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Research Article

Chaotic behaviour of maps possessing the almost average shadowing property

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Abstract

In this paper, we investigate the chaotic behaviour of maps having the almost average shadowing property by obtaining the relationship of the almost average shadowing property with different kinds of chaos. Moreover, we relate the notion of almost average shadowing property with some other types of shadowing properties, for instance, ergodic shadowing, $\mathcal{F}_{\underline{d}}$-shadowing and $\underline{d}$-shadowing. We also study the notion of almost average shadowing property for maps induced on hyperspaces. Our study is supported by providing counter-examples wherever necessary.

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1. Introduction

Computer simulation of the trajectory of a point in discrete dynamical systems often produces pseudo-orbits. So to study the dynamical properties of the underlying dynamical system, it is important to determine the condition under which these numerically obtained pseudo-orbits can be traced by real trajectories of the system. Shadowing property makes it possible to approximate pseudo-orbits by real orbits of the system and this feature of shadowing property has made it an extremely significant and useful dynamical property in dynamical systems. There are several types of shadowing properties studied in the literature by now [8, 18, 31, 32]. Unlike the classical shadowing property, the notion of average shadowing property [5] and the notion of almost average shadowing property [6] deal with pseudo-orbits with average errors being very small whereas the notion of asymptotic average shadowing property [11] deals with pseudo-orbits with average errors tending to zero. These shadowing properties are also related to some useful dynamical properties [16, 17, 20, 22, 23, 29].

Another important concept in the theory of dynamical systems is chaos. The first mathematical definition of chaos was given by Li and Yorke in 1975 [25]. Since then, various types of chaos have been proposed and studied by many authors. The idea of chaos lies in the unpredictability of the behaviour of trajectories of the system. Each version of

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chaos represents complexity of the system in its own sense. Transitivity and sensitivity are two important ingredients of some variants of chaos. For some interesting results on sensitivity and chaos, kindly refer [1, 14, 19, 21, 24]. Another highly recognized ingredient of chaos is positivity of topological entropy. Topological entropy basically measures the chaotic behaviour of a map on a space. It is well known that for graph maps, positivity of topological entropy is equivalent to the existence of a Devaney chaotic subsystem [26].

A common approach to study the dynamics of the underlying system is to study the dynamics of its corresponding hyperspace. In 1975, Bauer et al. were the first to initiate the study of the connections between the dynamical properties of the base system and its induced system [3]. In recent times, there has been increasing interest in the study of the interplay of the dynamics of a discrete dynamical system (individual dynamics) with the corresponding hyperspace (collective dynamics). Many interesting dynamical properties of the original system are found to be inherited by the corresponding induced system and vice-versa which include weak mixing, topological mixing, shadowing, just to name a few [12, 30, 35]. In [30], it is shown that the weak mixing property of the base system is equivalent to the topological transitivity of the corresponding hyperspace. Also, in [13], authors have studied interplay of the variants of chaos between the base system and its induced system. Interestingly, in [15], author has introduced the notion of a general approximation property which unifies many types of shadowing and has also studied this property for induced maps on hyperspaces.

The notion of almost average shadowing property (ALASP) is defined in [6] and its relation with proximality and various kinds of transitivities is studied in [10]. In the present paper, we relate the notion of ALASP with various other notions in dynamical systems and study the notion of ALASP for maps induced on hyperspaces. The paper is organised as follows. Section 2 is devoted to some basic definitions and known results required for further sections. In Section 3, we prove that every contraction map has the ALASP, which gives us a large class of maps possessing the ALASP. In Section 4, we relate the notion of ALASP with various kinds of chaos. We focus mainly on topological chaos, Li-Yorke chaos, Auslander-Yorke chaos, Devaney chaos and P-chaos. In Section 5, we obtain relationships of the ALASP with different kinds of shadowing properties, for instance, ergodic shadowing, $\mathcal{F}_{\underline{d}}$-shadowing and $\underline{d}$-shadowing. Finally in Section 6, we find the relation between the ALASP of the base map on a space and that of its induced map on the corresponding hyperspace. The paper is also furnished with many interesting examples to gloss the study done.

2. Preliminaries

We denote the set of real numbers by $\mathbb{R}$, the set of positive integers by $\mathbb{N}$ and the set of non-negative integers by $\mathbb{Z}_+$. For us, a dynamical system is a pair (X, f), where X is a compact metric space with metric d and $f : X \to X$ is a continuous map. Note that for definitions given in this section we do not need compactness of the phase space, it is needed mainly for further sections. We say that a dynamical system (X, f) is non-trivial if X consists of more than one point. A dynamical system (Y, g) is a *factor* of a dynamical system (X, f) if there is a continuous surjection called *factor map* $\pi : X \to Y$ such that $\pi \circ f = g \circ \pi$. Let (X, f) be a dynamical system. The f*-orbit* of a point $x \in X$ is given by the set $\{f^n(x) : n \geq 0\}$ and is denoted by $O_f(x)$. The ω*-limit set* of a point $x \in X$ is the set of limit points of $O_f(x)$ and is denoted by $\omega_f(x)$.

