}{Sup}\ k(\theta_1,\theta_2) < 1,$$

which leads to

$$1-\frac{||g_1(.)||_{L^{\frac{q}{\alpha-1}}[0,T]}}{\Gamma(\alpha)}\underset{\theta_1\leq a}{Sup}\ k(\theta_1,0) > 0,$$

hence $\quad m > 0$.
We deduce, from the Banach fixed point theorem, that $\phi_{\tilde{y}_0}$ has a unique fixed point in $\overline{B(0,a)}$.

2 . Let x and y be two solutions of (2) with initial states in ω, respectively, $\tilde{x}_0$ and $\tilde{y}_0$, we have :

$$h(\tilde{x}_0) - h(\tilde{y}_0) = x(.) - y(.) = S_\alpha(.)\chi^*_\omega(\tilde{x}_0 - \tilde{y}_0) + L_\alpha(.)(Nx - Ny),$$

then

$$\begin{aligned}
||h(\tilde{x}_0) - h(\tilde{y}_0)||_{L^r(0,T;X^\alpha)} &\leq ||S_\alpha(.)\chi^*_\omega(\tilde{x}_0 - \tilde{y}_0)||_{L^r(0,T;X^\alpha)} \\
&\quad +||L_\alpha(.)(Nx - Ny)||_{L^r(0,T;X^\alpha)} \\
&\leq ||\tilde{x}_0 - \tilde{y}_0||_{X(\omega)}.||g_1(.)||_{L^r[0,T]} \\
&\quad +D_1||x-y||_{L^r(0,T;X^\alpha)} \\
&\leq \kappa||\tilde{x}_0 - \tilde{y}_0||_V.||g_1(.)||_{L^r[0,T]} \\
&\quad +D_1||h(\tilde{x}_0) - h(\tilde{y}_0)||_{L^r(0,T;X^\alpha)} \qquad ,
\end{aligned}$$

which gives

$$||h(\tilde{x}_0) - h(\tilde{y}_0)||_{L^r(0,T;X^\alpha)} \leq \frac{\kappa ||g_1(.)||_{L^r[0,T]}}{1 - D_1} ||\tilde{x}_0 - \tilde{y}_0||_V.$$

Finally, h is Lipschitz continuous.

□

In the next result we show that the initial state in ω $(\tilde{y}_0)$ is a solution of a fixed point problem, keeping in mind that the measurements are in a ball of $L^2(0,T;\mathcal{O})$.
The solution of (2) can be written as

$$y(.) = S_\alpha(.)\chi_\omega^* \tilde{y}_0 + L_\alpha(.)Ny,$$

applying the observation operator, we get

$$z(.) = CS_\alpha(.)\chi_\omega^* \tilde{y}_0 + CL_\alpha(.)Ny,$$

or equivalently,

$$K_\alpha^\omega \tilde{y}_0 = z(.) - CL_\alpha(.)Ny,$$

and since the system (4)-(3) is approximately ω-observable, we obtain

$$\tilde{y}_0 = \left(K_\alpha^\omega\right)^\dagger (z(.) - CL_\alpha(.)Ny),$$

We set

$$\Phi_z(\tilde{y}_0) = \left(K_\alpha^\omega\right)^\dagger (z(.) - CL_\alpha(.)h(\tilde{y}_0)), \tag{19}$$

then $\tilde{y}_0$ can be seen as a fixed point of $\Phi_z(.)$.

If the system (4) is approximately ω-observable and (15) hold, we have the following result

Theorem 5.8. *Let's assume that*

H1 . $\forall \tilde{y}_0 \in \overline{B(0,m)}$, $\quad CL_\alpha(.)(Ny) \in \mathcal{I}m(K_\alpha^\omega)$, *where* $y = h(\tilde{y}_0)$.

H2 . $\exists \delta > 0$, *such that*

$$||CL_\alpha(.)Ny||_{L^2(0,T;\mathcal{O})} \leq \delta ||Ny||_{L^p(0,T;L^2(\Omega))}.$$

Then we have the following assertions :

1 . $\exists\ l$ *and* $\rho = \rho(a,l,m) > 0$, $\ \forall z \in \overline{B(0,\rho)} \subset L^2(0,T;\mathcal{O})$, $\ \Phi_z(.)$ *has a unique fixed point in* $\overline{B(0,m)}$.

