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Asymptotics of Singularly Perturbed Volterra Type Integro-Differential Equation

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Abstract

This paper addresses the asymptotic behaviors of a linear Volterra type integro-differential equation. We study a singular Volterra integro equation in the limiting case of a small parameter with proper choices of the unknown functions in the equation. We show the effectiveness of the asymptotic perturbation expansions with an instructive model equation by the methods in superasymptotics. The methods used in this study are also valid to solve some other Volterra type integral equations including linear Volterra integro-differential equations, fractional integro-differential equations, and system of singular Volterra integral equations involving small (or large) parameters.

Keywords: *Singular perturbation, Volterra integro-differential equations, asymptotic analysis, singularity*
2010 Mathematics Subject Classification: *41A60; 45M05; 34E15*

1. Introduction

A systematic approach to approximation theory can find in the subject of asymptotic analysis which deals with the study of problems in the appropriate limiting regimes. Approximations of some solutions of differential equations, usually containing a small parameter ε, are essential in the analysis. The subject has made tremendous growth in recent years and has a vast literature. Developed techniques of asymptotics are successfully applied to problems including the classical long-standing problems in mathematics, physics, fluid mechanics, astrodynamics, engineering and many diverse fields, for instance, see [1, 2, 3, 4, 5, 6, 7]. There exists a plethora of examples of asymptotics which includes a wide variety of problems in rich results from the very practical to the highly theoretical analysis. It is well known that there are many problems which are not amenable to exact solutions. Thus, the only recourse is to develop methods within the concept of asymptotic analysis to approximate their asymptotic behaviors and extract as much information as possible from such problems that are not exactly solvable. Sometimes obtaining exact solutions of such equations may be impracticable. Numerically accurate approximate solutions are often attainable even with the first couple of terms with the techniques in asymptotics. One may obtain numerical solutions to specific problems which may be analytically achievable. Therefore, as it is sufficient in some cases, only a few terms of the asymptotic expansions are sought, and usually, no attempt is performed to generate all perturbative terms of the asymptotic expansion. To remark that, in addition to providing numerically good accuracies in limiting cases, asymptotic analysis is also important in describing the asymptotic behaviors of the problems hidden inside that are invisible to classical asymptotics. Since approximations are widely used, especially in the last decades, these hidden features must be included in computing the solutions of the equations as they contain a wealth of information. Good use of the powerful techniques of asymptotics accurately provides meaningful results.

Asymptotics is a powerful tool in addressing the problems in applied mathematics and theoretical physics. It composes a large class of approximations to the differential equations and integrals. Approximate solutions of mathematical problems may be derived in various ways. When the methods are misapplied or unable display the hidden features, it may lead to erroneous conclusions, especially when the Stokes phenomenon is involved. Likewise, true asymptotic behaviors and common features of most problems can be represented by the methods in asymptotic analysis. Stirling [8] (also see [9]) who gave an asymptotic representation of the factorial (gamma) function, Euler [10], MacLaurin [11], and Stieltjes [12] were among the firsts who used asymptotics series. However, the first rigorous foundation of the asymptotic analysis and their interpretation was established and demonstrated by Poincaré in 1886 [13]. He suggested representing the solution of a differential equation as a series in powers of a small parameter. The practical success of his technique depends on the convergence of the power series representations; when his approach works, it leads to good approximations. However, when the series representation is not convergent, his technique, unfortunately, may fail to generate a good approximation. In particular, random truncation of a divergent series any earlier or later than the optimal truncation point may increase the truncation error due to the additional terms. Therefore, it is notoriously inadequate with such series. However, in many physical models, the first few terms of the series expansion often

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provide satisfactory outcomes to the functions. Thence, his technique is still useful. Later, Dingle [14] observed that the Poincaré's definition was not unique and discovered that later order terms of the expansion occur in the same standard form, particularly, factorial divided by power. Berry [15] is the first person truncating the divergent series optimally (near the least term). Upon truncation of the divergent series optimally and ignoring the rest of the terms, he remarkably achieved the smallest error with an exponential accuracy and named this as superasymptotics [15]. Later, hyperasymptotics which deals with repeated expansion(s) of the truncated remainder(s) is introduced in [16] and exponentially small terms are extracted from these. Analysis of the exponentially small terms which may change their behaviors as certain lines are crossed via the growing subject of asymptotics has been studied by many. For further details regarding these, we refer the reader to see, for example, [1, 17, 18, 19, 20, 21, 22] and references therein.