For a subset A of $\mathbb{Z}_+$, we define the *upper density* of A by

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{1}{n} |A \cap \{0, 1, \dots, n-1\}|$$

and the *lower density* of A by

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{1}{n} |A \cap \{0, 1, \ldots, n-1\}|,$$

where $|C|$ denotes the cardinality of the set $C \subseteq \mathbb{Z}_+$. We say that A is *syndetic* if it has bounded gaps, that is, there exists $N \in \mathbb{N}$ such that $[n, n+N] \cap A \neq \emptyset$ for every $n \in \mathbb{Z}_+$ and A is *cofinite* if $\mathbb{Z}_+ \setminus A$ is finite.

For $\delta > 0$, a sequence $\{x_i\}_{i\geq 0}$ in X is called a δ-pseudo-orbit of f if $d(f(x_i), x_{i+1}) < \delta$ for every $i \geq 0$. We say that the map f has the *shadowing property* if for every $\epsilon > 0$, there is a $\delta > 0$ such that every δ-pseudo-orbit $\{x_i\}_{i\geq 0}$ of f is ϵ-shadowed by a point $z \in X$, that is, $d(f^i(z), x_i) < \epsilon$ for every $i \geq 0$. The map f has the *asymptotic average shadowing property* (AASP) [11] if every asymptotic average pseudo-orbit $\{x_i\}_{i\geq 0}$ of f, that is, $\{x_i\}_{i\geq 0}$ satisfying

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f(x_i), x_{i+1}) = 0,$$

is asymptotically shadowed in average by a point $z \in X$, that is,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z), x_i) = 0.$$

For $\delta > 0$, a sequence $\{x_i\}_{i\geq 0}$ in X is called an almost δ-average-pseudo-orbit of f if

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f(x_i), x_{i+1}) < \delta.$$

We say that the map f has the *almost average shadowing property* (ALASP) [6] if for every $\epsilon > 0$, there is a $\delta > 0$ such that every almost δ-average-pseudo-orbit $\{x_i\}_{i\geq 0}$ of f is ϵ-shadowed in average by a point $z \in X$, that is,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z), x_i) < \epsilon.$$

For $\delta > 0$, a sequence $\{x_i\}_{i\geq 0}$ in X is called a δ-average-pseudo-orbit of f if there is an integer $N = N(\delta) > 0$ such that for all $n \geq N$ and for all $k \geq 0$,

$$\frac{1}{n} \sum_{i=0}^{n-1} d(f(x_{i+k}), x_{i+k+1}) < \delta.$$

The map f is said to have the *average-shadowing property* (ASP) [5] if for every $\epsilon > 0$, there is a $\delta > 0$ such that every δ-average-pseudo-orbit $\{x_i\}_{i\geq 0}$ of f is ϵ-shadowed in average by a point $z \in X$, that is,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z), x_i) < \epsilon.$$

For $\delta > 0$, a sequence $\{x_i\}_{i\geq 0}$ in X is called a δ-ergodic pseudo-orbit of f if $\underline{d}(\{i \in \mathbb{Z}_+ : d(f(x_i), x_{i+1}) < \delta\}) = 1$. The map f has the *ergodic shadowing property* if for every $\epsilon > 0$, there is a $\delta > 0$ such that every δ-ergodic pseudo-orbit of f can be ϵ-ergodic shadowed by a point $z \in X$, that is, $\underline{d}(\{i \in \mathbb{Z}_+ : d(f^i(z), x_i) < \epsilon\}) = 1$ [9]. We say that the map f has the *$\underline{d}$-shadowing property* if for every $\epsilon > 0$, there is a $\delta > 0$ such that for every δ-ergodic pseudo-orbit $\{x_i\}_{i\geq 0}$ of f, there exists $z \in X$ such that $\underline{d}(\{i \in \mathbb{Z}_+ : d(f^i(z), x_i) < \epsilon\}) > 0$ [7]. The map f has the *$\mathcal{F}_{\underline{d}}$-shadowing property* if for every $\epsilon > 0$, there is a $\delta > 0$ such that for every δ-pseudo-orbit $\{x_i\}_{i\geq 0}$ of f, there exists $z \in X$ such that $\underline{d}(\{i \in \mathbb{Z}_+ : d(f^i(z), x_i) < \epsilon\}) > 0$ [29].

For non-empty subsets U, V of X, we define the *hitting time set* as $N(U, V) = \{k \in \mathbb{Z}_+ : f^k(U) \cap V \neq \emptyset\}$ and for $x \in X$ and a neighborhood U of x, we define the *visiting time set* as $N(x, U) = \{n \in \mathbb{Z}_+ : f^n(x) \in U\}$. We say that the map f is *transitive* if for any pair of non-empty open subsets U, V of X, the set $N(U, V)$ is non-empty and f is *totally transitive* if each iterate f^k, $k \in \mathbb{N}$, is transitive. We say that f is *weakly mixing* if $f \times f$ is transitive and f is *mixing* if for any pair of non-empty open subsets U, V of X, the set $N(U,V)$ is cofinite. The map f is *minimal* if every point of X has dense orbit, equivalently, if X does not admit any non-empty, proper, closed, f-invariant subset. By f-invariance of a subset A of X we mean $f(A) \subseteq A$.

