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On the Idempotents of Semigroup of Partial Contractions of a Finite Chain

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ABSTRACT

Let [n] = {1, 2, . . . , n} be a finite chain. Let $\mathcal{P}_n$ and $\mathcal{T}_n$ be the semigroups of partial and full transformations on [n], respectively. Let $\mathcal{CP}_n = \{\alpha \in \mathcal{P}_n : |x\alpha - y\alpha| \leq |x-y| \ \forall x, y \in \text{dom } \alpha\}$ and $\mathcal{CT}_n = \{\alpha \in \mathcal{T}_n : |x\alpha - y\alpha| \leq |x - y| \ \forall x, y \in [n]\}$. Then $\mathcal{CP}_n$ and $\mathcal{CT}_n$ are subsemigroups of $\mathcal{P}_n$ and $\mathcal{T}_n$, respectively. In this paper, we characterize the idempotent elements and compute the number of idempotents of height *n-1* and *n-2* for the semigroups $\mathcal{CP}_n$ and $\mathcal{CT}_n$, respectively.

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Introduction

Let $[n] = \{1, 2, . . . , n\}$ be a finite chain, we adopt as in the literature, the notations $\mathcal{P}_n$ and $\mathcal{T}_n$ to denote, partial and full transformation semigroups on $[n]$, respectively. A map $\alpha \in \mathcal{P}_n$ is said to be a contraction if and only if $|x\alpha - y\alpha| \leq |x - y| \ \forall x, y$ in dom α.. Let

$$\mathcal{CP}_n = \{\alpha \in \mathcal{P}_n : |x\alpha - y\alpha| \leq |x - y| \ \forall x, y \in \text{dom } \alpha\}$$

and

$$\mathcal{CT}_n = \{\alpha \in \mathcal{T}_n : |x\alpha - y\alpha| \leq |x - y| \ \forall x, y \in [n]\}.$$

Then $\mathcal{CP}_n$ and $\mathcal{CT}_n$ are subsemigroups of $\mathcal{P}_n$ and $\mathcal{T}_n$, respectively. They are known to be the semigroups of partial and full contraction of $[n]$, respectively. For basic concept in semigroup theory, we refer the reader to Howie (1995).

Let α be element of S, where S is any of the transformation semigroups $\mathcal{CP}_n$ and $\mathcal{CT}_n$. Then dom α, im α , $h(\alpha) = |\text{im } \alpha|$ and $b(\alpha) = |\text{dom } \alpha|$ denote the *domain, image, height and width* of α, respectively. Also, let fix $\alpha = \{x \in \text{dom } \alpha \mid x\alpha = x\}$, $f(\alpha) = |\text{fix } \alpha|$, shift$(\alpha) = \{x \in \text{dom } \alpha \mid x\alpha = x\}$, def$(\alpha)$ = |shift(α)| = $|\text{dom } \alpha| - f(\alpha)$ and $Jr = \{\alpha \in S | h(\alpha) = r\}$. For $\alpha, \beta \in S$, the composition of α and β is defined as $x(\alpha \circ \beta) = ((x)\alpha)\beta$ for all x in dom α. Without ambiguity, we shall be using the notation $\alpha\beta$ to denote $\alpha \circ \beta$.

The algebraic study of various semigroups of contraction were initiated recently. For example; Zhao and Yang (2012) characterized regularity and Green's equivalences for the semigroup of order preserving partial contractions $\mathcal{OCP}_n$. Recently, Ali et al., (2018) extend this study to the general semigroup of partial contractions $\mathcal{CP}_n$. These semigroups were shown to be left abundant, for example see the work of Umar and Zubairu (Umar and Zubairu, 2018a; Umar and Zubairu, 2018b). In another development, Garba et al., (2017) characterized the starred Green's equivalences on the semigroup of full contraction $\mathcal{CT}_n$. Moreover, the ranks of the subsemigroups of order preserving or order reversing full contraction, $\mathcal{ORCT}_n$ (also known as the semigroup of monotone or anti-tone full contraction maps) and order preserving full contractions, $\mathcal{OCT}_n$ (also known as the semigroup of monotone full contraction maps) were computed by Toker (2020). Furthermore, as an extension of the work of Bugay (2020); Toker (2020) computes the ranks of certain ideals of the subsemigroups $\mathcal{ORCT}_n$ and $\mathcal{OCT}_n$, respectively. Most of these algebraic properties leads to many combinatorial problems.