In this paper, we consider a simple mathematical problem of a first-order linear ordinary Volterra type integro-differential equation whose behavior is to be determined asymptotically in the limit $\varepsilon \to 0$, where small parameter ε is the coefficient of the first derivative term. It is used to illustrate some of the principles of asymptotology and to provide a cautionary example of some of the pitfalls. The type of problems arises in many scientific and engineering problems. Deriving the exact solutions of the type of problems sometimes encounters considerable difficulty. In such cases, asymptotic approximation techniques are essential. It is precisely this class of problems that is amenable to asymptotic methods, for example, superasymptotics that will be discussed later. The outline of the paper is organized as follows. Section 2 introduces the type of the Volterra integro-differential equation which will be taken into consideration in the paper with arbitrarily selected functions it involves. Asymptotic approximations usually involve power series. Hence, Section 3 is devoted to the expansion of the model problem including the formulation of the asymptotic expansions in this sense. In Section 4, an asymptotic representation of the solution of the equation is truncated at its least. Section 5 investigates the remainder differential equation and shows that it is exponentially small. Finally, Section 6 discusses the conclusion.

The objective of the present study is to illustrate the asymptotic behavior of solutions of the Volterra type integro-differential equation with the powerful tool of asymptotics. The form of the equation investigated in this paper is addressed in [23, 24]. Moreover, some asymptotic properties of the Volterra integro-differential equation can be found in, for example, [25, 26, 27, 28]. In this paper, we are interested in the behavior of following singularly perturbed linear Volterra type integro-differential equations for small ε in an asymptotic sense

$$\varepsilon \frac{dy(z,\varepsilon)}{dz} + \int_0^z k(z,t,\varepsilon,y(t,\varepsilon))dt = f(z,\varepsilon), \tag{1.1}$$

where $0 < \varepsilon \ll 1$ is a perturbation parameter, and it gives rise to the singular nature of the equation. The asymptotic behavior can be obtained by assuming when the first term $\varepsilon \frac{dy(z,\varepsilon)}{dz}$ of the equation (1.1) is not negligible. Therefore, we will be mainly concerned with the asymptotic behavior of (1.1) as the perturbation parameter ε approaches to zero.

2. Model Problem

Herein, we briefly address the integro type singular differential equation with an expository example. To do this, we first assume the unknown functions of the equation (1.1) as

$$k(z,t,\varepsilon,y(t,\varepsilon)) = ty(t,\varepsilon) \quad \text{and} \quad f(z,\varepsilon) = z.$$

These pose that a model Volterra integro-differential equation whose asymptotic properties we investigate in the next sections is

$$\varepsilon \frac{dy(z,\varepsilon)}{dz} + \int_0^z ty(t,\varepsilon)dt = z. \tag{2.1}$$

Our main approaches are to generate approximation to the solution $y(z,\varepsilon)$ of integro-differential equation by expansion and to address its asymptotic behavior in the limiting regimes. In the following section, we expand the solution of the model equation (2.1) by the power series representation in ε in the sense of Poincaré. This suggests using the truncated series to approximately compute the unknown quantity $y(z,\varepsilon)$ of the integro-differential equation.

3. The Asymptotic Expansion for the Solution

Since asymptotic approximations usually involve power series representations, we investigate the integro-differential equation with a usual perturbation technique. We will assume the asymptotic series solution of the integro-differential equation exists in the small power series. Once the perturbation coefficient of the expansion is known, we then can address the exact behavior of the equation. Hence, the first step of the investigation of obtaining an accurate approximation is principally to expand $y(z,\varepsilon)$ in powers of ε. Let us assume the asymptotic solution of $y(z,\varepsilon)$ we seek in (2.1) has the series representation in the limit $\varepsilon \to 0$ as

$$y(z,\varepsilon) \sim \sum_{n=0}^{\infty} \varepsilon^n y_n(z). \tag{3.1}$$

Substitution of this expansion back into (2.1) gives

$$\sum_{n=0}^{\infty} \varepsilon^{n+1} \frac{dy_n(z)}{dz} + \int_0^z t \sum_{n=0}^{\infty} \varepsilon^n y_n(t)dt = z. \tag{3.2}$$

After grouping terms of like powers of ε in (3.2), we have

$$\sum_{n=0}^{\infty} \varepsilon^{n+1} \left(\frac{dy_n(z)}{dz} + \int_0^z t y_{n+1}(t)dt \right) + \int_0^z t y_0(t)dt = z. \tag{3.3}$$

