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Almost Para-Contact Metric Structures on 5-dimensional Nilpotent Lie Algebras

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Abstract

In this manuscript, almost para-contact metric structures on 5 dimensional nilpotent Lie algebras are studied. Some examples of para-Sasakian and para-contact structures on five-dimensional nilpotent Lie algebras are given.

1. Introduction

Almost paracontact structures were first studied by [1] and after the work of Zamkovoy in [2], many authors have made contribution. For recent studies, see [3]-[8]. In [9], almost paracontact metric structures were classified into 2^{12} classes taking into consideration the Levi-Civita covariant derivative of the fundamental 2-form of the structure. In this work, we study almost paracontact metric structures on 5-dimensional nilpotent Lie algebras.

In the literature, there are many researches on five dimensional Lie algebras equipped with an almost contact metric structure. Andrada et al., studied Sasakian structures on five dimensional Lie algebras [10]. Calvaruso and Fino introduced an approach on five dimensional K-contact Lie algebras [11]. Nilpotent Lie algebras having dimension 5 were classified in [12]. According to this classification, we examined the Lie algebras equipped with quasi-Sasakian structures in [13]. Also in [14], we studied some certain classes, such as $\alpha-$ Sasakian, $\beta-$ Kenmotsu, cosymplectic, nearly cosymplectic, on five dimensional nilpotent Lie algebras and obtained some results on the corresponding Lie groups. This paper is organised in a similar vein with almost paracontact metric structure. Under the light of the classifications given in [9] and [12], we investigate the existence of left invariant para-cosymplectic, nearly para-cosymplectic, α-para-Sasakian, β-para-Kenmotsu and paracontact structures on 5 dimensional nilpotent Lie algebras.

2. Preliminaries

A $2n+1$ dimensional differentiable manifold M has an almost paracontact structure (ϕ,ξ,η), if it has an endomorphism ϕ, a vector field ξ and a 1-form η such that

$$\phi^2 = I-\eta\otimes\xi, \qquad \eta(\xi)=1, \phi(\xi)=0,$$

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$$\text{there exists a disribution } \mathbb{D}: p \in M \longrightarrow \mathbb{D}_p = Ker\eta.$$

An almost paracontact manifold is one which has an almost paracontact structure and if in addition M has a semi-Riemannian metric g satisfying

$$g(\phi(X),\phi(Y)) = -g(X,Y)+\eta(X)\eta(Y)$$

for all vector fields X,Y, then M is called an almost paracontact metric manifold with an almost paracontact metric structure and a compatible metric g. The 2-form

$$\Phi(X,Y) = g(\phi(X),Y)$$

for all $X,Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the set of smooth vector fields on M, is defined to be the fundamental 2-form of M. In [2], a classification of almost paracontact metric manifolds was obtained by using the covariant derivative of Φ.

In this work we focus on following almost paracontact structures.

Let M be a differentiable manifold with an almost paracontact metric structure (ϕ,ξ,η,g) and the fundamental 2-form Φ.

(ϕ,ξ,η,g) is said to be

- para-cosymplectic if $\nabla_X\Phi(Y,Z) = 0$,
- nearly para-cosymplectic if $\nabla_X\Phi(X,Y) = 0$, or equivalently, $(\nabla_X\phi)(Y)+(\nabla_Y\phi)(X) = 0$,
- α-para-Sasakian if $\nabla_X\phi(Y) = \alpha(g(X,Y)\xi-\eta(Y)X)$ for a constant α,
- β-para-Kenmotsu if $\nabla_X\phi(Y) = \beta(g(X,\phi(Y))\xi+\eta(Y)\phi(X))$ for a constant β,
- α-paracontact if $\Phi = \alpha d\eta$, where $d\eta$ is the exterior derivative of η and α is a constant,
- paracontact if $\Phi = d\eta$

for all vector fields X, Y, Z on M.

An almost paracontact metric structure (ϕ,ξ,η,g) on a connected Lie group G uniquely induces an almost paracontact metric structure (ϕ,ξ,η,g) on the corresponding Lie algebra $\mathfrak{g}$.

In this manuscript, we investigate almost paracontact metric structures on 5-dimensional nilpotent Lie algebras. Nilpotent Lie algebras with dimensions ≤ 5 were classified in [12], see also [15, 16]. These are algebras denoted by $\mathfrak{g}_i$ with the corresponding basis $\{e_1,\ldots,e_5\}$ and non-zero brackets:

$$\begin{aligned}
\mathfrak{g}_1 &: [e_1,e_2]=e_5,[e_3,e_4]=e_5\\
\mathfrak{g}_2 &: [e_1,e_2]=e_3,[e_1,e_3]=e_5,[e_2,e_4]=e_5\\
\mathfrak{g}_3 &: [e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_5,[e_2,e_3]=e_5\\
\mathfrak{g}_4 &: [e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_5\\
\mathfrak{g}_5 &: [e_1,e_2]=e_4,[e_1,e_3]=e_5\\
\mathfrak{g}_6 &: [e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_2,e_3]=e_5
\end{aligned}$$

3. Almost paracontact metric structures on $\mathfrak{g}_i$

Let (ϕ,ξ,η,g) be a left invariant a.p.c.m.s. (almost paracontact metric structure) on a connected Lie group G with corresponding Lie algebra $\mathfrak{g}_i$. We use the same notation for the corresponding a.p.c.m.s. on $\mathfrak{g}_i$.

We study each algebra $\mathfrak{g}_i$ seperately:

The algebra $\mathfrak{g}_1$: Consider the basis $\{e_1,\ldots,e_5\}$ with non-zero brackets

$$[e_1,e_2]=e_5,[e_3,e_4]=e_5.$$

- There is no para-cosymplectic structure on $\mathfrak{g}_1$.
  To see this, we show that $\mathfrak{g}_1$ does not have a non-zero parallel vector field. Let $\xi = \sum a_ie_i$ be a parallel vector field on $\mathfrak{g}_1$. Then for all basis elements, we have $g(\nabla_{e_i}\xi,e_j) = 0$. By Kozsul's formula,

  $$2g(\nabla_{e_1}\xi,e_2) = -g(e_1,[\xi,e_2])+g(\xi,[e_2,e_1]+g(e_2,[e_1,\xi])) = -a_5g(e_5,e_5) = 0,$$

  implying $a_5 = 0$. Similarly, $2g(\nabla_{e_1}\xi,e_5) = a_2g(e_5,e_5) = 0$ gives $a_2 = 0$, $2g(\nabla_{e_2}\xi,e_5) = -a_1g(e_5,e_5) = 0$ yields $a_1 = 0$. From the equation $2g(\nabla_{e_3}\xi,e_5) = a_4g(e_5,e_5) = 0$, we get $a_4 = 0$ and $2g(\nabla_{e_4}\xi,e_5) = -a_3g(e_5,e_5) = 0$ gives $a_3 = 0$. Thus, a vector field $\xi = \sum a_ie_i$ is parallel if and only if $a_i = 0$. Since for a para-cosymplectic structure the characteristic vector field is parallel, there is no para-cosymplectic structure on $\mathfrak{g}_1$.