2 . *The mapping*

$$\begin{array}{cccc} h' & : & \overline{B(0,\rho)} & \longrightarrow & \overline{B(0,m)} \\ & & z & \longmapsto & \tilde{y}_0, \end{array} \tag{20}$$

"which, for every measurement (z) *in* $\overline{B(0,\rho)}$, *associates the unique fixed point of* $\Phi_z(.)$*" is Lipschitzian.*

Proof. :
1 . As $k(\theta_1,\theta_2) \xrightarrow[\theta_1,\theta_2 \to 0]{} 0$, $\ \exists\ l, \exists \nu > 0$, such that :

$$k(\theta_1,\theta_2) < \nu < \frac{1 - D_1}{\delta\kappa ||g_1(.)||_{L^r[0,T]}} \qquad \forall \theta_1,\theta_2 \leq l,$$

which gives

$$\underset{\theta_i \leq l}{Sup}\ k(\theta_1,\theta_2) \leq \nu < \frac{1 - D_1}{\delta\kappa ||g_1(.)||_{L^r[0,T]}},$$

then

$$D_2 = \delta \underset{\theta_i \leq l}{Sup}\ k(\theta_1,\theta_2)\frac{\kappa||g_1(.)||_{L^r[0,T]}}{1-D_1} < 1.$$

If $a \leq l$, let $\tilde{x}_0, \tilde{y}_0 \in \overline{B(0,m)} \subset V$, then

$$\Phi_z(\tilde{y}_0) - \Phi_z(\tilde{x}_0) = (K_\alpha^\omega)^\dagger \left[CL_\alpha(.)(Ny - Nx)\right],$$

hence, we have

$$||\Phi_z(\tilde{y}_0) - \Phi_z(\tilde{x}_0)||_V = \left|\left|(K_\alpha^\omega)\,(K_\alpha^\omega)^\dagger \left[CL_\alpha(.)(Ny - Nx)\right]\right|\right|_{L^2(0,T;\mathcal{O})},$$

using **(H1)**, we obtain

$$||\Phi_z(\tilde{y}_0) - \Phi_z(\tilde{x}_0)||_V = ||CL_\alpha(.)(Ny - Nx)||_{L^2(0,T;\mathcal{O})},$$

and from **(H2)**, we deduce that

$$\begin{aligned} ||\Phi_z(\tilde{y}_0) - \Phi_z(\tilde{x}_0)||_V &\leq \delta\|Ny - Nx\|_{L^p(0,T;L^2(\Omega))} \\ &\leq \delta \underset{\theta_i \leq l}{Sup}\ k(\theta_1,\theta_2)\frac{\kappa||g_1(.)||_{L^r[0,T]}}{1-D_1}\|\tilde{y}_0 - \tilde{x}_0\|_V \\ &\leq D_2\|\tilde{y}_0 - \tilde{x}_0\|_V. \end{aligned}$$

Thus $\Phi_z(.)$ is a strict contraction on $\overline{B(0,m)}$.
We show now that $\Phi_z\left(\overline{B(0,m)}\right) \subset \overline{B(0,m)}$. Let $\tilde{y}_0 \in \overline{B(0,m)}$, we have

$$\begin{aligned} \|\Phi_z(\tilde{y}_0)\|_V &= ||z(.) - CL_\alpha(.)Ny||_{L^2(0,T;\mathcal{O})} \\ &\leq \|z(.)\|_{L^2(0,T;\mathcal{O})} + \delta.k\left(\|y\|,0\right)\|y\|_{L^r(0,T;X^\alpha)}, \end{aligned}$$

and since $a \leq l$, we have

$$\|\Phi_z(\tilde{y}_0)\|_V \leq \|z(.)\|_{L^2(0,T;\mathcal{O})} + a.\delta \underset{\theta \leq l}{Sup} k(\theta,0).$$

Let's set $\rho := m - a.\delta \underset{\theta \leq l}{Sup} k(\theta,0)$, which is positive. In fact

$$\delta \underset{\theta \leq l}{Sup}\ k(\theta,0)\frac{\kappa||g_1(.)||_{L^r[0,T]}}{1-D_1} \leq D_2 = \delta \underset{\theta_i \leq l}{Sup}\ k(\theta_1,\theta_2)\frac{\kappa||g_1(.)||_{L^r[0,T]}}{1-D_1} < 1,$$