For $\delta > 0$ and x, $y \in X$, a δ-*chain* of f from x to y of *length* $n \in \mathbb{N}$ is a finite sequence $x_0 = x, x_1, \ldots, x_n = y$ satisfying $d(f(x_i), x_{i+1}) < \delta$ for $0 \leq i \leq n-1$. We say that the map f is *chain transitive* if for any $\delta > 0$ and any pair of points x, $y \in X$, there is a δ-chain of f from x to y and f is *chain mixing* if for any $\delta > 0$ and any pair of points x, $y \in X$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, there is a δ-chain of f from x to y of length n. The map f is said to have the *specification property* if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \in \mathbb{N} \setminus \{1\}$, any finite sequence $y_1, y_2, \ldots, y_n \in X$ and any sequence $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_n \leq b_n$ of natural numbers with $a_{i+1} - b_i \geq N$ for $i = 1, 2, \ldots, n-1$, there is a point $z \in X$ such that $d(f^j(z), f^j(y_k)) < \epsilon$ for every k with $1 \leq k \leq n$ and every j with $a_k \leq j \leq b_k$.

The map f is said to be *equicontinuous* if for every $\epsilon > 0$, there is a $\delta > 0$ such that whenever x, $y \in X$ with $d(x,y) < \delta$ then $d(f^n(x), f^n(y)) < \epsilon$ for each $n \in \mathbb{N}$. A factor (Y,g) of (X,f) is said to be *maximal equicontinuous factor* of (X,f) if (Y,g) is equicontinuous and every equicontinuous factor of (X,f) is a factor of (Y,g). A point $x \in X$ is called a *sensitive point* of f if there exists $\epsilon > 0$ such that for any neighborhood U of x we have $\mathrm{diam}(f^n(U)) > \epsilon$ for some $n \in \mathbb{N}$, where $\mathrm{diam}(A)$ denotes the diameter of A. We denote the set of sensitive points of f by $Sen(f)$. We say that the map f is *sensitive* if $Sen(f) = X$ and a uniform $\epsilon > 0$ works for all $x \in X$. A point $x \in X$ is called a *periodic point* of f if $f^k(x) = x$ for some $k \in \mathbb{N}$ and it is called a *minimal point* of f if $N(x,U)$ is syndetic for every neighborhood U of x. We denote the set of periodic points of f by $P(f)$ and the set of minimal points of f by $M(f)$.

We recall Bowen's definition of topological entropy. For $n \in \mathbb{N}$ and $\epsilon > 0$, a subset $A \subseteq X$ is said to be an (n,ϵ)-separated set for f if for each pair a, b of distinct points in A, there is a $k \in \{0, 1, \ldots, n-1\}$ such that $d(f^k(a), f^k(b)) > \epsilon$. Let $s_n(f,\epsilon)$ denote the maximal cardinality of an (n,ϵ)-separated set for f. Then the *topological entropy* of f is defined as

$$h(f) = \lim_{\epsilon \to 0} \left[\limsup_{n \to \infty} \frac{\log s_n(f,\epsilon)}{n} \right].$$

We also recall the *asymptotic* and *proximal* relations on X which are given by $Asym(f) = \{(x,y) \in X \times X : \lim_{n\to\infty} d(f^n(x), f^n(y)) = 0\}$ and $Prox(f) = \{(x,y) \in X \times X : \liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0\}$ respectively. Elements in $Asym(f)$ and $Prox(f)$ are called asymptotic and proximal pairs of f respectively. A dynamical system (X,f) is *proximal* if $Prox(f) = X \times X$. We say that the map f is *distal* if $Prox(f) = \Delta_X$, where $\Delta_X = \{(x,x) : x \in X\}$ is the diagonal of $X \times X$ and f is called *Li-Yorke chaotic* if there is an uncountable set $\mathcal{U} \subseteq X$, called as scrambled set (for f), such that $(x,y) \in Prox(f) \setminus Asym(f)$ for any two distinct points x, $y \in \mathcal{U}$. We say that the map f is *densely Li-Yorke chaotic* if it admits a dense, uncountable scrambled set [14]. We say that the map f is *topologically chaotic* if $h(f) > 0$ and it is *Auslander-Yorke chaotic* if it is transitive and sensitive. Also, f is *Devaney chaotic* if it is transitive, sensitive and $P(f)$ is dense in X and it is *P-chaotic* if it has the shadowing property and $P(f)$ is dense in X [2].

3. Relating contraction with the ALASP

In the following result we prove that every contraction map has the ALASP.

Theorem 3.1. *Let (X,d) be a metric space. If $f : X \to X$ is a contraction map, then f has the ALASP.*

Proof. Since f is a contraction map, there exists $\alpha \in \mathbb{R}$, $0 < \alpha < 1$, such that $d(f(x), f(y)) \leq \alpha d(x, y)$ for all $x, y \in X$. Let $\epsilon > 0$ be a real number. Choose $\delta > 0$ such that $\delta/(1-\alpha) < \epsilon$. Let $\{x_i\}_{i\geq 0}$ be an almost δ-average-pseudo-orbit of f. Then

$$\limsup_{n\to\infty} \sum_{i=0}^{n-1} \beta_i < \delta, \tag{3.1}$$

where $\beta_i = d(f(x_i), x_{i+1})$ for every $i \geq 0$. Let $y_0 = x_0$ and $y_{i+1} = f(y_i)$, $i \geq 0$. Then $d(f(x_0), f(y_0)) = 0$. Consider,

$$\begin{aligned} d(f(x_1), f(y_1)) &\leq \alpha d(x_1, y_1) \\ &\leq \alpha(d(x_1, f(x_0)) + d(f(x_0), f(y_0))) \\ &= \alpha\beta_0. \end{aligned}$$