Similarly, there are no nonzero parallel vector fields and no para-cosymplectic structures on remaining Lie algebras $\mathfrak{g}_i$. Now we calculate covariant derivatives of basis elements as follows:

$$\nabla_{e_1} e_2 = \sum \varepsilon_i g(\nabla_{e_1} e_2, e_i) e_i, \quad \text{where} \quad \varepsilon_i = g(e_i, e_i)$$

We write $g(\nabla_{e_1} e_2, e_i)$ by Kozsul's formula. The nonzero covariant derivatives are:

$$\nabla_{e_1} e_2 = \frac{1}{2} e_5, \;\; \nabla_{e_1} e_5 = -\frac{1}{2} \varepsilon_2 \varepsilon_5 e_2,$$

$$\nabla_{e_2} e_1 = -\frac{1}{2} e_5, \;\; \nabla_{e_2} e_5 = \frac{1}{2} \varepsilon_1 \varepsilon_5 e_1,$$

$$\nabla_{e_3} e_4 = \frac{1}{2} e_5, \;\; \nabla_{e_3} e_5 = -\frac{1}{2} \varepsilon_4 \varepsilon_5 e_4,$$

$$\nabla_{e_4} e_3 = -\frac{1}{2} e_5, \;\; \nabla_{e_4} e_5 = \frac{1}{2} \varepsilon_3 \varepsilon_5 e_3,$$

$$\nabla_{e_5} e_1 = -\frac{1}{2} \varepsilon_2 \varepsilon_5 e_2, \;\; \nabla_{e_5} e_2 = \frac{1}{2} \varepsilon_1 \varepsilon_5 e_1, \;\; \nabla_{e_5} e_3 = -\frac{1}{2} \varepsilon_4 \varepsilon_5 e_4, \;\; \nabla_{e_5} e_4 = \frac{1}{2} \varepsilon_3 \varepsilon_5 e_3$$

- There is no nearly para-cosymplectic structure on $\mathfrak{g}_1$.
  Assume that (ϕ, ξ, η, g) is a nearly para-cosymplectic structure. Then we have $\nabla_{e_i} \phi(e_j) + \nabla_{e_j} \phi(e_i) = 0$.
  Let

$$\phi(e_1) = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5,$$

$$\phi(e_2) = b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 e_4 + b_5 e_5,$$

$$\phi(e_3) = c_1 e_1 + c_2 e_2 + c_3 e_3 + c_4 e_4 + c_5 e_5,$$

$$\phi(e_4) = d_1 e_1 + d_2 e_2 + d_3 e_3 + d_4 e_4 + d_5 e_5,$$

$$\phi(e_5) = f_1 e_1 + f_2 e_2 + f_3 e_3 + f_4 e_4 + f_5 e_5.$$

Since $0 = \Phi(e_i, e_i) = g(\phi(e_i), e_i)$, we have $a_1 = b_2 = c_3 = d_4 = f_5 = 0$.
From the equation $(\nabla_{e_1} \Phi)(e_1, e_5) = 0$, we obtain $0 = -\Phi(e_1, \nabla_{e_1} e_5) = -g(\phi(e_1), -\frac{1}{2}\varepsilon_2 \varepsilon_5 e_2)$, which implies $g(\phi(e_1), e_2) = -g(\phi(e_2), e_1) = 0$, thus $a_2 = b_1 = 0$.
Similarly, from $(\nabla_{e_1} \Phi)(e_1, e_2) = -\Phi(e_1, \nabla_{e_1} e_2) = -\frac{1}{2} g(\phi(e_1), e_5) = 0$, which implies $a_5 = f_1 = 0$.
$(\nabla_{e_4} \Phi)(e_4, e_3) = 0$ gives $g(\phi(e_4), e_5) = -g(\phi(e_5), e_4) = 0$ and so $d_5 = f_4 = 0$.
$(\nabla_{e_4} \Phi)(e_4, e_5) = 0$ gives $g(\phi(e_4), e_3) = -g(\phi(e_3), e_4) = 0$ and so $d_3 = c_4 = 0$.
$(\nabla_{e_2} \Phi)(e_2, e_1) = 0$ gives $g(\phi(e_2), e_5) = -g(\phi(e_5), e_2) = 0$ and so $b_5 = f_2 = 0$.
$(\nabla_{e_3} \Phi)(e_3, e_4) = 0$ gives $g(\phi(e_3), e_5) = -g(\phi(e_5), e_3) = 0$ and so $c_5 = f_3 = 0$.
Thus,

$$\phi(e_1) = a_3 e_3 + a_4 e_4,$$

$$\phi(e_2) = b_3 e_3 + b_4 e_4,$$

$$\phi(e_3) = c_1 e_1 + c_2 e_2,$$

$$\phi(e_4) = d_1 e_1 + d_2 e_2,$$

$$\phi(e_5) = 0.$$

Since

$$\begin{aligned} 0 &= (\nabla_{e_1} \phi)(e_5) + (\nabla_{e_5} \phi)(e_1) \\ &= e_3\{-\varepsilon_2 \varepsilon_5 b_3 + \tfrac{1}{2} \varepsilon_3 \varepsilon_5 a_4\} \\ &\quad + e_4\{-\varepsilon_2 \varepsilon_5 b_4 - \tfrac{1}{2} \varepsilon_4 \varepsilon_5 a_3\} \end{aligned}$$

and e_3, e_4 are linearly independent, we have

$$2\varepsilon_2 b_3 + \varepsilon_3 a_4 = 0,$$

$$2\varepsilon_2 b_4 - \varepsilon_4 a_3 = 0.$$

Similarly, from $(\nabla_{e_1} \phi)(e_5) + (\nabla_{e_5} \phi)(e_1) = 0$, we get

$$2\varepsilon_1 a_3 + \varepsilon_3 b_4 = 0,$$

$$-2\varepsilon_1 a_4 - \varepsilon_4 b_3 = 0.$$

Now, we have

$$2\varepsilon_2 b_3 + \varepsilon_3 a_4 = 0,$$

$$-\varepsilon_4 b_3 - 2\varepsilon_1 a_4 = 0.$$

Multiply the first equation by $2\varepsilon_3$ and the second equation by ε_1. Then we get $b_3 = 0$ and $a_4 = 0$. Similarly $a_3 = 0$ and $b_4 = 0$.
From the equation $(\nabla_{e_1}\phi)(e_3)+(\nabla_{e_3}\phi)(e_1) = 0$, we obtain $(c_2+a_4)e_5 = 0$, that is, $c_2 = -a_4$ and since $a_4 = 0$, we have $c_2 = 0$.
From the equation $(\nabla_{e_2}\phi)(e_3)+(\nabla_{e_3}\phi)(e_2) = 0$, we obtain $c_1 = b_4$ and since $b_4 = 0$, we have $c_1 = 0$.
Similarly, $(\nabla_{e_2}\phi)(e_4)+(\nabla_{e_4}\phi)(e_2) = 0$ implies $d_1 = -b_3 = 0$ and $(\nabla_{e_1}\phi)(e_4)+(\nabla_{e_4}\phi)(e_1) = 0$ yields $d_2 = a_3 = 0$.
Therefore $\phi(e_i) = 0$ and there is no non-zero nearly para-cosymplectic structure on $\mathfrak{g}_1$.