hence

$$\delta \underset{\theta \leq l}{Sup}\ k(\theta,0)\kappa||g_1(.)||_{L^r[0,T]} < 1 - D_1 \leq 1 - \frac{||g_1(.)||_{L^q_{\alpha-1}[0,T]}}{\Gamma(\alpha)}\underset{\theta_1 \leq a}{Sup}\ k(\theta_1,0),$$

thus

$$\delta \underset{\theta \leq l}{Sup}\ k(\theta,0) < \frac{1}{\kappa||g_1(.)||_{L^r[0,T]}}\left[1 - \frac{||g_1(.)||_{L^q_{\alpha-1}[0,T]}}{\Gamma(\alpha)}\underset{\theta_1 \leq a}{Sup}\ k(\theta_1,0)\right],$$

which gives

$$\frac{1}{\kappa||g_1(.)||_{L^r[0,T]}}\left[1 - \frac{||g_1(.)||_{L^q_{\alpha-1}[0,T]}}{\Gamma(\alpha)}\underset{\theta_1 \leq a}{Sup}\ k(\theta_1,0)\right] - \delta \underset{\theta \leq l}{Sup}\ k(\theta,0) > 0.$$

Then $\quad m - a.\delta \underset{\theta \leq l}{Sup}\, k(\theta, 0) = \delta > 0.$

If $\|z(.)\|_{L^2(0,T;\mathcal{O})} \leq \rho$, then $\|\Phi_z(\tilde{y}_0)\|_V \leq m.$

Thus

$$\Phi_z\left(\overline{B(0,m)}\right) \subset \overline{B(0,m)}.$$

Therefore, by the Banach's fixed theorem, if $z \in \overline{B(0,\rho)}$, the function $\Phi_z(.)$ has a unique fixed point in $\overline{B(0,m)}$, which is the initial state in ω.

If $l \leq a$, we use the remark (5.7) and reprove the proposition (5.6), for another $a \leq l$

2 . Let's consider $z_1, z_2 \in \overline{B(0,\rho)}$, we have

$$h'(z_1) - h'(z_2) = \Phi_{z_1}\left(h'(z_1)\right) - \Phi_{z_2}\left(h'(z_2)\right) = \tilde{y}_{01} - \tilde{y}_{02}.$$

Then

$$\begin{aligned} \|h'(z_1) - h'(z_2)\|_V &= \|\Phi_{z_1}\left(h'(z_1)\right) - \Phi_{z_2}\left(h'(z_2)\right)\|_V \\ &\leq \|\Phi_{z_1}\left(h'(z_1)\right) - \Phi_{z_1}\left(h'(z_2)\right)\|_V \\ &\quad + \|\Phi_{z_1}\left(h'(z_2)\right) - \Phi_{z_2}\left(h'(z_2)\right)\|_V \\ &\leq D_2\|h'(z_1) - h'(z_2)\|_V + \|z_1 - z_2\|_{L^2(0,T;\mathcal{O})}. \end{aligned}$$

Which gives $\|h'(z_1) - h'(z_2)\|_V \leq \dfrac{1}{1-D_2}\|z_1 - z_2\|_{L^2(0,T;\mathcal{O})}$, thus h' is Lipschitzian. □

6. Numerical Approach

For this section we adopt the same assumptions of the fourth section.
Let's consider the following sequence,

$$\begin{cases} y_0^0 &= 0, \\ y_0^{n+1} &= \left(K_\alpha^\omega\right)^\dagger \left(z(.) - CL_\alpha(.)Nh(y_0^n)\right), \end{cases}$$

and for every n in $\mathbb{N}$, we consider the following fractional system,

$$\begin{cases} {}^C D_{0^+}^\alpha y^n(x,t) = Ay^n(x,t) + Ny^n(x,t) & in\ Q\ , \quad \alpha \in \,]0,1[\,, \\ y^n(\xi,t) = 0 & in\ \Sigma, \\ y^n(x,0) = y_0^n(x) & in\ \Omega, \end{cases} \tag{21}$$

with the output equation,

$$z_n(t) = Cy^n(t).$$

Theorem 6.1. *The sequence $\{y_0^n\}_{n\geq 0}$ converges to the desired initial state y_0 in ω.*