Also,

$$\begin{aligned} d(f(x_2), f(y_2)) &\leq \alpha d(x_2, y_2) \\ &\leq \alpha(d(x_2, f(x_1)) + d(f(x_1), f(y_1))) \\ &\leq \alpha\beta_1 + \alpha^2\beta_0. \end{aligned}$$

Therefore by induction we get

$$d(f(x_i), f(y_i)) \leq \alpha\beta_{i-1} + \alpha^2\beta_{i-2} + \alpha^3\beta_{i-3} + \cdots + \alpha^{i-1}\beta_1 + \alpha^i\beta_0$$

for all $i \geq 1$. We claim that y_0 is the required tracing point of $\{x_i\}_{i\geq 0}$. It is clear that $d(y_0, x_0) = 0$ and $d(f(y_0), x_1) = d(f(x_0), x_1) = \beta_0$. Also,

$$d(f^2(y_0), x_2) = d(f(y_1), x_2) \leq d(f(y_1), f(x_1)) + d(f(x_1), x_2) \leq \alpha\beta_0 + \beta_1.$$

Therefore by induction we have

$$d(f^i(y_0), x_i) \leq \alpha^{i-1}\beta_0 + \alpha^{i-2}\beta_1 + \cdots + \alpha\beta_{i-2} + \beta_{i-1}$$

for all $i \geq 1$. Thus for fixed $n \in \mathbb{N}$, we get

$$\begin{aligned} \frac{1}{n}\sum_{i=0}^{n-1} d(f^i(y_0), x_i) \leq \frac{1}{n}[&\beta_0(1 + \alpha + \alpha^2 + \cdots + \alpha^{n-2}) \\ &+ \beta_1(1 + \alpha + \alpha^2 + \cdots + \alpha^{n-3}) + \cdots + \beta_{n-3}(1+\alpha) + \beta_{n-2}]. \end{aligned}$$

This gives for all $n \in \mathbb{N}$,

$$\frac{1}{n}\sum_{i=0}^{n-1} d(f^i(y_0), x_i) \leq \frac{1}{1-\alpha}\frac{1}{n}\sum_{i=0}^{n-1}\beta_i.$$

Therefore using (3.1) we have

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} d(f^i(y_0), x_i) \leq \frac{1}{1-\alpha}\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\beta_i < \frac{1}{1-\alpha}\delta < \epsilon.$$

Thus f has the ALASP. □

Remark 3.2. It is clear that a map having ALASP need not be contracting, for instance, the usual tent map has the ALASP [10], however, it is not contracting.

Remark 3.3. One can verify that the notion of ALASP is preserved under topological conjugacy on compact spaces. Following example justifies that the condition of compactness can not be relaxed in this case.

Example 3.4. Let $X = \mathbb{N}$, $f : X \to X$ be defined as $f(x) = x + 1$, $x \in \mathbb{N}$. Also, let $Y = \{\frac{1}{2^n} : n \in \mathbb{N}\}$ and $g : Y \to Y$ be defined as $g(1/2^n) = 1/2^{n+1}$, $n \in \mathbb{N}$. Then (X, f) is topologically conjugate to (Y, g) with topological conjugacy $h : X \to Y$ given by $h(n) = 1/2^n$, $n \in \mathbb{N}$. Note that f being an isometry does not have the ALASP, however, g being contracting has the ALASP.

4. Variants of chaos and the ALASP

Note that the tent map $f_1(x) = 1-|1-2x|$, $x \in [0, 1]$, the logistic map $f_2(x) = 4x(1-x)$, $x \in [0, 1]$ and the doubling map $f_3(x) = (2x) \bmod 1$, $x \in [0, 1]$ are some examples of maps having the ALASP with positive topological entropy. Interestingly, we have the following:

Theorem 4.1. *Let (X, f) be a non-trivial dynamical system. If f is a surjection having the ALASP and the shadowing property, then it is topologically chaotic.*

Proof. Since f is a surjection having the ALASP and the shadowing property, we have $f \times f$ is a surjection having the ALASP and the shadowing property [6], which gives $\overline{M(f \times f)} = X \times X$ [10]. Also, f being a surjection having the ALASP, it can not be equicontinuous implying that $Sen(f) \neq \emptyset$. Let $z \in Sen(f)$. Then there exists $\epsilon > 0$ such that for any neighborhood $U \subseteq X$ of z, there is a $k \in \mathbb{N}$ such that $\mathrm{diam}(f^k(U)) > 2\epsilon$. Suppose δ with $0 < \delta < \epsilon/2$ is obtained corresponding to $\epsilon/2$ by the shadowing property of f. Let $U \subseteq X$ be a neighborhood of z with $\mathrm{diam}(U) < \delta/2$. Then $\mathrm{diam}(f^k(U)) > 2\epsilon$. Since $\overline{M(f \times f)} = X \times X$ and U is open, using continuity of f, we can choose $(a, b) \in M(f \times f) \cap (U \times U)$ such that $d(f^k(a), f^k(b)) > 2\epsilon$. Since $N((a, b), (B(a, \delta/2) \times B(b, \delta/2)))$ (under $f \times f$) is syndetic, choose an integer $p > k$ such that $(f \times f)^p(a, b) \in B(a, \delta/2) \times B(b, \delta/2)$, which gives $d(a, f^p(a)) < \delta/2$ and $d(b, f^p(b)) < \delta/2$.