- A vector field ξ on $\mathfrak{g}_1$ is Killing if and only if $\xi \in \langle e_5\rangle$: For a Killing vector field $\xi = \sum_i \xi_i e_i$, we have $g(\nabla_{e_i}\xi, e_j) = -g(\nabla_{e_j}\xi, e_i)$. From $g(\nabla_{e_2}\xi, e_5) = -g(\nabla_{e_5}\xi, e_2)$, we have $\xi_1 = 0$.
  $g(\nabla_{e_1}\xi, e_5) = -g(\nabla_{e_5}\xi, e_1)$ gives $\xi_2 = 0$.
  $g(\nabla_{e_4}\xi, e_5) = -g(\nabla_{e_5}\xi, e_4)$ yields $\xi_3 = 0$.
  $g(\nabla_{e_3}\xi, e_5) = -g(\nabla_{e_5}\xi, e_3)$ implies $\xi_4 = 0$ and we have no any other restriction on the coefficients of ξ.
  By similar calculations, in $\mathfrak{g}_2$, $\mathfrak{g}_3$ and $\mathfrak{g}_4$, a vector field ξ is Killing if and only if $\xi = \xi_5 e_5$.
  A vector field ξ in $\mathfrak{g}_5$ or $\mathfrak{g}_6$ is Killing on each of these algebras if and only if $\xi = \xi_4 e_4 + \xi_5 e_5$.
- There are α-para-Sasakian structures on $\mathfrak{g}_1$, where $\alpha = \pm\frac{1}{2}$.
  For $y = \xi$, we get $-\phi(\nabla_x\xi) = \alpha\{g(x,\xi)\xi - x\}$. Thus, $\nabla_x\xi = \alpha\phi(x)$. The characteristic vector field of an α-para-Sasakian structure is Killing. Thus $\xi = \xi_5 e_5$ and

$$\phi(e_1) = \frac{1}{\alpha}\nabla_{e_1}\xi = \frac{1}{\alpha}\xi_5(-\frac{1}{2}\varepsilon_2\varepsilon_5 e_2),$$

$$\phi(e_2) = \frac{1}{\alpha}\nabla_{e_2}\xi = \frac{1}{\alpha}\xi_5(\frac{1}{2}\varepsilon_1\varepsilon_5 e_1),$$

$$\phi(e_3) = \frac{1}{\alpha}\nabla_{e_3}\xi = \frac{1}{\alpha}\xi_5(-\frac{1}{2}\varepsilon_4\varepsilon_5 e_4),$$

$$\phi(e_4) = \frac{1}{\alpha}\nabla_{e_4}\xi = \frac{1}{\alpha}\xi_5(\frac{1}{2}\varepsilon_3\varepsilon_5 e_3),$$

$$\phi(e_5) = 0.$$

Now we check the defining relation of an α-para-Sasakian structure (ϕ,ξ,η,g)

$$(\nabla_x\phi)(y) = \alpha\{g(x,y)\xi - \eta(y)x\}$$

for each pair of basis elements. For $x = y = e_1$, we should have

$$(\nabla_{e_1}\phi)(e_1) = \alpha\{g(e_1,e_1)\xi_5 e_5\},$$

which implies

$$-\frac{1}{4\alpha}\varepsilon_2\varepsilon_5 e_5 = \alpha\varepsilon_1 e_5.$$

Multiply both sides of the above equation by ε_1, we obtain $\varepsilon_1\varepsilon_2\varepsilon_5 = -4\alpha^2$. Thus $\varepsilon_1\varepsilon_2\varepsilon_5 = -1$ and $\alpha = \pm\frac{1}{2}$.
Similarly, for $x = y = e_3$, we get $\varepsilon_3\varepsilon_4\varepsilon_5 = -4\alpha^2$, which gives $\varepsilon_3\varepsilon_4\varepsilon_5 = -1$ and $\alpha = \pm\frac{1}{2}$. There is no any other restriction on ε_i or on α.
We have $\varepsilon_1\varepsilon_2\varepsilon_5 = -1$ and $\varepsilon_3\varepsilon_4\varepsilon_5 = -1$.
Case 1: If $\varepsilon_5 = -1$, then $\varepsilon_1\varepsilon_2 = 1$. Either $\varepsilon_1 = 1$ and $\varepsilon_2 = 1$; or $\varepsilon_1 = -1$ and $\varepsilon_2 = -1$. Also, since $\varepsilon_3\varepsilon_4 = 1$, $\varepsilon_3 = 1$ and $\varepsilon_4 = 1$; or $\varepsilon_3 = -1$ and $\varepsilon_4 = -1$. In these cases the signature is not $(3,2)$. Thus $\varepsilon_5 \neq -1$.
Case 2: If $\varepsilon_5 = 1$, then $\varepsilon_1\varepsilon_2 = -1$ and $\varepsilon_3\varepsilon_4 = -1$. There are four possibilities for the signature of the metric.

$$\varepsilon_1 = 1, \varepsilon_2 = -1, \varepsilon_3 = 1, \varepsilon_4 = -1, \varepsilon_5 = 1$$

$$\varepsilon_1 = 1, \varepsilon_2 = -1, \varepsilon_3 = -1, \varepsilon_4 = 1, \varepsilon_5 = 1$$

$$\varepsilon_1 = -1, \varepsilon_2 = 1, \varepsilon_3 = -1, \varepsilon_4 = 1, \varepsilon_5 = 1$$

$$\varepsilon_1 = -1, \varepsilon_2 = 1, \varepsilon_3 = 1, \varepsilon_4 = -1, \varepsilon_5 = 1.$$

One can check that (ϕ,ξ,η,g), where $\phi(e_i)$ are given as above and g has one of the signatures above, are α-para-Sasakian structures, where $\alpha = \pm\frac{1}{2}$.