Proof. All we need to show is that y_0^n converges to $h'(z)$ in V. Firstly we will show that $(y_0^n)_{n\geq 0}$ is a Cauchy sequence.
We have,

$$\begin{aligned} \|y_0^{n+1} - y_0^n\|_V &= ||\Phi_z(y_0^n) - \Phi_z(y_0^{n-1})||_V, \\ &\leq D_2\|y_0^n - y_0^{n-1}\|_V, \\ &\vdots \\ &\leq D_2^n\|y_0^1\|_V, \end{aligned}$$

hence, $\forall m > n > 0$

$$\begin{aligned}
\|y_0^m - y_0^n\|_V &\leq \|y_0^m - y_0^{m-1}\|_V + \|y_0^{m-1} - y_0^{m-2}\|_V + \cdots + \|y_0^{n+1} - y_0^n\|_V, \\
&\leq D_2^{m-1}\|y_0^1\|_V + D_2^{m-2}\|y_0^1\|_V + \cdots + D_2^n\|y_0^1\|_V, \\
&\leq \frac{D_2^n}{1-D_2}\|y_0^1\|_V,
\end{aligned}$$

which gives that $(y_0^n)_{n\geq 0}$ is a Cauchy sequence and eventually convergent.
Remark that, from the sequence definition, we have

$$\forall n \geq 1,\ K_\alpha^\omega y_0^n = z - CL_\alpha(.)Nh(y_0^{n-1}).$$

Hence,

$$\begin{aligned}
\|y_0^n - h'(z)\|_V &= \|h'(z_n) - h'(z)\|_V, \\
&\leq \frac{1}{1-D_2}\|z_n - z\|_{L^2(0,T;\mathcal{O})}, \\
&\leq \frac{1}{1-D_2}\|z - CL_\alpha(.)Nh(y_0^n) - K_\alpha^\omega y_0^n\|_{L^2(0,T;\mathcal{O})}, \\
&\leq \frac{1}{1-D_2}\|K_\alpha^\omega y_0^{n+1} - K_\alpha^\omega y_0^n\|_{L^2(0,T;\mathcal{O})}, \\
&\leq \frac{1}{1-D_2}\|y_0^{n+1} - y_0^n\|_V, \\
&\leq \frac{D_2^n}{1-D_2}\|y_0^1\|_V.
\end{aligned}$$

We deduce that y_0^n converges to desired initial state in ω.

□

We set $r_n(.) = z(.) - CL_\alpha(.)Nh(y_0^{n-1})$.
we have

$$\begin{aligned}
z(.) - z_n(.) &= z(.) - CL_\alpha(.)Nh(y_0^n) - K_\alpha^\omega y_0^n \\
&= CL_\alpha(.)Nh(y_0^{n-1}) - CL_\alpha(.)Nh(y_0^n) \\
&= r_n(.) - r_{n+1}(.),
\end{aligned}$$

which gives

$$r_{n+1}(.) = (z_n(.) - z(.)) + r_n(.).$$

Thus we obtain the following algorithm,

$$\begin{cases}
\blacktriangleright\ \text{Initialization of}: \varepsilon,\ \alpha,\ \omega,\ \text{Sensors...} \\
\blacktriangleright\ r(.) = z(.) - CL_\alpha(.)h(0). \\
\blacktriangleright\ \textit{Repeat}: \\
\qquad \begin{cases}
\bullet\ \tilde{y}_0 = \left(K_\alpha^\omega\right)^\dagger r(.). \\
\bullet\ \text{Solve}\ {}^C D_{0^+}^\alpha y = Ay + Ny\ ;\quad y(0) = \chi_\omega^* \tilde{y}_0. \\
\bullet\ \bar{z}(.) = Cy. \\
\bullet\ r(.) = (\bar{z}(.) - z(.)) + r(.).
\end{cases} \\
\blacktriangleright\ \text{Until}\ \|z(.) - \bar{z}(.)\|_{L^2(0,Y;\mathcal{O})} \leq \varepsilon.
\end{cases}$$

Finally $\tilde{y}_0$ corresponds with the initial state on ω.