Let $Z = \{a, f(a), f^2(a), \ldots, f^{p-1}(a)\}$ and $Z^* = \{b, f(b), f^2(b), \ldots, f^{p-1}(b)\}$ with placing of elements fixed. For $n \in \mathbb{N}$, take $\{w_i\}_{0\leq i\leq np-1} = Z_1Z_2 \ldots Z_n$, where $Z_j \in \{Z, Z^*\}$, $j = 1, 2, \ldots, n$ (for example, if $Z_j = Z$ for every j, then $\{w_i\} = \{a, f(a), f^2(a), \ldots, f^{p-1}(a), a, f(a), f^2(a), \ldots, f^{p-1}(a), \ldots\}$). Since $d(x, f^p(y)) < \delta$ for $x, y \in \{a, b\}$, we have $\{w_i\}_{0\leq i\leq np-1}$ is a finite δ-pseudo-orbit of f. Therefore there exists $w \in X$ such that $d(f^i(w), w_i) < \epsilon/2$ for all $0 \leq i \leq np-1$. Let $\{c_j\}, \{d_j\} \in Z_1Z_2 \ldots Z_n$ be two distinct δ-pseudo-orbits of f consisting of np elements with $\epsilon/2$-tracing points c, d respectively. Since $d(f^k(a), f^k(b)) > 2\epsilon$ and $p > k$, there exists j with $0 \leq j \leq np - 1$ such that $d(c_j, d_j) > 2\epsilon$. This gives $2\epsilon < d(c_j, d_j) \leq d(c_j, f^j(c)) + d(f^j(c), f^j(d)) + d(f^j(d), d_j) < \epsilon/2 + d(f^j(c), f^j(d)) + \epsilon/2$ so that $d(f^j(c), f^j(d)) > \epsilon$. Therefore the set $A = \{z : z$ is an $\epsilon/2$-tracing point of some $\{z_i\} \in Z_1Z_2 \ldots Z_n\}$ is (np, ϵ)-separated and hence (np, η)-separated for every $\eta \in (0, \epsilon)$. It is clear that $|A| = 2^n$, which gives $s_{np}(f, \eta) \geq 2^n$ for every $\eta \in (0, \epsilon)$ and every $n \in \mathbb{N}$. Thus $h(f) \geq (\log 2)/p > 0$ and hence f is topologically chaotic. □

Following example justifies that the condition of ALASP is necessary in Theorem 4.1.

Example 4.2. Consider $X = \{a, b\}$ with discrete metric and f as the cycle permutation on X given by $f(a) = b$, $f(b) = a$. It is shown in [6] that f does not have the ALASP. Also, it is clear that f has the shadowing property (every $0 < \delta < 1$ will work). Moreover, f being an isometry, $h(f) = 0$.

It is known that the AASP implies the ALASP [34, Theorem 4.3]. So using [17, Corollary 3.12], we have the following remark:

Remark 4.3. If (X, f) is a dynamical system and there is a point $y \in X$ such that $\omega_f(x) = \{y\}$ for every $x \in X$, then f has the ALASP.

Next example justifies that the condition of shadowing property is necessary in Theorem 4.1.

Example 4.4. Consider the unit circle S^1 and the map $f : S^1 \to S^1$ defined as $f(e^{2\pi\iota t}) = e^{2\pi\iota t^2}$, $t \in [0,1)$. By Remark 4.3, f has the ALASP, indeed $\omega_f(x) = \{(1,0)\}$ for every $x \in S^1$. Since f is not mixing, it does not have the shadowing property [10]. Also, on S^1 the positivity of topological entropy of a map is equivalent to that of being it Devaney chaotic, which gives $h(f) = 0$ since f is not transitive.

In general, topological chaos need not imply the ALASP as exhibited by the following example.

Example 4.5. Let Y be a proximal subshift of $(\{0,1\}^{\mathbb{Z}}, \sigma)$ having positive topological entropy and unique fixed point 0^∞. One can refer [28] for the construction of such subshift. Let $A = \{(\dots x_{-2}x_{-1}x_0x_1x_2\dots) \in \{0,1\}^{\mathbb{Z}} :$ there exists $y \in Y$ with $x_{2i} = y_i$ and $x_{2i+1} = 0\ \forall\ i \in \mathbb{Z}\}$. Then $X = A \cup \sigma(A)$ is also a proximal subshift having positive topological entropy and a fixed point 0^∞. However, $\sigma|_X$ does not have the ALASP since it does not have the ASP, by [29, Example 20].

Observe that using Theorem 4.1 and [4, Corollary 2.4], we get that if (X, f) is a non-trivial dynamical system and f is a surjection having the ALASP and the shadowing property, then f is Li-Yorke chaotic. Moreover, we have the following result:

Theorem 4.6. *Let (X, f) be a dynamical system and f be minimal having the ALASP.*
(a) *If X is infinite, then f is densely Li-Yorke chaotic.*
(b) *If X is non-trivial, then f is Auslander-Yorke chaotic.*

Proof. (a) Since the ALASP implies the ASP, (X, f) has only trivial maximal equicontinuous factor, by [16, Theorem 6.1]. Using [14, Proposition A.4] and the minimality of f, we get that f is weakly mixing. Thus the result follows by [14, Corollary 3.6].