- There is no β-para-Kenmotsu structure on $\mathfrak{g}_1$.
  The characteristic vector field ξ of a β-para-Kenmotsu structure satisfies the property $g(\nabla_x\xi, y) = g(\nabla_y\xi, x)$. Checking for basis elements, we obtain that $\xi = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 + \xi_4 e_4$. The definition of a β-para-Kenmotsu structure (ϕ, ξ, η, g) is
  $$(\nabla_x\phi)(y) = -\beta\{g(x, \phi(y))\xi + \eta(y)\phi(x)\}.$$
  For $y = \xi$, we get $\nabla_x\xi = \beta\phi^2(x) = \beta\{x - \eta(x)\xi\}$. Now
  $$\nabla_{e_1}\xi = \nabla_{e_1}(\xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 + \xi_4 e_4) = \frac{\xi_2}{2}e_5 = \beta\{e_1 - \varepsilon_1\xi_1(\xi_1 e_1 + ... + \xi_4 e_4)\}.$$
  Since basis elements are linearly independent, we have $\xi_2 = 0$, $1 - \varepsilon_1\xi_1^2 = 0$, $\xi_1\xi_3 = 0$, $\xi_1\xi_4 = 0$. If $\xi_1 = 0$, then $1 - \varepsilon_1\xi_1^2 = 1 = 0$ and thus $\xi_1 \neq 0$. Therefore, $\xi_3 = \xi_4 = 0$ and $\xi = \xi_1 e_1$. Now for $x = e_2$, we have
  $$\nabla_{e_2}\xi = -\frac{\xi_1}{2}e_5 = \beta\{e_2 - \eta(e_2)\xi\} = \beta e_2,$$
  which implies $\xi_1 = 0$, that is $\xi = 0$.
- There are paracontact structures on $\mathfrak{g}_1$.
  More generally, consider an α-paracontact structure (ϕ, ξ, η, g) with the fundamental 2-form Φ. Since $\Phi = \alpha d\eta$, we have
  $$0 = \Phi(\xi, x) = \alpha d\eta(\xi, x) = \frac{1}{2}\{(\nabla_\xi\eta)(x) - (\nabla_x\eta)(\xi)\} = -(\nabla_\xi\eta)(x),$$
  that is, $(\nabla_\xi\eta)(x) = g(\xi, \nabla_\xi x) = 0$. By Kozsul's formula,
  $$0 = 2g(\nabla_\xi x, \xi) = -2g(\xi, [x, \xi]).$$
  Thus for a paracontact structure, the characteristic vector field ξ satisfies $g(\xi, [x, \xi]) = 0$ for all vector fields x.
  Let $\xi = \sum \xi_i e_i$. We have
  $$0 = g(\xi, [e_1, \xi]) = g(\xi, \xi_2 e_5) = \xi_2\xi_5\varepsilon_5,$$
  $$0 = g(\xi, [e_2, \xi]) = g(\xi, -\xi_1 e_5) = -\xi_1\xi_5\varepsilon_5,$$
  $$0 = g(\xi, [e_3, \xi]) = g(\xi, \xi_4 e_5) = \xi_4\xi_5\varepsilon_5,$$
  $$0 = g(\xi, [e_4, \xi]) = g(\xi, -\xi_3 e_5) = -\xi_3\xi_5\varepsilon_5.$$
  It is easy to observe that the structure (ϕ, ξ, η, g), where $\xi = e_5$, g has signature $+, -, +, -, +$, $\phi(e_1) = e_2$, $\phi(e_2) = e_1$, $\phi(e_3) = e_4$, $\phi(e_4) = e_3$, $\phi(e_5) = 0$ is paracontact.

The algebra $\mathfrak{g}_2$:

The nonzero brackets of basis elements are:

$$[e_1, e_2] = e_3, [e_1, e_3] = e_5, [e_2, e_4] = e_5.$$

Assume that g is a semi Riemannian metric with signature $g(e_i, e_i) = \varepsilon_i$. Nonzero covariant derivatives of g calculated by the Kozsul's formula are:

$$\begin{array}{lll}
\nabla_{e_1}e_2 = \frac{1}{2}e_3, & \nabla_{e_1}e_3 = -\frac{1}{2}\varepsilon_2\varepsilon_3 e_2 + \frac{1}{2}e_5, & \nabla_{e_1}e_5 = -\frac{1}{2}\varepsilon_3\varepsilon_5 e_3, \\
\nabla_{e_2}e_1 = -\frac{1}{2}e_3, & \nabla_{e_2}e_3 = \frac{1}{2}\varepsilon_1\varepsilon_3 e_1, & \nabla_{e_2}e_4 = \frac{1}{2}e_5, \\
\nabla_{e_2}e_5 = -\frac{1}{2}\varepsilon_4\varepsilon_5 e_4, & \nabla_{e_3}e_1 = -\frac{1}{2}\varepsilon_2\varepsilon_3 e_2 - \frac{1}{2}e_5, & \nabla_{e_3}e_2 = \frac{1}{2}\varepsilon_1\varepsilon_3 e_1, \\
\nabla_{e_3}e_5 = \frac{1}{2}\varepsilon_1\varepsilon_5 e_1, & \nabla_{e_4}e_2 = -\frac{1}{2}e_5, & \nabla_{e_4}e_5 = \frac{1}{2}\varepsilon_2\varepsilon_5 e_2, \\
\nabla_{e_5}e_1 = -\frac{1}{2}\varepsilon_3\varepsilon_5 e_3, & \nabla_{e_5}e_2 = -\frac{1}{2}\varepsilon_4\varepsilon_5 e_4, & \\
\nabla_{e_5}e_3 = \frac{1}{2}\varepsilon_1\varepsilon_5 e_1, & \nabla_{e_5}e_4 = \frac{1}{2}\varepsilon_2\varepsilon_5 e_2. &
\end{array}$$

- There exists no nearly-para-cosymplectic structure.
  Assume that (ϕ,ξ,η,g) is a nearly para-cosymplectic structure. Then we have $\nabla_{e_i}\phi(e_j)+\nabla_{e_j}\phi(e_i)=0$.
  Let

$$\phi(e_1)=a_1e_1+a_2e_2+a_3e_3+a_4e_4+a_5e_5,$$
$$\phi(e_2)=b_1e_1+b_2e_2+b_3e_3+b_4e_4+b_5e_5,$$
$$\phi(e_3)=c_1e_1+c_2e_2+c_3e_3+c_4e_4+c_5e_5,$$
$$\phi(e_4)=d_1e_1+d_2e_2+d_3e_3+d_4e_4+d_5e_5,$$
$$\phi(e_5)=f_1e_1+f_2e_2+f_3e_3+f_4e_4+f_5e_5.$$

  Since $0=\Phi(e_i,e_i)=g(\phi(e_i),e_i)$, we have $a_1=b_2=c_3=d_4=f_5=0$.
  From the equation

$$\begin{aligned} 0 &= (\nabla_{e_2}\varphi)(e_2)=b_1\nabla_{e_2}e_1+b_3\nabla_{e_2}e_3+b_4\nabla_{e_2}e_4+b_5\nabla_{e_2}e_5\\ &= b_1(-\tfrac{1}{2}e_3)+b_3(\tfrac{1}{2}\varepsilon_1\varepsilon_3)e_1+b_4(\tfrac{1}{2}e_5)+b_5(-\tfrac{1}{2}\varepsilon_4\varepsilon_5)e_4\end{aligned}$$

  we have, $b_1=b_3=b_4=b_5=0$. The equation $0=(\nabla_{e_5}\varphi)(e_5)$ gives $f_1=f_2=f_3=f_4=0$. Since $(\nabla_x\Phi)(x,y)=0$, we have

$$0=(\nabla_{e_1}\Phi)(e_1,e_2)=-\Phi(e_1,\nabla_{e_1}e_2)=-\frac{1}{2}g(\phi(e_1),e_3)=\frac{1}{2}g(\phi(e_3),e_1),$$