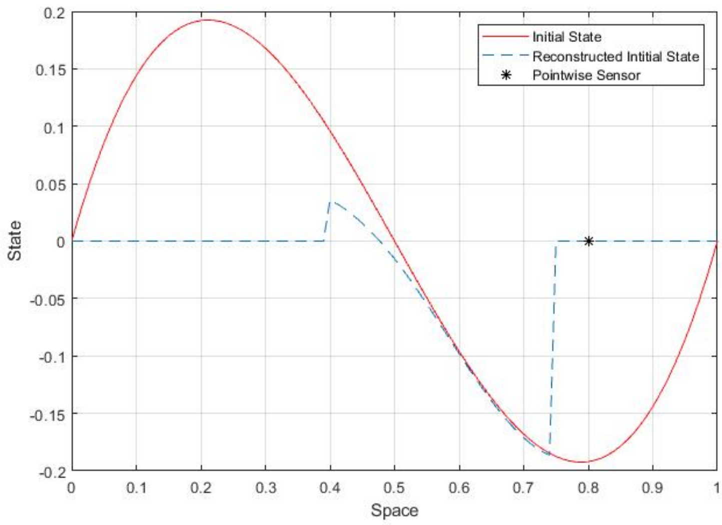

Figure 1: Initial and estimated initial state in $\omega = [0.4 \;\; 0.75]$.

Simulations

We give here numerical illustrations for the obtained algorithm. We show, with the same fractional system, both types of sensors, zonal and pointwise.

Remark 6.2. *The results are related to the choice of ω, the sensor's location also as the initial state of the system.*

Let $\Omega = [0, 1]$, $T = 2$ and $\alpha \in]0, 1[$, we consider the following time-fractional diffusion system,

$$\begin{cases} {}^{C}D^{\alpha}_{0^+} y(x,t) - \dfrac{\partial^2}{\partial x^2} y(x,t) = \displaystyle\sum_{i=1}^{\infty} \langle y(.,t), \varphi_i \rangle^2 \varphi_i(x) & in \ \Omega \times]0, T], \\ y(0,t) = y(1,t) = 0 & in [0,T], \\ y(x,0) = y_0(x) \in \mathcal{D}(A) & in \ \Omega, \end{cases}$$

where $\{\varphi_i(x) = \sin(i\pi x), \ i = 1, 2, ...\}$ is an orthonormal basis of the state space $L^2([0, 1])$.

Pointwise Sensor

For this case we consider the order of derivation $\alpha = 0.2$ and that :
♦ The subregion $\omega = [0.4 \;\; 0.75]$.
♦ The measurements are given by means of a pointwise sensor located in $b = 0.8$, which means $z(t) = y(0.8, t)$.
♦ The initial state supposedly unknown in ω is $y_0(x) = 2x(x - 1)(2x - 1)$.
After 5 iterations of the proposed algorithm we obtain the figure (1).

The reconstruction error is : $\|y_0 - \tilde{y}_0\|_{L^2(\omega)} = 1.21 \times 10^{-2}$.
We remark in figure (1) that the reconstructed initial state is very close to the initial one in the desired subregion ω.
The figure (2) shows the evolution of the reconstruction error in function of the sensor's location, and it is very clear that the reconstruction error is sensitive to the position of the sensor.

Zonal Sensor

For this case we consider that :
♦ The order of derivation $\alpha = 0.8$.
♦ The subregion $\omega = [0.2 \;\; 0.6]$.

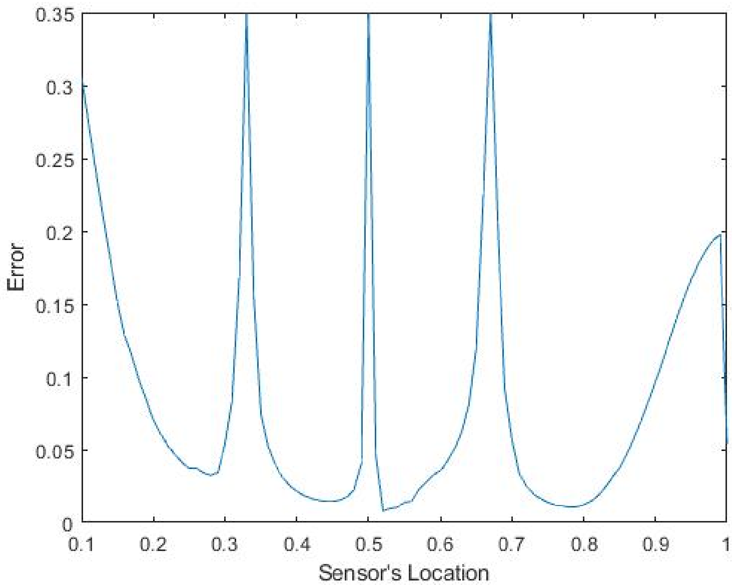

Figure 2: Error evolution.

♦ The measurements are given by means of a zonal sensor, with spatial distribution equal to 1, located in $D = [0.1 \;\; 0.3]$, which means

$$z(t) = \int_{0.1}^{0.3} y(x,t)dx$$

.