An element *a* in a semigroup *S* is said to be an *idempotent* if and only if $a^2=a$. The set of all idempotents in any semigroup *S* is denoted by $E(S)$. The cardinality of idempotents of many semigroups of transformation on chain have been found. For example; in 1961, Clifford and Preston (1961) study the idempotents in $\mathcal{L}-$class and $\mathcal{R}-$class of $\mathcal{T}_n$ and gave under Exercise 2.2(2a) the number of idempotents in each $\mathcal{L}-$class of height *r* in $\mathcal{T}_n$ is r^{n-1} and that of $R-$class correspond to the product $n_1n_2 \ldots n_r$, where $n = n_1 + n_2 + \ldots + n_r$ is a partition of $[n]$. In 1968, similar study was carried out by Tainter (1968). He characterized and computes the number of idempotents of the semigroup of full transformations $\mathcal{T}_n$. Moreover, Garba (1990), computes the number of idempotents in the semigroup of partial transformations $\mathcal{P}_n$. In fact the number of idempotents of many subsemigroups of $\mathcal{P}_n$ and $\mathcal{T}_n$ were readily available in the existing literature. For example, Howie (1971) computes the number of idempotents in the semigroup of order preserving full transformation on $[n]$. Later on, similar question was raised by Gomes and Howie (1992) for the semigroup of order preserving partial transformations on $[n]$, which was later answered by Laradji and Umar (2004). For most of these existing combinatorial results, we refer the reader to Umar (2014) and Gayushkin and Mazorchuk (2009). However, it seems like, the number of idempotents in the new semigroups of partial and full contractions on a chain have not been found. In this paper, we characterize the idempotents elements and compute the number of idempotents of height n-1 and n - 2 for the semigroups $\mathcal{CP}_n$ and $\mathcal{CT}_n$, respectively.

Idempotents and their characterizations

In this section, we characterize the idempotent elements in $\mathcal{CP}_n$. At the end of this section, we give as a corollary that $E(\mathcal{CP}_n)$ is not a semigroup.

Now let $\alpha \in \mathcal{CP}_n$ of height r $(0 \leq r \leq n)$. Then dom $\alpha \subset [n]$ can always be partitioned into blocks as dom $\alpha = A_1 \cup A_2 \cup \cdots \cup A_r$ and α is expressible as

$$\begin{pmatrix} A_1 & A_2 \cdots & A_r \\ a_1 & a_2 \cdots & a_r \end{pmatrix},$$

where $A_i\alpha = a_i$ *and* $|a_i - a_j| \leq |x - y|$ for all $x \in A_i$ and $y \in A_j$ $(i, j \in \{1, 2, . . . , r\})$. A subset B of $[n]$ is called *convex* if whenever $a, b \in B$ with $a \leq b$ and $a < c < b$ $(c \in [n])$ then $c \in B$. Now we expand this definition a little bit. Let $\varnothing \neq A \subset [n]$ and let $\varnothing \neq C \subset A$. Then C is called convex subset of A if whenever $a, b \in C$ with $a \leq b$ and $a < c < b$ $(c \in A)$ then $c \in C$. For example, $B = \{3, 4, 5, 6\}$ is a convex subset of Howie (1995) and, for $A = \{1, 3, 5, 6, 7\}$, $B = \{1, 3, 5\}$ is a convex subset of A.

The following results are about image of a contraction. These are found in (Ali et al., 2018; Adeshola and Umar, 2018).