Equating the coefficients of like powers of ε in (3.3), we may deduce the integral equation of the leading order term $y_0(z)$ at $O(1)$ as following

$$\int_0^z t y_0(t)dt = z. \tag{3.4}$$

Likewise, we may deduce the recurrence relationship of the recursive set of successive terms at $O(\varepsilon^{n+1})$ from (3.3), that is,

$$\frac{dy_n(z)}{dz} + \int_0^z t y_{n+1}(t)dt = 0, \tag{3.5}$$

from which the perturbation coefficient $y_{n+1}(z)$ can be computed recursively from the preceding terms. Upon solving the integration (3.4), we obtain the leading order solution $y_0(z)$ as

$$y_0(z) = \frac{1}{z}. \tag{3.6}$$

The leading order solution has a singularity at $z = 0$ which must be secured for all the terms of the expansion due to the successive relations of the terms in (3.5). For example, the recurrence relation may write down for $n = 0$ as

$$\frac{dy_0(z)}{dz} + \int_0^z t y_1(t)dt = 0.$$

Employing the leading order solution in (3.6) yields to the following

$$\begin{aligned} \int_0^z t y_1(t)dt &= -\frac{dy_0(z)}{dz} \\ &= \frac{1}{z^2}. \end{aligned}$$

Solving this integration leads to the succeeding solution $y_1(z)$

$$y_1(z) = -\frac{2}{z^4}.$$

Repeating this iterative process for $n = 1$ and $n = 2$ in the recurrence relation (3.5) and solving each of the equations, after doing some computations, respectively provide

$$y_2(z) = \frac{40}{z^7} \quad \text{and} \quad y_3(z) = -\frac{2240}{z^{10}}.$$

The above successive terms of the expansion illustrate the singular behavior of the expansion of $y(z,\varepsilon)$ at $z = 0$. Notice that the power of the singularity so does the strength of the singularity at zero increases by three for each calculated term. The coefficients of an asymptotic expansion to n terms may be calculated by using the recurrence relationship. Particularly, repeatedly doing this process for the successive terms of the expansion via the recurrence relation in succession enables us to derive the form of the tail of the expansion such that, upon simplification,

$$y_n(z) = (-1)^n \frac{(3n-1)!}{z^{3n+1} 3^{n-1} (n-1)!}. \tag{3.7}$$

Notice that it is in the nature of factorial divided by power and it is naturally divergent which can be found directly by the ratio test as n increases. Deriving the perturbative coefficient also shows the series representation indeed appears in this nature as discussed earlier. Back substitution of (3.7) into the naive expansion (3.1) represents the solution by the series

$$y(z,\varepsilon) \sim \sum_{n=0}^{\infty} (-1)^n \varepsilon^n \frac{(3n-1)!}{z^{3n+1} 3^{n-1} (n-1)!}. \tag{3.8}$$

Hence, the model Volterra integro-differential equation (2.1) associates with this divergent sum.

4. Truncating y_n at its least

The series representation we extract from the integro-differential equation suggests the possibility of calculating the equation approximately. As mentioned before, the traditional perturbation method works rather well with some series, but it is inadequate and misleading with some other series. In particular, sometimes when the series representations are convergent and truncated at a random point, expansions derives a good approximation, even with first few terms. In contrast, when the series representation is divergent or convergent but the rate of convergence is extremely slow, randomly truncated series may not lead a good approximation. In our case, clearly, for sufficiently small values of ε, the magnitude of the terms of the expansion at first decreases. It can be shown that first a couple of terms provides a good approximation. However, as the number of the terms increases in the expansion, the series eventually diverges. Therefore, to the approximation be useful, the error must be small. When the representation is divergent, the expansion must be truncated to obtain a good approximation from the integro type differential equation. Let us truncate the expansion (3.8) after $N \gg 1$ terms and introduce the remainder, that is,

$$y(z,\varepsilon) = \sum_{n=0}^{N-1} \varepsilon^n y_n(z) + R_N(z,\varepsilon), \tag{4.1}$$

where $R_N(z,\varepsilon)$ is the resultant remainder at $\mathrm{O}\big(\varepsilon^N y_N(z)\big)$. Once (4.1) is back substituted to (2.1), the remainder satisfies the Volterra integro-differential equation

$$\varepsilon \frac{dR_N(z,\varepsilon)}{dz} + \int_0^z tR_N(t,\varepsilon)dt = -\varepsilon^N \frac{dy_{N-1}(z)}{dz}.$$