(b) It is well known that a minimal map is either equicontinuous or sensitive [1]. Since f is surjective and has the ALASP, it can not be equicontinuous (using the fact that a surjective equicontinuous map does not have the ALASP [10]). Thus f is Auslander-Yorke chaotic. □

The condition of minimality is necessary in Theorem 4.6 (a) as shown by the following example.

Example 4.7. Let $Y = \{(1 - \frac{1}{n}) : n \in \mathbb{N}\}$ and $X = Y \cup \{1\}$. Then X is clearly a compact metric space with usual metric of $\mathbb{R}$. Define $f : X \to X$ as

$$f(x) = \begin{cases} x^+ & \text{if } x \in Y, \\ x & \text{if } x = 1, \end{cases}$$

where x^+ denotes the element of X which is immediate to the right of x. It is clear that $\omega_f(x) = \{1\}$ for every $x \in X$ so that f has the ALASP, by Remark 4.3. Also, f is neither minimal nor Li-Yorke chaotic.

The condition of ALASP is necessary in Theorem 4.6 (a) is justified by following two examples.

Example 4.8. Consider the Morse shift defined as follows. First define the sequence of *Morse blocks* inductively as $M_0 = 0$, $M_n = M_{n-1}M^c_{n-1}$ for all $n \geq 1$, where M^c_n is the complement of M_n, obtained by interchanging the 1's and 0's in M_n. The *Morse sequence* $m \in \Sigma_2 = \{0,1\}^{\mathbb{Z}_+}$ is the limit of the sequence of Morse blocks, that is, $m = \lim_{n\to\infty} M_n$, so the first few terms of the Morse sequence are

$$m = 0110100110010110100101100110100 1\dots.$$

Let σ be the one-sided shift map on Σ_2. Then the sequence m generates the infinite *Morse set* $M = \overline{O_\sigma(m)}$, closure of $O_\sigma(m)$, and $(M, \sigma|_M)$ is called the *Morse system*, which is minimal but not weakly mixing [33] and hence does not have the ALASP. Since

$Asym(\sigma|_M) = Prox(\sigma|_M)$, the Morse shift has no Li-Yorke pairs which implies that it is not Li-Yorke chaotic.

Example 4.9. Consider an odometer defined as follows. Let $s = (s_j)_{j=1}^{\infty}$ be a sequence of positive integers which is strictly increasing and satisfies $s_1 \geq 2$, s_j divides s_{j+1}. Take $X(j) = \{0, 1, \dots, s_j - 1\}$ and $X_s = \{x \in \Pi_{j=1}^{\infty} X(j) : x_{j+1} \equiv (x_j) \bmod s_j\}$. Define $f : X_s \to X_s$ as $f((x_j)) = (y_j)$, where $y_j = (x_j + 1) \bmod s_j$, $j = 1, 2, \dots$. Then the dynamical system (X_s, f) is called an *odometer* (or an *adding machine*) defined by s. It is clear that f is minimal and equicontinuous which gives that it does not have the ALASP since a surjective equicontinuous map does not have the ALASP [10]. Also, f being distal, is not Li-Yorke chaotic.

The condition of minimality is necessary in Theorem 4.6 (b) as shown by the following example.

Example 4.10. Consider the space and the map as given in Example 4.4. Here f has the ALASP. However, f is neither minimal nor Auslander-Yorke chaotic since f is not transitive.

The condition of ALASP is necessary in Theorem 4.6 (b) as exhibited by the following example.

Example 4.11. Let S^1 denote the unit circle, identified with $\mathbb{R}/\mathbb{Z}$, or with the interval $[0, 1)$, by the function $e^{2\pi\iota x}$ and f_α be the rotation by an irrational number $\alpha \in (0, 1)$ given by $f_\alpha : S^1 \to S^1$, $f_\alpha(x) = (x + \alpha) \bmod 1$. The map f_α being minimal is transitive but not sensitive which implies that it is not Auslander-Yorke chaotic. Also, f_α being distal does not have the ALASP [10].

In general, the Auslander-Yorke chaos need not imply the ALASP as justified by the following example.

Example 4.12. We consider the Sturmian system defined as follows. First consider the irrational rotation on the unit circle S^1 defined as $f_\alpha : S^1 \to S^1$, $f_\alpha(x) = (x + \alpha) \bmod 1$, where $\alpha \in (0, 1)$ is an irrational number. Take $I_0 = [0, 1 - \alpha)$ and $I_1 = [1 - \alpha, 1)$ and consider the *coding function* ν given by $\nu(x) = 0$ if $x \in I_0$, $\nu(x) = 1$ if $x \in I_1$. Then the coding of the orbit of $t \in [0, 1)$ under the rotation f_α, that is, $\nu(f_\alpha^n(t))$, provides the rotation sequence which is a *Sturmian sequence*. We call $X(\alpha)$, the closure of the set of all such bi-infinite sequences and σ be the shift map on $\Sigma_2 = \{0, 1\}^{\mathbb{Z}}$. Then the system $(X(\alpha), \sigma|_{X(\alpha)})$ is called a *Sturmian system*. It is clear that the Sturmian shift is minimal but not weakly mixing which gives that it does not have the ALASP, by Theorem 4.6 (a). Also, it is transitive and sensitive giving that it is Auslander-Yorke chaotic.