Lemma 1 (Ali et al., 2018, Lemma 1.8)**.** *Let* $\alpha \in \mathcal{CP}_n$ *and let A be a convex subset of dom* α*. Then* $A\alpha$ *is convex.*

Corollary 2 (Adeshola and Umar, 2018, Lemma 1.2)**.** *Let* $\alpha \in \mathcal{CT}_n$*. Then im* α *is convex.*

It is well known that $\alpha \in E(\mathcal{P}_n)$ if and only if im α =fix (α). In particular,

$$\alpha = \begin{pmatrix} A_1 & A_2 \cdots & A_r \\ a_1 & a_2 \cdots & a_r \end{pmatrix} \in J_r$$

is an idempotent if and only if $f(\alpha) = r$, in other words the blocks A_i $(1 \leq i \leq r)$ are stationary, i. e., $a_i \in A_i$ for all $1 \leq i \leq r$. These properties also holds for the semigroup $\mathcal{CP}_n \subset \mathcal{P}_n$.

Theorem 3. *If* $\alpha \in E(\mathcal{CP}_n)$ *with* $h(\alpha) = r$ *then* fix α *is a convex subset of* dom α.

Proof. Let $\alpha = \begin{pmatrix} A_1 & A_2 \cdots & A_r \\ a_1 & a_2 \cdots & a_r \end{pmatrix} \in E(CP_n)$. Suppose by way of contradiction that fix α is not convex subset of dom α. Then there exist $x, y \in$ fix α such that $x \leq y$ and $x < z < y$ ($z \in$ dom α) and $z \notin$ fix α. Thus there exists $j \in \{1, 2, . . . , r\}$ with $z \in A_j$. Since α is an idempotent then by definition $z\alpha \in$ fix α. Let $t = z\alpha$ and notice that the block A_j is stationary for each $1 \leq i \leq r$. Then it follows that $t \in A_j$ and therefore $t, z \in A_j$. Thus either $t < z$ or $t > z$. Now suppose $t < z$. Notice that $z \leq y$. Thus

$$|z\alpha - y\alpha| = |t - y| > |z - y|$$

and similarly if we assume that $z < t$, and notice also that $x \leq z$, then

$$|z\alpha - x\alpha| = |t - x| > |x - z|.$$

Thus the two cases contradict the fact that α is a contraction and hence the result follows.

Corollary 4. *If* α *is an idempotent in* $\mathcal{CP}_n$ *of height r then fixed points of* α *are tied together.*

Remark 5. *As a consequence of Theorem 3, Lemma 1 and Corollary 2 we have that for all idempotents* $\varepsilon \in \mathcal{CP}_n$*, there exists a subset A of* dom ε *such that A is convex.*

Lemma 6. *Let α, β be elements of* $E(\mathcal{CP}_n)$*. If* fix $\alpha \cap$ fix $\beta = \emptyset$ *then αβ is not necessary an idempotent.*

To see this, consider $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 1 & 1 & 2 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} 2 & 4 & 5 & 6 \\ 4 & 4 & 5 & 4 \end{pmatrix}$ elements of *CP*6.

The composition is $\alpha\beta = \begin{pmatrix} 2 & 5 \\ 4 & 4 \end{pmatrix}$. Which is not an idempotent.

Remark 7. *Let α, β be elements in* $E(\mathcal{CP}_n)$*, if* fix $\alpha \cap$ fix $\beta = \emptyset$ *then αβ is an idempotent if and only if* im $\beta = \{x\}$ *and there exists* $y \in$ dom β *such that* $y\beta = x$ *and* $x\alpha = y$ *or there exists* $y \in$ dom α *such that* $y\alpha = x$ *and* $x\beta = y$.

As a consequence of Lemma 6, we readily have the following:

Corollary 8. $E(\mathcal{CP}_n)$ *is not a semigroup.*

Number of Idempotents in the semigroups $\mathcal{CP}_n$ and $\mathcal{CT}_n$

In this section, we compute the number of idempotent elements of height n - 1 and n - 2 of the semigroups $\mathcal{CP}_n$ and $\mathcal{CT}_n$, respectively. We also compute the order of idempotents of height 2 and width greater than or equal to n-1 in $\mathcal{CP}_n$ and give as a corollary the order of idempotent elements in $\mathcal{CT}_n$ of height 2. At the end of the section, we give as a conjuncture, the number of idempotent elements in $\mathcal{CT}_n$ of height n - 3. The method of proof used in the results of this section were mainly combinatorial arguments.