As for the asymptotic expansion for $y(z,\varepsilon)$, the resultant remainder differential equation can be considered in a similar manner. Truncated series provides a good approximation of the function. For large values of the summation index, the truncated series does not convey to a good approximation. A random truncation of the perturbation expansion may or may not lead to a uniform approximation depending on the type of the series. Since the asymptotic expansion is divergent, it is important to know where you truncate the divergent expansion. Having a random number of terms in the expansion may lead to having an increased or decreased truncation error. Therefore, it is best to truncate the expansion where it changes its direction to infinity. Taking the derivative of the resultant remainder with respect to the truncation point such that

$$\frac{d}{dN}\,|R_N(z,\varepsilon)| = 0,$$

gives, after some computations for large N, that optimal truncation occurs at

$$N = \frac{1}{3}\sqrt{\frac{|z|^3}{\varepsilon}} + a, \tag{4.2}$$

where $0 \leq a < 1$ needs to be added in order to make sure that the truncation point is an integer. The terms in the series do not start increasing up to this point. This is the point at which the asymptotic series begins diverging to infinity. With respect to Kruskal's Principle of Maximal Balance "no term should be neglected without a good reason" [29], we truncate the asymptotic expansion at its least. The accuracy of the approximation at this point is maximum while the magnitude of the late terms is minimum for superasymptotics. In asymptotics, the accuracy of the approximation also depends on the perturbation parameter, particularly, it becomes better as the independent variable ε, in this case, approaches zero.

5. Remainder Analysis

Since the series expansion is truncated at its least, the resultant remainder is supposed to be exponentially small. As for the asymptotic expansion of $y(z,\varepsilon)$, our next step is to show this. To do so, we use useful Stirling's approximation of factorial functions [30] that is

$$N! \approx \sqrt{2\pi}\frac{N^{N+\frac{1}{2}}}{e^N}. \tag{5.1}$$

This provides a very accurate approximation of $N!$ even when N is not large. To address the exponentially small behavior as $\varepsilon \to 0$, we employ this into the first neglected term of the expansion for sufficiently large values of N. We then find that the order of the truncation error is asymptotic to

$$\begin{aligned} R_N(z,\varepsilon) &\sim \varepsilon^N y_N(z) \\ &= (-1)^N \varepsilon^N \frac{(3N-1)!}{z^{3N+1}3^{N-1}(N-1)!}. \end{aligned} \tag{5.2}$$

Using the Stirling's approximation (5.1) to factorial terms of (5.2) with the optimal truncation point in (4.2), after some computations for sufficiently large values of N, we have

$$\begin{aligned} R_N(z,\varepsilon) &\sim (-1)^N \frac{\varepsilon^N}{z^{3N+1}} \frac{(3N-1)^{3N-1/2}}{3^{N-1}(N-1)^{N-1/2}} \exp(-2N) \\ &= (-1)^N \frac{\varepsilon^N}{z^{3N+1}} 3^{2N+1/2} N^{2N} \exp(-2N). \end{aligned} \tag{5.3}$$

As it is convenient to use polar coordinates, we set $z=|z|e^{i\theta}$, where θ denotes the phase of z. Employing the optimal truncation point (4.2) to above (5.3), we find

$$R_N(z,\varepsilon)\sim\frac{\sqrt{3}}{|z|}\exp\left(\frac{i\pi}{3}\sqrt{\frac{|z|^3}{\varepsilon}}+i\pi a-i\theta\right)\exp\left(-\frac{2}{3}\sqrt{\frac{|z|^3}{\varepsilon}}-2a\right). \tag{5.4}$$

Hence, the truncation error vanishes exponentially in the usual form as it can be seen from the right most term of (5.4) as $\varepsilon\to 0$. Improved accuracy is achieved by optimally including later order terms of the expansion.

6. Conclusion

In this paper, upon expanding solution of the linear Volterra type integro-differential equation in powers of ε, we obtain the asymptotic series representation of the model differential equation via the methods of superasymptotics. Unlike the traditional asymptotic expansion, truncation of the divergent series made sure that the truncation error is at least and the resultant remainder is exponentially small. It permitted us to study the asymptotic behavior of the Volterra type integro-differential equations and enabled us to extract the exponentially small behaviors which are encoded in the divergent perturbative expansion. The model example illustrates Kruskal's Principles of Asymptotology that no term neglected without a reason. The process of re-expansion, similarly, can be repeated within the hyperasymptotics's framework and refined information can be obtained.

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