  and thus $a_3=c_1=0$. In addition,

$$\begin{aligned} 0 &= (\nabla_{e_1}\Phi)(e_1,e_3)=-\Phi(e_1,\nabla_{e_1}e_3)\\ &= -g(\phi(e_1),-\tfrac{1}{2}\varepsilon_2\varepsilon_3e_2+\tfrac{1}{2}e_5)\\ &= \tfrac{1}{2}g(\phi(e_1),\varepsilon_2\varepsilon_3e_2)+\tfrac{1}{2}g(\phi(e_5),e_1)\\ &= \tfrac{1}{2}\varepsilon_2\varepsilon_3g(\phi(e_1),e_2)\end{aligned}$$

  implies $a_2=b_1=0$. Since $\phi(e_5)=0$, we have $g(\phi(e_i),e_5)=-g(\phi(e_5),e_i)=0$ and as a result $a_5=b_5=c_5=d_5=0$.
  Since

$$\begin{aligned} 0 &= (\nabla_{e_2}\Phi)(e_2,e_5)=-\Phi(e_2,\nabla_{e_2}e_5)\\ &= \tfrac{1}{2}\varepsilon_4\varepsilon_5g(\phi(e_2),e_4),\end{aligned}$$

  we get $b_4=d_2=0$. Since

$$\begin{aligned} 0 &= (\nabla_{e_3}\Phi)(e_3,e_1)=-\Phi(\nabla_{e_3}e_1,e_3)=\Phi(e_3,\nabla_{e_3}e_1)\\ &= -\tfrac{1}{2}\varepsilon_2\varepsilon_3g(\phi(e_3),e_2),\end{aligned}$$

  we have $c_2=b_3=0$. Now,

$$0=(\nabla_{e_1}\phi)(e_2)+(\nabla_{e_2}\phi)(e_1)=a_4\nabla_{e_2}e4=\frac{a_4}{2}e_5=0$$

  gives $a_4=0$. Since $\phi(e_1)=0$, we obtain $0=g(\phi(e_1),e_4)=-g(\phi(e_4),e_1)$, that is $d_1=0$. From

$$0=(\nabla_{e_1}\phi)(e_5)+(\nabla_{e_5}\phi)(e_1)=\varepsilon_3\varepsilon_5c_4\phi(e_3)=0,$$

  we get $c_4=0$ and this implies also $d_3=0$. To sum up $\phi(e_i)=0$ for all $i=1,\cdots,5$.
  By similar calculations, there are no nearly-para-cosymplectic structures on the remaining Lie algebras $\mathfrak{g}_i$.
- There is no α-para-Sasakian structure.
  Let (ϕ,ξ,η,g) be such a structure. We have $\xi=\xi_5e_5$, since the characteristic vector field is Killing. Since $g(\xi,\xi)=\xi_5^2\varepsilon_5=1$, $\varepsilon_5=1$. $\nabla_x\xi=\alpha\phi(x)$,

$$\phi(e_1)=\frac{1}{\alpha}\nabla_{e_1}e_5=-\frac{1}{2\alpha}\varepsilon_3\varepsilon_5e_3.$$

  Now we check the defining relation $(\nabla_x\phi)(y)=\alpha\{g(x,y)\xi-\eta(y)x\}$ for basis elements.
  Let $x=y=e_1$. In this case, the equation $(\nabla_{e_1}\phi)(e_1)=\alpha g(e_1,e_1)e_5$ implies

$$\frac{1}{4\alpha}\varepsilon_2\varepsilon_5e_2-\{\frac{1}{4\alpha}\varepsilon_3\varepsilon_5+\alpha\varepsilon_1\}e_5=0,$$

  this is not possible since e_2 and e_5 are linearly independent.
- This algebra does not have a β-para-Kenmotsu structure.
  From the equation $g(\nabla_x\xi,y)=g(\nabla_y\xi,x)$ in a β-para-Kenmotsu structure, ξ is obtained in the form $\xi=\xi_1e_1+\xi_2e_2+\xi_3e_3+\xi_4e_4$. Also for $x=e_3$ in the equation

$$\nabla_x\xi=\beta\phi^2(x)=\beta\{x-\eta(x)\xi\},$$

we get

$$\{\frac{\xi_2}{2}\varepsilon_1\varepsilon_3+\beta\varepsilon_3\xi_1\xi_3\}e_1-\{\frac{\xi_2}{2}\varepsilon_2\varepsilon_3+\beta\varepsilon_3\xi_2\xi_3\}e_2+\beta(\varepsilon_3\xi_3^2-1)e_3+\beta\varepsilon_3\xi_3\xi_4e_4-\frac{\xi_1}{2}e_5=0.$$

Linear independence of basis element yields $\xi_1=0$, $\xi_2=0$, $\xi_3=1$ and $\xi_4=0$. Thus $\xi=\xi_3e_3$. However, in this case,

$$\nabla_{e_2}\xi=\frac{\xi_3}{2}\varepsilon_1\varepsilon_3e_1\neq\beta\{e_2-\eta(e_2)\xi\}=\beta e_2.$$

- There are paracontact structures. Consider a paracontact structure (ϕ,ξ,η,g) with the fundamental 2-form Φ. Since $\Phi=d\eta$, the equation

$$g(\phi(e_i),e_j)=g(\nabla_{e_i}\xi,e_j)-g(\nabla_{e_j}\xi,e_i)$$

holds for all basis elements. Let $\xi=\sum\xi_ie_i$. Then,

$$g(\phi(e_1),e_2)=g(\nabla_{e_1}\xi,e_2)-g(\nabla_{e_2}\xi,e_1)=-\varepsilon_3\xi_3,$$

$$g(\phi(e_1),e_3)=g(\nabla_{e_1}\xi,e_3)-g(\nabla_{e_3}\xi,e_1)=-\varepsilon_5\xi_5,$$

$$g(\phi(e_1),e_4)=g(\nabla_{e_1}\xi,e_4)-g(\nabla_{e_4}\xi,e_1)=0,$$

$$g(\phi(e_1),e_5)=0,$$

$$g(\phi(e_2),e_3)=0,$$

$$g(\phi(e_2),e_4)=g(\nabla_{e_2}\xi,e_4)-g(\nabla_{e_4}\xi,e_2)=-\varepsilon_5\xi_5,$$

$$g(\phi(e_2),e_5)=g(\phi(e_3),e_4)=g(\phi(e_3),e_5)=g(\phi(e_4),e_5)=0.$$

Thus,

$$\phi(e_1)=-\xi_3\varepsilon_2\varepsilon_3e_2-\xi_5\varepsilon_3\varepsilon_5e_3,$$

$$\phi(e_2)=\xi_3\varepsilon_1\varepsilon_3e_1-\xi_5\varepsilon_4\varepsilon_5e_4,$$

$$\phi(e_3)=\xi_5\varepsilon_1\varepsilon_5e_1,$$

$$\phi(e_4)=\xi_5\varepsilon_2\varepsilon_5e_2,$$

$$\phi(e_5)=0.$$

Now the equation $\phi^2(e_3)=e_3-\eta(e_3)\xi$ and linear independence of basis elements imply

$$\xi_1\xi_3=0,\quad \xi_3\xi_4=0,\quad \xi_3\xi_5=0.$$

There are structures satisfying these properties. For example, the structure (ϕ,ξ,η,g), such that $\xi=e_5$, $\phi(e_1)=e_3$, $\phi(e_2)=e_4$, $\phi(e_3)=e_1$, $\phi(e_4)=e_2$ and the metric has signature $+,+,-,-,+$ is paracontact.