Now let $S = \mathcal{CP}_n$ and $E(J_r)$ be the set of idempotents of height r in $\mathcal{CP}_n$. We compute the order of idempotents in S of height n - 1 and height n - 2 in the following theorem.

Theorem 9. *Let* $S = \mathcal{CP}_n$*. Then*

(i) $|E(J_{n-1})| = n + 4$*, for* $n \geq 3$*;*

(ii) $|E(J_{n-2})| = \dfrac{n^2 + 7n + 28}{2}$*, for* $n \geq 5$.

Proof. (i) Let $\alpha \in E(J_{n-1})$ for $n \geq 3$. Then since α is a partial map, dom $\alpha \subseteq [n]$, and since $h(\alpha) \leq$ |dom α| and $h(\alpha) = n - 1$ then either |dom α| = n or |dom α| = n - 1. As such $E(J_{n-1}) = \{\alpha \in E(J_{n-1})$: |dom α| = $n\} \cup \{\alpha \in E(J_{n-1})$: |dom α| = $n - 1\}$.

(a) Now suppose |dom α| = n. Then dom α is of the form dom $\alpha = \{a_1, a_2, \ldots, a_n\}$ where $a_i < a_{i+1}$ and $a_i \in [n]$ for all i. Thus by Corollary 4, α is in of the following types:

$$\text{first type } \alpha = \begin{pmatrix} a_1 & a_2 \cdots a_{n-1} & a_n \\ a_1 & a_2 \cdots a_{n-1} & k \end{pmatrix} \textit{ or second type } \alpha = \begin{pmatrix} a_1 & a_2 \cdots a_{n-1} & a_n \\ k & a_2 \cdots a_{n-1} & a_n \end{pmatrix},$$

where $k \in \text{fix } \alpha = \{a_1, a_2, \ldots, a_{n-1}\}$.

If α is of the first type. Then α is a contraction if and only if $k = a_{n-1}$ or $k = a_{n-1}$ (i. e., there are only 2 choices for k in this case) and similarly if α is of the second type, then α is a contraction if and only if $k = a_2$ or $k = a_2$ (i. e., there are only 2 choices for k in this case). Thus by sum rule we have all together a total of 4 idempotent elements.

(b) Secondly, suppose $|\text{dom } \alpha| = n - 1$, then there are $\begin{pmatrix} n \\ n-1 \end{pmatrix}$ possible domains in $[n]$ of order $n - 1$, which turns out to be n after simplification. For a particular case, consider dom $\alpha = \{a_1, a_2, \ldots, a_{n-1}\}$. Then to form an idempotent of height equal $n - 1$, we first notice Corollary 4 that the fixed points are tied together. Thus α is of the form

$$\begin{pmatrix} a_1 & a_2 \cdots a_{n-2} & a_{n-1} \\ a_1 & a_2 \cdots a_{n-2} & a_{n-1} \end{pmatrix}$$

i. e., a partial identity on $[n]$. This can be done in only 1 way. Using product rule, there are total of $n \times 1$ ways of forming such idempotent elements. Hence all together from (a) and (b) we have a total of $n + 4$ idempotents.

(ii) Let α be an idempotent of height equal $n - 2$ in S, for $n \geq 5$. Since α is a partial map then α has 3 possible domains, i. e., domain of order n, $n - 1$ or $n - 2$.

(a) Suppose $|\text{dom } \alpha| = n$. Then dom $\alpha = \{a_1, a_2, \ldots, a_{n-2}, a_{n-1}, a_n\}$. Notice that $h(\alpha) = n-2$, therefore by Corollary 4, the $n - 2$ fixed point must be tied together. This can be done in three(3) ways, i. e., α is either

$$\begin{pmatrix} a_1 & a_2 \cdots a_{n-2} & a_{n-1} & a_n \\ a_1 & a_2 \cdots a_{n-2} & x & y \end{pmatrix} \textit{ or } \begin{pmatrix} a_1 & a_2 \cdots a_{n-2} & a_{n-1} & a_n \\ x & y \cdots a_{n-2} & a_{n-1} & a_n \end{pmatrix} \textit{ or } \begin{pmatrix} a_1 & a_2 \cdots a_{n-2} & a_{n-1} & a_n \\ x & a_2 \cdots a_{n-2} & a_{n-1} & y \end{pmatrix},$$

where $x, y \in \text{fix } \alpha$.