♦ The initial state supposedly unknown in ω is $y_0(x) = 2x(1-x)(2x-1)$.
After 7 iterations with the proposed algorithm we obtain the figure (3).

The reconstruction error is : $\|y_0 - \tilde{y}_0\|_{L^2(\omega)} = 9.4 \times 10^{-3}$.
As we can see in figure (3), the estimated initial state is quite near the actual one.
In order to show that the error changes with the choice of the geometric domain of the sensor, we give the following table :

Geometric domain of the sensor	**Error**
[0.3,0.5]	5.85×10^{-2}
[0.5,0.7]	1.3×10^{-1}
[0.7,0.9]	1.26×10^{-1}

Table 1: Some values of the reconstruction error for some different considerations of the geometric domain of the sensor.

Table (1) show that the reconstruction error is influenced by the placement of the geometric domain of the sensor.

7. Conclusion

In this paper we shed light on the concept of regional observability of semilinear Caputo type time-fractional diffusion systems, of order $\alpha \in]0,1]$. The two different methods that we gave are both based on fixed point techniques, and regarding future works, we intend to investigate the same problem with the

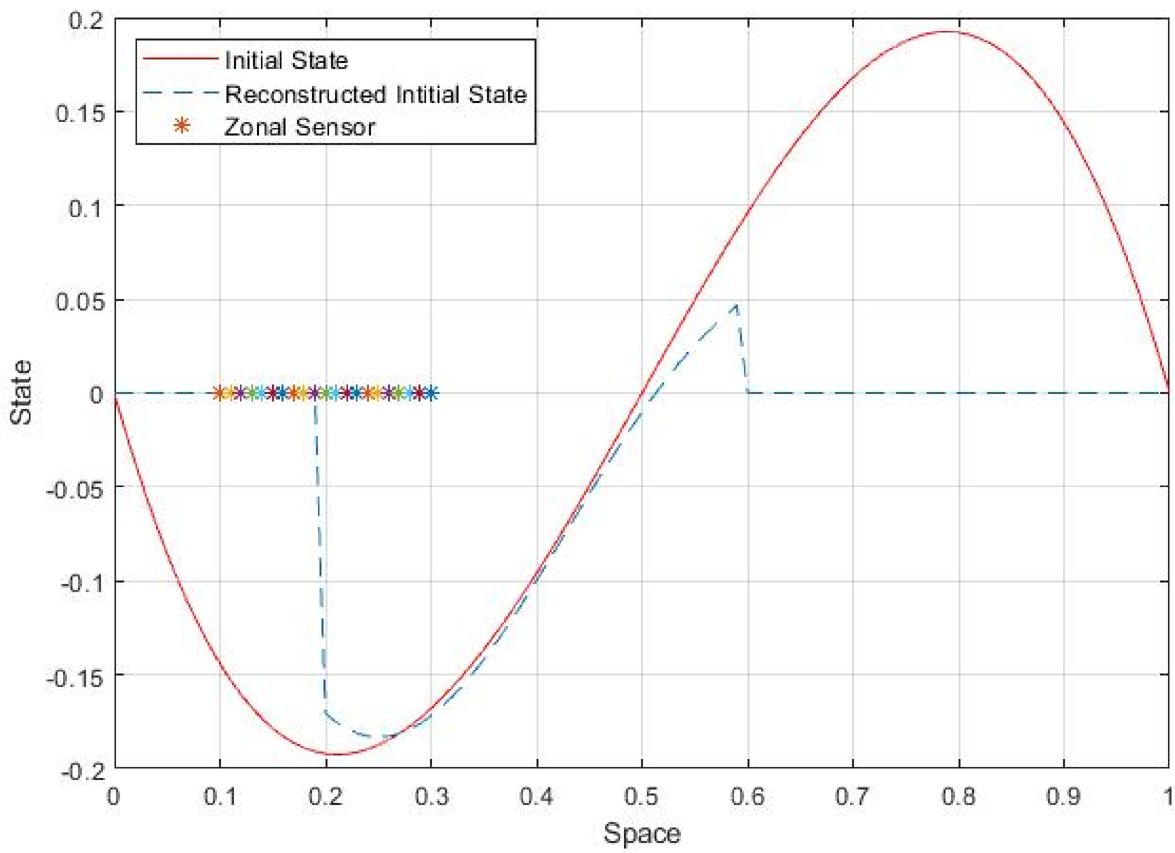

Figure 3: Initial and estimated initial state in $\omega = [0.2 \;\; 0.6]$.