The algebra $\mathfrak{g}_3$: The nonzero brackets and nonzero covariant derivatives are as follows:

$$[e_1,e_2]=e_3,\quad [e_1,e_3]=e_4,\quad [e_1,e_4]=e_5,\quad [e_2,e_3]=e_5$$

$$\begin{array}{lll}
\nabla_{e_1}e_2=\frac{1}{2}e_3, & \nabla_{e_1}e_3=-\frac{1}{2}\varepsilon_2\varepsilon_3e_2+\frac{1}{2}e_4, & \nabla_{e_1}e_4=-\frac{1}{2}\varepsilon_3\varepsilon_4e_3+\frac{1}{2}e_5,\\
\nabla_{e_1}e_5=-\frac{1}{2}\varepsilon_4\varepsilon_5e_4, & \nabla_{e_2}e_1=-\frac{1}{2}e_3, & \nabla_{e_2}e_3=\frac{1}{2}\varepsilon_1\varepsilon_3e_1+\frac{1}{2}e_5,\\
\nabla_{e_2}e_5=-\frac{1}{2}\varepsilon_3\varepsilon_5e_3, & \nabla_{e_3}e_1=-\frac{1}{2}\varepsilon_2\varepsilon_3e_2-\frac{1}{2}e_4, & \nabla_{e_3}e_2=\frac{1}{2}\varepsilon_1\varepsilon_3e_1-\frac{1}{2}e_5\\
\nabla_{e_3}e_4=\frac{1}{2}\varepsilon_1\varepsilon_4e_1, & \nabla_{e_3}e_5=\frac{1}{2}\varepsilon_2\varepsilon_5e_2, & \nabla_{e_4}e_1=-\frac{1}{2}\varepsilon_3\varepsilon_4e_3-\frac{1}{2}e_5\\
\nabla_{e_4}e_3=\frac{1}{2}\varepsilon_1\varepsilon_4e_1, & \nabla_{e_4}e_5=\frac{1}{2}\varepsilon_1\varepsilon_5e_1, & \nabla_{e_5}e_1=-\frac{1}{2}\varepsilon_4\varepsilon_5e_4,\\
\nabla_{e_5}e_2=-\frac{1}{2}\varepsilon_3\varepsilon_5e_3, & \nabla_{e_5}e_3=\frac{1}{2}\varepsilon_2\varepsilon_5e_2, & \nabla_{e_5}e_4=\frac{1}{2}\varepsilon_1\varepsilon_5e_1.
\end{array}$$

- There is no α-para-Sasakian structure.
  The characteristic vector field of an α-para-Sasakian is Killing. Thus if (ϕ,ξ,η,g) is an α-para-Sasakian structure, $\xi = e_5$. Then
$$\phi(e_1) = \frac{1}{\alpha}\nabla_{e_1}e_5 = -\frac{1}{2\alpha}\varepsilon_4\varepsilon_5 e_4$$
  and the equation
$$(\nabla_{e_1}\phi)(e_1) = \alpha\{g(e_1,e_1)e_5 - \eta(e_1)e_1\}$$
  result in the contradiction
$$\frac{1}{4\alpha}\varepsilon_3\varepsilon_5 e_3 - \{\frac{1}{4\alpha}\varepsilon_4\varepsilon_5 - \alpha\varepsilon_1\}e_5 = 0.$$
- There is no β-para-Kenmotsu structure.
  Since ξ satisfies $g(\nabla_x\xi,y) = g(\nabla_y\xi,x)$, checking this condition for basis elements, we get that ξ is of the form $\xi = \xi_1 e_1 + \xi_2 e_2$. For $x = e_1$, the equation $\nabla_x\xi = \beta\{x - \eta(x)\xi\}$ implies $(1-\beta\varepsilon_1\xi_1^2)e_1 - \beta\varepsilon_1\xi_1\xi_2 e_2 - \frac{\xi_2}{2}e_3 = 0$. From linear independence, we have $\xi_2 = 0$ and so $\xi = \xi_1 e_1$. Now for $x = e_2$, we get $\beta e_2 + \frac{\xi_1}{2}e_3 = 0$, a contradiction.
- There are paracontact structures.
  By using the defining equation of an α-paracontact structure
$$\Phi(e_i,e_j) = g(\phi(e_i),e_j) = \alpha d\eta = \alpha\{g(\nabla_{e_i}\xi,e_j) - g(\nabla_{e_j}\xi,e_i)\},$$
  we write
$$\begin{aligned}\phi(e_1) &= -\alpha\{\xi_3\varepsilon_2\varepsilon_3 e_2 + \xi_4\varepsilon_3\varepsilon_4 e_3 + \xi_5\varepsilon_4\varepsilon_5 e_4\},\\ \phi(e_2) &= \alpha\{\xi_3\varepsilon_1\varepsilon_3 e_1 - \xi_5\varepsilon_3\varepsilon_5 e_3\},\\ \phi(e_3) &= \alpha\{\xi_4\varepsilon_1\varepsilon_4 e_1 + \xi_5\varepsilon_2\varepsilon_5 e_2\},\\ \phi(e_4) &= \alpha\xi_5\varepsilon_1\varepsilon_5 e_1,\\ \phi(e_5) &= 0.\end{aligned}$$
  In addition, the relation $0 = g(\phi(e_5),\phi(e_i)) = -g(e_5,e_i) + \eta(e_5)\eta(e_i)$ gives $\xi_1\xi_5 = \xi_2\xi_5 = \xi_3\xi_5 = \xi_4\xi_5 = 0$. We can find structures with these properties. For instance, (ϕ,ξ,η,g), where $\xi = e_5$, $\phi(e_1)e_4$, $\phi(e_2) = e_3$, $\phi(e_3) = e_2$, $\phi(e_4) = e_1$, $\phi(e_5) = 0$ and g has the signature $+,+,-,-,+$ is a paracontact structure.