Notice that the first type and the second type elements have the same number of choices of x and y. Thus we may consider

$$\alpha = \begin{pmatrix} a_1 & a_2 \cdots a_{n-2} & a_{n-1} & a_{n-1} \\ a_1 & a_2 \cdots a_{n-2} & x & y \end{pmatrix}.$$

Thus α is a contraction if and only if x and y have the following choices, ($x = a_{n-1} = y$) or ($x = a_{n-3} = y$) or ($x = a_{n-2}$ and $y = a_{n-3}$) or ($x = a_{n-3}$ and $y = a_{n-2}$) or ($x = a_{n-3}$ and $y = a_{n-4}$). This gives five(5) idempotents. All together we have a total of 2×5 number of idempotents. Now for the third type α, the choices for x and y are either ($x = a_2$ and $y = a_{n-1}$) or ($x = a_2$ and $y = a_{n-2}$) or ($x = a_3$ and $y = a_{n-1}$) or

($x = a_3$ and $y = a_{n-2}$), which is a total of 4 ways. Now all together we have (2 × 5 + 4) number of idempotent elements which simplify to 14 number of idempotent elements.

(b) Suppose |dom α| = n - 1. Notice that there are n - 2 fixed points and by Corollary 4 these fixed points are tied together (i. e., convex). Thus the domain of α is of three types, each a subset of $[n]$ = $\{a_1, a_2, . . . , a_n\}$. *i. e.*,

Type(1): dom α is either $\{a_1, a_2, . . . , a_{n-2}, a_{n-1}\}$ or $\{a_2, a_3, . . . , a_n\}$ (i. e., a convex subsets of $[n]$ of order n - 1);

Type(2): dom α is either $\{a_1, a_2, . . . , a_{n-2}, a_n\}$, $\{a_1, a_3, a_4, . . . , a_n\}$ or $\{a_1, a_2, . . . , a_{n-3}, a_{n-1}, a_n\}$ or $\{a_1, a_2, a_4, . . . , a_n\}$ i. e., a subset with a_{n-1} missing in the first type, a_2 missing in the second type, a_{n-2} missing in the third type and a_3 missing in the fourth type, respectively.

Type(3): dom α is either $\{a_1, a_2, . . . , a_{n-r}, a_{n-r+2}, . . . , a_n\}$ or $\{a_1, a_2, . . . , a_{n-r-1}, a_{n-r+1}, . . . , a_n\}$, where $4 \leq r \leq n - 3$.

We treat each type separately:

Subtype(1): dom α is $\{a_1, a_2, . . . , a_{n-2}, a_{n-1}\}$ or $\{a_2, a_3, . . . , a_n\}$ (i. e., a convex subset of $[n]$ of order n - 1). In particular, consider dom $\alpha = \{a_1, a_2, . . . , a_{n-2}, a_{n-1}\}$. Then α is either $\begin{pmatrix} a_1 & a_2 \cdots a_{n-2} & a_{n-1} \\ a_1 & a_2 \cdots a_{n-2} & x \end{pmatrix}$ *or*

$$\begin{pmatrix} a_1 & a_2 \cdots a_{n-2} & a_{n-1} \\ x & a_2 \cdots a_{n-2} & a_{n-1} \end{pmatrix}.$$

So in each case if dom $\alpha = \{a_1, a_2, . . . , a_{n-2}, a_{n-1}\}$ we have 4 idempotents and similarly if dom $\alpha = \{a_2, a_3, . . . , a_{n-2}, a_n\}$ we also have 4 idempotents, therefore we have 8 idempotents.