Hilbert Uniqueness Method (HUM), we also plan to study the regional boundary observability for the same class of systems.

References

[1] A. Amara, S. Etemad, S. Rezapour, Approximate solutions for a fractional hybrid initial value problem via the Caputo conformable derivative. Adv. Differ. Equ. 2020(1) (2020) 608.

[2] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory And Applications of Fractional Differential Equations, Elsevier (2006).

[3] A. Boutoulout, H. Bourray, F.Z. El Alaoui, Boundary gradient observability for semilinear parabolic systems: Sectorial approach, Math. Sci. Lett. 2(1) (2013) 45-54.

[4] A. Boutoulout, H. Bourray, F.Z. El Alaoui, S. Benhadid, Regional observability for distributed semi-linear hyperbolic systems, Int. J. Control. 87(5) (2014) 898-910.

[5] A. Boutoulout, H. Bourray, F. Z. El Alaoui, Regional Boundary Observability for Semi-Linear Systems Approach and Simulation, Int. J. Math. Anal. 4(24) (2010) 1153-1173.

[6] A. Dzieliński, D. Sierociuk, Fractional Order Model of Beam Heating Process and Its Experimental Verification, In New Trends in Nanotechnology and Fractional Calculus Applications, D. Baleanu, Z. B. Guvenc, and J. A. T. Machado, Eds. Dordrecht: Springer Netherlands, (2010) 287-294.

[7] A. Dzieliński, D. Sierociuk, G. Sarwas, Some applications of fractional order calculus, Bull. Pol. Acad. Sci. 58(4) (2010) 583-592.

[8] A. Dzieliński, G. Sarwas, D. Sierociuk, Time domain validation of ultracapacitor fractional order model, In 49th IEEE Conference on Decision and Control (CDC), (2010) 3730-3735.

[9] A. El Jai, Eléments d'analyse et de contrôle des systemes, Perpignan: Presses Universitaires de Perpignan, (2005).

[10] A. El Jai, A.J. Pritchard, Capteurs et actionneurs dans l'analyse des systemes distribués, Elsevier Masson, (1997).

[11] A.J. Pritchard, A. Wirth, Unbounded Control and Observation Systems and Their Duality, SIAM J. Control Optim. 16(4) (1978) 535-545.

[12] A. L. Alaoui, E. Azroul, A. A. Hamou, Monotone Iterative Technique for Nonlinear Periodic Time Fractional Parabolic Problems, Adv. Theory Nonlinear Anal. Appl. 4(3) (2020) 197-213.

[13] A. Salim, M. Benchohra, E. Karapınar, J.E. Lazreg, Existence and Ulam stability for impulsive generalized Hilfer-type fractional differential equations, Adv. Differ. Equ. 2020(1) (2020) 601.

[14] D. Baleanu, S. Etemad, H. Mohammadi, S. Rezapour, A novel modeling of boundary value problems on the glucose graph, Commun. Nonlinear Sci. Numer. Simul. 100 (2021) 105844.

[15] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Berlin, Heidelberg: Springer, 840 (1981).

[16] D. Salamon, Infinite Dimensional Linear Systems With Unbounded Control and Observation: A Functional Analytic Approach, Trans. Am. Math. Soc. 300 (1987) 383-431.

[17] D. Xu, Q. Wang, Y. Li, Controllability and observability of fractional linear systems with multiple different orders. In 2016 31st Youth Academic Annual Conference of Chinese Association of Automation (YAC), Wuhan, Hubei Province, China, (2016) 286-291.

[18] E. Zerrik, H. Bourray, A. El Jai, Regional Observability for Semilinear Distributed Parabolic Systems, J. Dyn. Control Syst. 10(3) (2004) 413-430.

[19] F. Ge, Y. Chen, C. Kou, Regional Analysis of Time-Fractional Diffusion Processes, Springer International Publishing (2018).

[20] F.Z. El Alaoui, Regional observability of semilinear systems, Ph.D thesis, Faculty of Sciences, Moulay Ismail University, Meknes, (2011).

[21] I.I. Vrabie, C_0-semigroups and applications, 1st edition. Amsterdam, Boston: Elsevier Science, (2003).