The algebra $\mathfrak{g}_4$: The nonzero brackets and nonzero covariant derivatives are:
$$[e_1,e_2] = e_3, \quad [e_1,e_3] = e_4, \quad [e_1,e_4] = e_5$$
$$\begin{array}{lll}
\nabla_{e_1}e_2 = \frac{1}{2}e_3, & \nabla_{e_1}e_3 = -\frac{1}{2}\varepsilon_2\varepsilon_3 e_2 + \frac{1}{2}e_4, & \nabla_{e_1}e_4 = -\frac{1}{2}\varepsilon_3\varepsilon_4 e_3 + \frac{1}{2}e_5,\\
\nabla_{e_1}e_5 = -\frac{1}{2}\varepsilon_4\varepsilon_5 e_4, & \nabla_{e_2}e_1 = -\frac{1}{2}e_3, & \nabla_{e_2}e_3 = \frac{1}{2}\varepsilon_1\varepsilon_3 e_1,\\
\nabla_{e_3}e_1 = -\frac{1}{2}\varepsilon_2\varepsilon_3 e_2 - \frac{1}{2}e_4, & \nabla_{e_3}e_2 = \frac{1}{2}\varepsilon_1\varepsilon_3 e_1, & \nabla_{e_3}e_4 = \frac{1}{2}\varepsilon_1\varepsilon_4 e_1,\\
\nabla_{e_4}e_1 = -\frac{1}{2}\varepsilon_3\varepsilon_4 e_3 - \frac{1}{2}e_5 & \nabla_{e_4}e_3 = \frac{1}{2}\varepsilon_1\varepsilon_4 e_1, & \nabla_{e_4}e_5 = \frac{1}{2}\varepsilon_1\varepsilon_5 e_1,\\
\nabla_{e_5}e_1 = -\frac{1}{2}\varepsilon_4\varepsilon_5 e_4, & \nabla_{e_5}e_4 = \frac{1}{2}\varepsilon_1\varepsilon_5 e_1. &
\end{array}$$

- This algebra does not admit an α-para-Sasakian structure.
  Let (ϕ,ξ,η,g) be an α-para-Sasakian structure. Since ξ is Killing, we have $\xi = e_5$ in $\mathfrak{g}_4$. From the equation $\nabla_{e_2}\xi = \alpha\phi(e_2)$, we get $\phi(e_2) = 0$. On the other hand,
$$0 = g(\phi(e_2),\phi(e_2)) \neq -g(e_2,e_2) + \eta(e_2)\eta(e_2) = -\varepsilon_2.$$
- There exists no β-para-Kenmotsu structure.
  From the equation $g(\nabla_x\xi,y) = g(\nabla_y\xi,x)$, the Reeb vector field is obtained in the form $\xi = \xi_1 e_1 + \xi_2 e_2 + \xi_4 e_4 + \xi_5 e_5$. Since $\nabla_{e_3}\xi = \beta\phi^2(e_3)$, we have
$$\frac{1}{2}\varepsilon_1(\varepsilon_3\xi_2 + \varepsilon_4\xi_4)e_1 - \frac{1}{2}\varepsilon_2\varepsilon_3\xi_1 e_2 - \beta e_3 - \frac{\xi_1}{2}e_4 = 0.$$
  Since basis elements are linearly independent, there is no nonzero number β satisfying this equation.

- There is no paracontact structure. Since
$$\Phi(e_i,e_j)=g(\phi(e_i),e_j)=d\eta(e_i,e_j)=g(\nabla_{e_i}\xi,e_j)-g(\nabla_{e_j}\xi,e_i)$$
for a paracontact structure, we obtain $\phi(e_4)=\frac{\xi_5}{2}\varepsilon_1\varepsilon_5e_1$ and $\phi(e_5)=0$. On the other hand, $\phi^2(e_5)=e_5-\eta(e_5)\xi$ gives
$$\xi_1\xi_5\varepsilon_5e_1+\xi_2\xi_5\varepsilon_5e_2+\xi_3\xi_5\varepsilon_5e_3+\xi_4\xi_5\varepsilon_5e_4)+(\xi_5^2\varepsilon_5-1)e_5=0.$$
From linear independence of basis elements, we have
$$\xi_1\xi_5=\xi_2\xi_5=\xi_3\xi_5=\xi_4\xi_5=0,\quad \xi_5^2\varepsilon_5=1.$$
Since $\xi_5^2\neq 0$, we get $\xi_1=\xi_2=\xi_3=\xi_4$ and $\xi=\xi_5e_5$. Then, $0=\phi^2(e_4)\neq e_4-\eta(e_4)\xi=e_4$.

The algebra $\mathfrak{g}_5$:
$$[e_1,e_2]=e_4,\quad [e_1,e_3]=e_5$$

$$\begin{array}{lll}
\nabla_{e_1}e_2=\frac{1}{2}e_4, & \nabla_{e_1}e_3=\frac{1}{2}e_5, & \nabla_{e_1}e_4=-\frac{1}{2}\varepsilon_2\varepsilon_4e_2,\\
\nabla_{e_1}e_5=-\frac{1}{2}\varepsilon_3\varepsilon_5e_3, & \nabla_{e_2}e_1=-\frac{1}{2}e_4, & \nabla_{e_2}e_4=\frac{1}{2}\varepsilon_1\varepsilon_4e_1,\\
\nabla_{e_3}e_1=-\frac{1}{2}e_5, & \nabla_{e_3}e_5=\frac{1}{2}\varepsilon_1\varepsilon_5e_1, & \nabla_{e_4}e_1=-\frac{1}{2}\varepsilon_2\varepsilon_4e_2,\\
\nabla_{e_4}e_2=\frac{1}{2}\varepsilon_1\varepsilon_4e_1, & \nabla_{e_5}e_1=-\frac{1}{2}\varepsilon_3\varepsilon_5e_3\quad, & \nabla_{e_5}e_3=\frac{1}{2}\varepsilon_1\varepsilon_5e_1.
\end{array}$$

- There exists no α-para-Sasakian structure.
Let (ϕ,ξ,η,g) be an α-para-Sasakian structure. Since ξ is Killing, $\xi=\xi_4e_4+\xi_5e_5$. From the equation $\nabla_x\xi=\alpha\phi(x)$, we get $\phi(e_4)=\phi(e_5)=0$. In addition,
$$g(\phi(e_4),\phi(e_4))=-g(e_4,e_4)+\eta(e_4)\eta(e_4)$$
implies $0=-\varepsilon_4+\xi_4^2$. Thus, $\varepsilon_4=1$ and $\xi_4^2=1$. Similarly, $\xi_5^2=1$ and $\varepsilon_5=1$. However, in this case, $g(\xi,\xi)=\xi_4^2\varepsilon_4+\xi_5^2\varepsilon_5=2\neq 1$.
- There is no β-para-Kenmotsu structure.
The Reeb vector field ξ satisfies $g(\nabla_x\xi,y)=g(\nabla_y\xi,x)$. Checking for basis elements, ξ is obtained in the form $\xi=\xi_1e_1+\xi_2e_2+\xi_3e_3$. We also know that $\nabla_x\xi=\beta\phi^2(x)=\beta\{x-\eta(x)\xi\}$. For $x=e_4$, we have
$$\frac{\xi_2}{2}\varepsilon_1\varepsilon_4e_1-\frac{\xi_1}{2}\varepsilon_2\varepsilon_4e_2-\beta e_4=0.$$
Since basis elements are linearly independent, there is no nonzero number β satisfying this equation.
- There is no paracontact structure. Since
$$\Phi(e_i,e_j)=g(\phi(e_i),e_j)=d\eta(e_i,e_j)=g(\nabla_{e_i}\xi,e_j)-g(\nabla_{e_j}\xi,e_i)$$
for a paracontact structure, we obtain $\phi(e_4)=0$ and $\phi(e_5)=0$. On the other hand, $\phi^2(e_4)=e_4-\eta(e_4)\xi$ gives
$$\xi_1\xi_4\varepsilon_4e_1+\xi_2\xi_4\varepsilon_4e_2+\xi_3\xi_4\varepsilon_4e_3+(\xi_4^2\varepsilon_4-1)e_4)+\xi_5\xi_4\varepsilon_4e_5=0.$$
From linear independence of basis elements, we have
$$\xi_1\xi_4=\xi_2\xi_4=\xi_3\xi_4=\xi_5\xi_4=0,\quad \xi_4^2\varepsilon_4=1.$$
Since $\xi_4^2\neq 0$, we get $\xi_1=\xi_2=\xi_3=\xi_5$ and $\xi=\xi_4e_4$. In this case, $0=\phi^2(e_5)\neq e_5-\eta(e_5)\xi=e_5$.