Subtype(2): If (dom $\alpha = \{a_1, a_2, . . . , a_{n-2}, a_n\}$ or $\{a_1, a_3, a_4, . . . , a_n\}$) and (dom $\alpha = \{a_1, a_2, . . . , a_{n-3}, a_{n-1}, a_n\}$ or $\{a_1, a_2, a_4, . . . , a_n\}$) i. e., a subset with a_{n-1} or a_2 missing and a subset with a_{n-2} or a_3 missing, respectively.

Now suppose dom $\alpha = \{a_1, a_2, . . . , a_{n-2}, a_n\}$. Then α is either of the following forms:

$$\alpha = \begin{pmatrix} a_1 & a_2 \cdots a_{n-3} & a_{n-2} & a_{n-1} \\ a_1 & a_2 \cdots a_{n-3} & a_{n-2} & x \end{pmatrix} \text{ or } \begin{pmatrix} a_1 & a_2 \cdots a_{n-3} & a_{n-2} & a_n \\ y & a_2 \cdots a_{n-3} & a_{n-2} & a_n \end{pmatrix}.$$

Thus ($x = a_{n-2}$) or ($x = a_{n-3}$) or ($x = a_{n-4}$) and ($y = a_1$) or ($y = a_1$), which gives a total of 5 idempotents, therefore we have all together 2 × 5 idempotents.

Now for dom $\alpha = \{a_1, a_2, . . . , a_{n-3}, a_{n-1}, a_n\}$, α is either of the following forms:

$$\alpha = \begin{pmatrix} a_1 & a_2 \cdots a_{n-3} & a_{n-1} & a_n \\ a_1 & a_2 \cdots a_{n-3} & a_{n-1} & x \end{pmatrix} \text{ or } \begin{pmatrix} a_1 & a_2 \cdots a_{n-3} & a_{n-1} & a_n \\ y & a_2 \cdots a_{n-3} & a_{n-1} & a_n \end{pmatrix}.$$

Thus, x has only one choice, which is a_{n-1} and y has two choices, i. e., ($y = a_2$) or ($y = a_3$). This gives a total of 3 idempotents, therefore all together we have a total of 2×3 idempotents.

Subtype(3): If n = 5 or n = 6 anyone can check the result holds. Now suppose $n \geq 7$.

If dom $\alpha = \{a_1, a_2, \ldots, a_{n-r}, a_{n-r+2}, \ldots, a_n\}$ or dom $\alpha = \{a_1, a_2, \ldots, a_{n-r-1}, a_{n-r+1}, \ldots, a_n\}$ for $4 \leq r \leq n - 3$, it is easy to see that there are 4n-24 idempotents.

(c) Suppose $|\text{dom } \alpha| = n - 2$. Notice that $h(\alpha) = n - 2$ and α is an idempotent, then dom α = im α = fix α. Therefore we have $\binom{n}{n-2} = \frac{n(n-1)}{2}$ number of idempotents. Hence the result follows by summing the results of (a), (b) and (c).

Corollary 10. *Let* $S = \mathcal{CT}_n$. *Then*

(i) $|E(J_{n-1})| = 4$, *for* $n \geq 3$;

(ii) $|E(J_{n-2})| = 14$, *for* $n \geq 5$.

Proof. Since $S = \mathcal{CT}_n$, then $|\text{dom } \alpha| = n$ for all α in $E(\mathcal{CT}_n)$. Thus the proof follows from (ia) and (iia) of the proof of Theorem 9.

Remark 11. *Since* J_1 *in* $\mathcal{P}_n$ *is the same as* J_1 *in* $\mathcal{CP}_n$ *then* $|E(J1)| = \sum_{r=1}^{n} \binom{n}{r}(r)$ *and if* $S = \mathcal{CT}_n$ *it is clear that* $|E(J_1)| = n$..

The next lemma gives the number of idempotents in $\mathcal{CP}_n$ of height 2 and width k greater than or equal to $n - 1$. Let us denote this number by $N(E(n, 2, k \geq n - 1))$, where k is the width of α and 2 is the height of α.

Theorem 12. *For* $n \geq 4$, *the number of idempotents in* $\mathcal{CP}_n$ *of height 2 and width greater than or equal to* $n - 1$ *is* $N(E(n, 2, k \geq n - 1)) = (n - 1)^{n-2} + (n - 2)^2 2^{n-3}$.