[22] H. Mohammadi, S. Kumar, S. Rezapour, S. Etemad, A theoretical study of the Caputo-Fabrizio fractional modeling for hearing loss due to Mumps virus with optimal control, Chaos Solitons Fractals, 144(2021) 110668.

[23] H. Zouiten, F.Z. El Alaoui, A. Boutoulout, Regional Boundary Observability with Constraints: a Numerical Approach, Int. Rev. Autom. Control. 8(5) (2015) 354-361.

[24] J.E. Lazreg, S. Abbas, M. Benchohra, E. Karapınar, Impulsive Caputo-Fabrizio fractional differential equations in b-metric spaces, Open Math. 19(1) (2021) 363-3752.

[25] J. Mu, B. Ahmad, S. Huang, Existence and regularity of solutions to time-fractional diffusion equations, Comput. Math. with Appl. 73(6) (2017) 985-996.

[26] K. Fujishiro, M. Yamamoto, Approximate controllability for fractional diffusion equations by interior control, Appl. Anal. 93(9) (2014) 1793-1810.

[27] K.G. Magnusson, Observability of Nonlinear Systems, IMA J. Math. Control Inf. 1(4) (1984) 339-358.

[28] K. Taira, Analytic semigroups and semilinear initial boundary value problems, Cambridge: Cambridge Univ. Press, (2004).

[29] K. Zguaid, F.Z. El Alaoui, A. Boutoulout, Regional Observability of Linear Fractional Systems Involving Riemann-Liouville Fractional Derivative, In Nonlinear Analysis: Problems, Applications and Computational Methods, Z. Hammouch, H. Dutta, S. Melliani, and M. Ruzhansky, Eds. Springer International Publishing, (2021) 164-178.

[30] K. Zguaid, F.Z. El Alaoui, A. Boutoulout, Regional observability for linear time fractional systems, Math. Comput. Simul. 185 (2021) 77-87.

[31] M. Amouroux, A. El Jai, E. Zerrik, Regional observability of distributed systems, Int. J. Syst. Sci. 25(2) (1994) 301-313.

[32] M. Farman, A. Akgul, A. Ahmad, Analysis and Simulation of Fractional-Order Diabetes Model, Adv. Theory Nonlinear Anal. Appl. 4(4) (2020) 483-497.

[33] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Applied Mathematical Sciences. New York: Springer. 144 (1983).

[34] R.F. Curtain, H. Zwart, An Introduction to Infinite-Dimensional Linear Systems Theory, New York: Springer-Verlag, (1995).

[35] R.S. Adiguzel, U. Aksoy, E. Karapınar, I.M. Erhan, On the Solutions of Fractional Differential Equations via Geraghty Type Hybrid Contractions, Appl. Comput. Math. 20(2) (2021) 313-333.

[36] R.S. Adiguzel, U. Aksoy, E. Karapınar, I.M. Erhan, Uniqueness of solution for higher-order nonlinear fractional differential equations with multi-point and integral boundary conditions, RACSAM. REV. R. ACAD. A. 115(3) (2021) 155.

[37] R.S. Adiguzel, U. Aksoy, E. Karapınar, I.M. Erhan, On the solution of a boundary value problem associated with a fractional differential equation, Math. Methods Appl. Sci. (2020) 1-12.

[38] R. Xue, Observability for fractional diffusion equations by interior control, Fract. Calc. Appl. Anal. 20(2) (2017) 537-552.

[39] S. Rezapour, S. Etemad, H. Mohammadi, A mathematical analysis of a system of Caputo-Fabrizio fractional differential equations for the anthrax disease model in animals, Adv. Differ. Equ. 2020(1) (2020) 481.

[40] S. Rezapour, S. Etemad, B. Tellab, P. Agarwal, J.L. Garcia Guirao, Numerical Solutions Caused by DGJIM and ADM Methods for Multi-Term Fractional BVP Involving the Generalized Ψ-RL-Operators, Symmetry. 13 (2021) 532.

[41] W. Arendt, C.J.K. Batty, M. Hieber, F. Neubrander, Vector-valued Laplace transforms and Cauchy problems, 2nd edition, Basel: Birkhauser (2011)

[42] Y. Zhou, F. Jiao, Existence of mild solutions for fractional neutral evolution equations, Comput. Math. with Appl. 59(3) (2010) 1063-1077.

[43] Y. Zhou, L. Zhang, X.H. Shen, Existence of mild solutions for fractional evolution equations, J. Integral Equ. Appl. 25(4) (2013) 557-586.