Remark 4.13. Note that in Example 4.4, f is not Devaney chaotic. Therefore in general, the ALASP need not imply Devaney chaos. Moreover, observe that if (X, f) is an infinite dynamical system, where f is a surjection having the ALASP and the shadowing property, then f being having the specification property [10], is Devaney chaotic.

A *continuum* is a non-trivial compact connected metric space.

Remark 4.14. Observe that if (X, f) is a dynamical system, where X is a continuum and f is P-chaotic, then f has the ALASP (using the result that a P-chaotic map on a continuum is mixing, by [2, Corollary 3.4] and the fact that the shadowing property and mixing together imply the ALASP [10]).

In general, P-chaos need not imply the ALASP which follows by the following example.

Example 4.15. Consider the Cantor set $X = \{0, 1\}^{\mathbb{N}}$ with metric $d(x, y) = \inf\{2^{-m} : x_i = y_i \text{ for all } i < m\}$ for every $x = (x_1, x_2, \dots)$, $y = (y_1, y_2, \dots) \in X$. Then the identity map I_X on X has the shadowing property and $P(I_X) = X$. However, I_X being distal does not have the ALASP [10].

5. The ALASP and variants of shadowing

In this section, we relate the notion of ALASP with ergodic shadowing, $\mathcal{F}_{\underline{d}}$-shadowing and $\underline{d}$-shadowing.

Theorem 5.1. *Let (X, f) be a dynamical system. If f has the ergodic shadowing property, then it has the ALASP. The converse holds if f is a surjection having the shadowing property.*

Proof. Since f has the ergodic shadowing property, it has the shadowing property and it is mixing implying that f has the ALASP [10]. Conversely, under the above hypothesis, we have f is mixing [10]. So using the shadowing property of f, we get that f has the ergodic shadowing property [9]. □

Remark 5.2. Observe that the Example 4.4 justifies that in general, the ALASP need not imply the ergodic shadowing property.

Theorem 5.3. *Let (X, f) be a dynamical system. If f has the ALASP, then it has $\mathcal{F}_{\underline{d}}$-shadowing property.*

Proof. Let $\epsilon > 0$ be a real number and $\delta > 0$ be obtained for $\epsilon/2$ by the ALASP of f. Suppose $\{x_i\}_{i\geq 0}$ is a δ-pseudo-orbit of f, then clearly it is also an almost δ-average-pseudo-orbit of f so that there exists $z \in X$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z), x_i) < \epsilon/2.$$

Let $\Lambda = \{i \in \mathbb{Z}_+ : d(f^i(z), x_i) < \epsilon\}$. Then

$$\begin{aligned} \epsilon/2 &> \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z), x_i) \\ &\geq \limsup_{n\to\infty} \frac{\epsilon}{n}(n - |\Lambda \cap \{0, 1, \dots, n-1\}|) \\ &= \epsilon - \epsilon \underline{d}(\Lambda). \end{aligned}$$

This gives $\underline{d}(\Lambda) > 1/2$. Thus f has the $\mathcal{F}_{\underline{d}}$-shadowing property. □

One can observe using [7, Lemma 4.1] that if f is chain mixing and has the $\mathcal{F}_{\underline{d}}$-shadowing property, then it has the $\underline{d}$-shadowing property. Using this we have the following:

Corollary 5.4. *Let (X, f) be a dynamical system. If f is surjective and has the ALASP, then it has the $\underline{d}$-shadowing property.*

Proof. Since f is surjective and has the ALASP, it is chain mixing [6]. Thus the result follows by Theorem 5.3. □

Remark 5.5. Note that the Example 4.15 justifies that in general, the $\mathcal{F}_{\underline{d}}$-shadowing property need not imply the ALASP.

6. The ALASP on hyperspaces

Let (X, f) be a dynamical system and $\mathscr{K}(X)$ denotes the collection of all non-empty compact subsets of X. The map f induces the map $\bar{f} : \mathscr{K}(X) \to \mathscr{K}(X)$ defined as $\bar{f}(K) = f(K) = \{f(x) : x \in K\}$ for every $K \in \mathscr{K}(X)$. Consider the Hausdorff metric H_d on $\mathscr{K}(X)$ given by

$$H_d(K_1, K_2) = \max\{\sup\{d(x_1, K_2) : x_1 \in K_1\}, \sup\{d(K_1, x_2) : x_2 \in K_2\}\}$$

for all K_1, $K_2 \in \mathscr{K}(X)$. So (X, f) induces the dynamical system $(\mathscr{K}(X), \bar{f})$ with Hausdorff metric. Note that the system (X, f) is a subsystem of the induced system $(\mathscr{K}(X), \bar{f})$, where each point $x \in X$ is identified as a subset $\{x\} \in \mathscr{K}(X)$ [27].

The following result shows that the ALASP of the induced map implies the ALASP of the base map.