Proof. Let α be idempotent in $\mathcal{CP}_n$ of height 2 such that $|\text{dom } \alpha| \geq n - 1$. Thus $|\text{dom } \alpha| = n - 1$ or $|\text{dom } \alpha| = n$. We treat differently the two cases:

case 1. Suppose $|\text{dom } \alpha| = n$. Then dom $\alpha = \{a_1, a_2 \ldots, a_n\}$, and we can select 2 convex images from this domain in $(n - 1)$ ways. Now fixing 2 images in n space, reduce the space to $n - 2$. The empty $n - 2$ space can be filled with one of the images or the other or both in a total of $(1 + 1)^{n-2} = 2^{n-2}$ ways. Thus by product rule all together we have $(n - 1)2^{n-2}$ ways.

case 2. Now suppose $|\text{dom } \alpha| = n - 1$. Notice that there are $\binom{n}{n-1}$ possible combination of this type of domains. Two out of them are convex while the remaining with 1 gap. If the domain is convex, then it is of the form dom $\alpha = \{a_1, a_2, \ldots, a_{n-1}\}$ or dom $\alpha = \{a_2, a_3, \ldots, a_{n-2}\}$, and by case 1 we have $2(n - 2)2^{n-3}$

possibilities. And if the domain has 1-gap, then is of the form dom $\alpha = \{a_1, a_2, \ldots, a_{n-2}, a_n\}$ or dom $\alpha = \{a_1, a_2, \ldots, a_{n-3}, a_{n-1}, a_n\}$ or · · · or dom $\alpha = \{a_1, a_3, a_4, \ldots, a_n\}$, which is a total of $\binom{n}{n-1} - 2$ of such domains or simply (n - 2) of them. If we consider dom $\alpha = \{a_1, a_3, a_4, \ldots, a_n\}$ and we fixed a_1 and a_3, then the remaining elements have to be map to only one element, which is a_3, meaning that $a_i\alpha = a_3$ for $4 \leq i \leq n$, i. e.,

$$\alpha = \begin{pmatrix} a_1 & a_3 & a_4 \cdots a_n \\ a_1 & a_3 & a_3 \cdots a_3 \end{pmatrix},$$

which can be done in just 1 way. Now the remaining elements $\{a_3, a_4, \ldots, a_n\}$ form a convex set. We can tie two convex images from this set in n - 3 ways. Therefore, there will be n - 3 remaining space. Thus the 2 images can be map to the remaining space in 2^{n-3} ways. Now by sum and product rule, all together we have $2(n - 2)2^{n-3} + (n - 2)((n - 3)2^{n-3} + 1)$ idempotents. Now the result follows by summing all the two cases.

As a consequence, we deduce the following corollary.

Corollary 13. *Let $S = \mathcal{CT}_n$. Then $|E(J_2)| = (n - 1)2^{n-2}$ for $n \geq 2$.*

Proof. Let $S = \mathcal{CT}_n$, since |dom α| = n for all α in $\mathcal{CT}_n$, then we can apply case 1 of the proof of Theorem 12. Then the result follows.

The order of idempotents of height r for $3 \leq r \leq n$-3 for the semigroups $\mathcal{CP}_n$ and $\mathcal{CT}_n$ remain an open problem. Next, we conclude this section with the following conjuncture, i. e., the number of idempotents of height 3 in CT_n is given as the following recurrence relation.

Conjecture 14. Let $S = \mathcal{CT}_n$. Then $|E(J_3)| = 2(a_n + (n - 4)a_{n-1})$ for $n \geq 4$, where $a_3 = 1$ and $a_n = \left(1 + \frac{3\sqrt{2}}{4}\right)\left(1+\sqrt{2}\right)^{n-4} + \left(1 - \frac{3\sqrt{2}}{4}\right)\left(1-\sqrt{2}\right)^{n-4}$.

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Statement of Conflict of Interest

Authors have declared no conflict of interest.

Author's Contributions

The contribution of the authors is equal.

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