The algebra $\mathfrak{g}_6$:
$$[e_1,e_2]=e_3,\quad [e_1,e_3]=e_4,\quad [e_2,e_3]=e_5$$

$$\begin{array}{lll}
\nabla_{e_1}e_2=\frac{1}{2}e_3, & \nabla_{e_1}e_3=-\frac{1}{2}\varepsilon_2\varepsilon_3e_2+\frac{1}{2}e_4, & \nabla_{e_1}e_4=-\frac{1}{2}\varepsilon_3\varepsilon_4e_3,\\
\nabla_{e_2}e_1=-\frac{1}{2}e_3, & \nabla_{e_2}e_3=\frac{1}{2}\varepsilon_1\varepsilon_3e_1+\frac{1}{2}e_5, & \nabla_{e_2}e_5=-\frac{1}{2}\varepsilon_3\varepsilon_5e_3,\\
\nabla_{e_3}e_1=-\frac{1}{2}\varepsilon_2\varepsilon_3e_2-\frac{1}{2}e_4, & \nabla_{e_3}e_2=\frac{1}{2}\varepsilon_1\varepsilon_3e_1-\frac{1}{2}e_5, & \nabla_{e_3}e_4=\frac{1}{2}\varepsilon_1\varepsilon_4e_1,\\
\nabla_{e_3}e_5=\frac{1}{2}\varepsilon_2\varepsilon_5e_2 & \nabla_{e_4}e_1=-\frac{1}{2}\varepsilon_3\varepsilon_4e_3, & \nabla_{e_4}e_3=\frac{1}{2}\varepsilon_1\varepsilon_4e_1,\\
\nabla_{e_5}e_2=-\frac{1}{2}\varepsilon_3\varepsilon_5e_3, & \nabla_{e_5}e_3=\frac{1}{2}\varepsilon_2\varepsilon_5e_2. &
\end{array}$$

- There exists no α-para-Sasakian structure.
  Since ξ is Killing, we have $\xi=\xi_4e_4+\xi_5e_5$. From the equation $\nabla_x\xi=\alpha\phi(x)$ implies $\phi(e_4)=\phi(e_5)=0$. In addition, $g(\phi(e_4),\phi(e_4))=-g(e_4,e_4)+\eta(e_4)\eta(e_4)$ yields $\varepsilon_4=1$ and $\xi_4^2=1$. Similarly we have $\varepsilon_5=1$ and $\xi_5^2=1$, which contradicts with $g(\xi,\xi)=1$.
- There is no β-para-Kenmotsu structure.
  The characteristic vector field of a β-para-Kenmotsu structure satisfies $g(\nabla_x\xi,y)=g(\nabla_y\xi,x)$. Then ξ should be of the form $\xi=\xi_1e_1+\xi_2e_2$. Now since $\nabla_{e_4}\xi=\beta\phi^2(e_4)=\beta\{e_4-\eta(e_4)\xi\}$, we have
  $$\frac{\xi_1}{2}\varepsilon_3\varepsilon_4e_3+\beta e_4=0,$$
  and there is no nonzero β with this property.
- There is no paracontact structure.
  Since
  $$\Phi(e_i,e_j)=g(\phi(e_i),e_j)=d\eta(e_i,e_j)=g(\nabla_{e_i}\xi,e_j)-g(\nabla_{e_j}\xi,e_i)$$
  for a paracontact structure, we obtain $\phi(e_4)=0$ and $\phi(e_5)=0$. On the other hand, $\phi^2(e_4)=e_4-\eta(e_4)\xi$ gives
  $$\xi_1\xi_4\varepsilon_4e_1+\xi_2\xi_4\varepsilon_4e_2+\xi_3\xi_4\varepsilon_4e_3+(\xi_4^2\varepsilon_4-1)e_4)+\xi_5\xi_4\varepsilon_4e_5=0.$$
  From linear independence of basis elements, we have
  $$\xi_1\xi_4=\xi_2\xi_4=\xi_3\xi_4=\xi_5\xi_4=0,\quad \xi_4^2=\varepsilon_4=1.$$
  Since $\xi_4^2\neq 0$, we get $\xi_1=\xi_2=\xi_3=\xi_5$ and $\xi=\xi_4e_4$. In this case, $0=\phi^2(e_5)\neq e_5-\eta(e_5)\xi=e_5$.

After all, we state followings.

Theorem 3.1. *An almost paracontact metric structure on a five dimensional nilpotent Lie algebra $\mathfrak{g}$ is para-cosymplectic if and only if $\mathfrak{g}$ is abelian.*

Thus we may state

Corollary 3.2. *There is no para-cosymplectic left invariant almost paracontact metric structure on a five dimensional connected Lie group whose corresponding Lie algebra is nilpotent.*

In addition we deduce followings.

Theorem 3.3. *There is no left-invariant nearly para-cosymplectic structure on a five dimensional nilpotent Lie group.*

Theorem 3.4. *A 5-dimensional nilpotent Lie algebra has an α-para-Sasakian structure if it is isomorphic to $\mathfrak{g}_1$.*

Corollary 3.5. *A five dimensional nilpotent Lie group has a left-invariant α-para-Sasakian structure if its Lie algebra is isomorphic to $\mathfrak{g}_1$.*

Theorem 3.6. *There exists no β-para-Kenmotsu structure on a five dimensional nilpotent Lie algebra.*

Corollary 3.7. *There is no left-invariant β-para-Kenmotsu structure on a five dimensional nilpotent Lie group.*

Theorem 3.8. *A 5-dimensional nilpotent Lie algebra has a paracontact structure if it is isomorphic to $\mathfrak{g}_1$, $\mathfrak{g}_2$ or $\mathfrak{g}_3$.*

Corollary 3.9. *A 5-dimensional nilpotent Lie group has a left invariant paracontact structure if its Lie algebra is isomorphic to $\mathfrak{g}_1$, $\mathfrak{g}_2$ or $\mathfrak{g}_3$.*

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