Theorem 6.1. *Let (X, f) be a dynamical system. If $\bar{f}$ has the ALASP, then so does f.*

Proof. Let $\epsilon > 0$ and $\delta > 0$ be obtained corresponding to ϵ by the ALASP of $\bar{f}$. Let $\{x_i\}_{i\geq 0}$ be an almost δ-average-pseudo-orbit of f, that is,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f(x_i), x_{i+1}) < \delta.$$

Since $H_d(\{\bar{f}(\{x_i\})\}, \{x_{i+1}\}) = d(f(x_i), x_{i+1})$ for every $i \geq 0$, we have

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} H_d(\{\bar{f}(\{x_i\})\}, \{x_{i+1}\}) < \delta.$$

Therefore $\{\{x_i\}\}_{i\geq 0}$ is an almost δ-average-pseudo-orbit of $\bar{f}$ so that there exists $Y \in \mathscr{K}(X)$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} H_d((\bar{f})^i(Y), \{x_i\}) < \epsilon.$$

Let $y_0 \in Y$. Then one can obtain that $H_d((\bar{f})^i(Y), \{x_i\}) \geq d(f^i(y_0), x_i)$ for every $i \geq 0$, which gives

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(y_0), x_i) < \epsilon.$$

Thus f has the ALASP. □

For the converse, we have the following result:

Theorem 6.2. *Let (X, f) be a dynamical system. Suppose that for every $\epsilon > 0$, there exists $\eta > 0$ such that if for every $j \geq 0$, a sequence $\{a_{i,j}\}_{i\geq 0} \subseteq X$ is η-shadowed in average by some $c_j \in X$, then $\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \sup_{j\geq 0} d(f^i(c_j), a_{i,j}) < \epsilon$. If f has the ALASP, then $\bar{f}$ has the ALASP.*

Proof. Let $\epsilon > 0$ and $\eta > 0$ be obtained corresponding to ϵ by the assumption in the hypothesis of the Theorem. Suppose $\delta > 0$ is provided corresponding to η by the ALASP of f. Let $\{Y_i\}_{i\geq 0}$ be an almost δ-average-pseudo-orbit of $\bar{f}$, that is,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} H_d(\bar{f}(Y_i), Y_{i+1}) < \delta. \tag{6.1}$$

Since each Y_i, $i \geq 0$, is compact, fixing $i \geq 0$ and $y_i \in Y_i$, we can obtain $y_{i+1} \in Y_{i+1}$ such that

$$H_d(\bar{f}(Y_i), Y_{i+1}) \geq d(f(y_i), y_{i+1}). \tag{6.2}$$

Each Y_k, $k \geq 0$, being a compact metric space is separable. Therefore each Y_k, $k \geq 0$, has a countable dense subset, say $\{x_k^l : l \geq 0\}$. Using (6.1) and (6.2), for each x_k^l, k, $l \geq 0$, we can form an almost δ-average-pseudo-orbit of f, say $\{z_{kr}^l : r \geq 0\}$, with $z_{kk}^l = x_k^l$. This way we get countably many almost δ-average-pseudo-orbits of f and we can rearrange these countable union of countably many elements into a sequence $\{\{y_i^j\}_{i\geq 0}\}_{j\geq 0}$ such that

$\overline{\{y_i^j : j \geq 0\}} = Y_i$ for every $i \geq 0$ and $\{y_i^j : i \geq 0\}$ is an almost δ-average-pseudo-orbit of f for every $j \geq 0$.

$$\begin{matrix} y_0^0 & y_0^1 & y_0^2 & y_0^3 & y_0^4 \cdots \\ y_1^0 & y_1^1 & y_1^2 & y_1^3 & y_1^4 \cdots \\ y_2^0 & y_2^1 & y_2^2 & y_2^3 & y_2^4 \cdots \\ y_3^0 & y_3^1 & y_3^2 & y_3^3 & y_3^4 \cdots \\ & & \cdots & & \end{matrix}$$

(Note that in the above table, for $i \geq 0$, ith row is dense in Y_i and each column is an almost δ-average-pseudo-orbit of f). Therefore for every $j \geq 0$, there exists $z_j \in X$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z_j), y_i^j) < \eta. \tag{6.3}$$

Let $Z = \overline{\{z_j : j \geq 0\}}$. Then each f^i, $i \geq 1$, being a closed map, $\overline{f^i(Z)} = f^i(Z)$ for every $i \geq 0$. One can observe that if $A = \overline{\{a_n : n \geq 1\}}$ and $B = \overline{\{b_n : n \geq 1\}}$, then $H_d(A, B) \leq \sup_{n\geq 1} d(a_n, b_n)$. This gives $H_d((\bar{f})^i(Z), Y_i) \leq \sup_{j\geq 0} d(f^i(z_j), y_i^j)$ for every $i \geq 0$. Using (6.3) and the condition in the hypothesis of the theorem we get

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \sup_{j\geq 0} d(f^i(z_j), y_i^j) < \epsilon.$$

This in turn gives

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} H_d((\bar{f})^i(Z), Y_i) < \epsilon.$$

Thus $\{Y_i\}_{i\geq 0}$ is ϵ-shadowed in average by $Z \in \mathscr{K}(X)$ and hence $\bar{f}$ has the ALASP. □

Example 6.3. It is clear that if f is a constant map, then $\bar{f}$ is also a constant map and hence both have the ALASP.

Example 6.4. Note that all the maps f_1, f_2, f_3, $\bar{f}_1$, $\bar{f}_2$ and $\bar{f}_3$ given in the beginning of Section 4 are topologically mixing and have the shadowing property. Thus they have the ALASP [10].

Example 6.5. In Example 4.11, f_α being an isometry does not have the ALASP [10] and hence by Theorem 6.1, $\bar{f}_\alpha$ does not have the ALASP.

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