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ESTIMATION OF STRESS-STRENGTH RELIABILITY OF A PARALLEL SYSTEM WITH COLD STANDBY REDUNDANCY AT COMPONENT LEVEL

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Abstract: In this paper, we consider the estimation problem of stress-strength reliability of a parallel system with cold standby redundancy. The reliability of the system is estimated when both strength and stress variables follow the exponential distribution and associated approximate confidence interval is constructed. Two different maximum likelihood and Bayes estimates are obtained. Lindley's approximation method has been utilized for Bayesian calculations. A real-life data set is analysed for illustrative purposes of the findings.

Key words: Stress-strength reliability; Parallel system; Cold standby redundancy

1. Introduction

In reliability analysis, stress-strength probability is a major concern for some experiments in the fields including industrial engineering, hydrology, economics and survival analysis. Stress-strength models are introduced originally by Birnbaum [2] then developed by Birnbaum and McCarty [3] and Church and Harris [6]. When X is the random strength of a system under the random stress Y, the probability $R = P(X > Y)$ indicates the measure of the performance of the system. If the stress exceeds the strength, the system stops operating, otherwise it continues to work. In 2003, Kotz et al. [13] interpreted the concept of stress-strength combined with the theory and applications.

A k-out-of-n: G system consists of n independent and identically distributed strength components and a common stress, and functions at least k out of the n components operate. When $k = 1$ and $k = n$, the k-out-of-n: G system becomes a parallel and series systems, respectively. A parallel system fails if and only if its each component fails so that this system works whenever at least one component works. Multicomponent stress-strength reliability has been of great interest among researchers in recent years. In this context, we can refer to Eryilmaz [8], Pakdaman and Ahmadi [18], Kızılaslan [12], Akgül [1] and Dey et al. [7].

Standby redundancy allocation to a system or components makes a great impact on system lifetime. Hence, it is widely used to improve system reliability. Different types of standby redundancy have been introduced and studied in the reliability literature. The component is said to be in the case of cold standby if it does not fail while in standby. When the components of a system fail, cold standby redundancy puts into operation and system operates until the standby component fails.

The cold standby redundancy can be implemented to a system at system and component levels. At component level, the standby components are connected to the original components one by one. At system level, the standby components construct an alternative system for the system of

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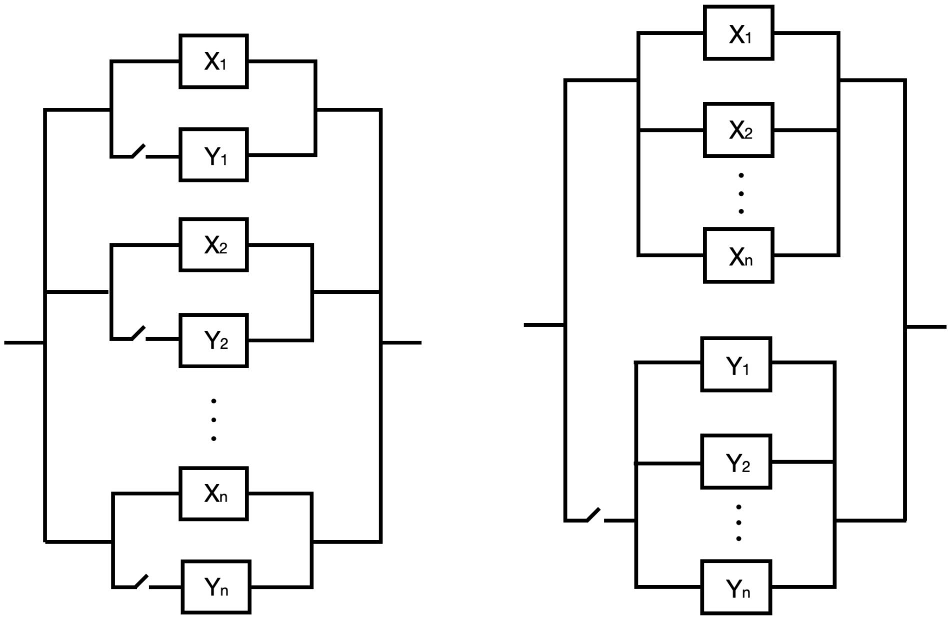
original components. When standby components are added to a n-component parallel system at the component or system level, new systems are obtained as in Figure 1.

We can consider the following example for the parallel system with standby components at component level. Computer systems are used in many areas. Saving customer transactions data automatically in banking system is one of them. For example, six computers in parallel design be used for data saving. The random variables $X_1, \ldots, X_6$ represent the lifetime of these computers. We add additional six computers to the parallel system as standby components at component level. The random variables $Y_1, \ldots, Y_6$ represent the lifetime of these new standby computers. Some factors such as lifetime of the subcomponents, cyber attacks, density of the data, etc. can be considered as stress variables of this system. The random variable T represents the stress variable. In this scenario, a probability for the total lifetime of the parallel system under the stress, that is $P(\max(X_1+Y_1, X_2+Y_2, \ldots, X_6+Y_6) > T)$, is the probability of saving data successfully.



(a) Component level (b) System level

FIGURE 1. Standby redundancy of parallel systems

In the literature, the stochastic comparisons of coherent systems when the cold standby redundancy applied at component level and system level have been investigated by many researchers. Chen and Xie [23] studied the effect of adding standby redundancy at system and component levels in series and parallel systems, and compared the superiority of the levels. Boland and El-Neweihi [4] extended the stochastic comparison results of the cold standby levels for the series and parallel systems from the usual stochastic ordering to the hazard rate ordering. They also obtained some results about the hazard rate ordering of the k-out-of-n: G system with cold standby redundancy. Zhao et al. [27] presented the likelihood ratio ordering result for the series system with n exponential components when the active components and standby components are identical. Similar results for the parallel system was proved for $n=2$ case. Eryilmaz and Tank [9] considered a series system with two dependent components and a single cold standby component. Eryilmaz [10] investigated the effect of adding cold standby redundancy to a general coherent structure at system and component levels. Tuncel [25] studied the residual lifetime of a single component system with a cold standby component when the lifetimes of these components were dependent. Chen et al. [5] discussed the problem of optimal allocation methods for two standby spares in a two-component series/parallel structure. Yan et al. [26] studied series and parallel systems of two components with one standby redundancy component when all components having exponential distribution. Roy and Gupta [21, 22] considered the reliability of a k-out-of-n and coherent systems equipped with two cold standby components, respectively.

The stress-strength reliability estimation of the k-out-of-n: G system has been paying great attention by researchers in decades. Many studies are available in the literature about this topic. Since adding standby component(s) to the system increases the system reliability, to investigate the reliability of this type of a system is useful for practitioners. Some recent studies are mentioned in the next references. Siju and Kumar [24] considered the estimation of reliability of a parallel system under the hybrid of active, warm and cold standby components by applying maximum likelihood method. The reliability estimation for the standby redundancy system consists of a certain number of same subsystems with series structure was considered by Liu et al. [15] for the generalized half-logistic distribution based on progressive Type-II censoring sample.

In this study, we consider a parallel system of n components with n standby components whenever the standby components connected to working components at component level. The estimation problem for the stress-strength reliability of this parallel system has been studied.

The rest of this article is organized as follows: Section 2 presents the formulation of the model that includes standby components. Section 3 contains the maximum likelihood estimate (MLE) and Bayes estimate of the reliability of the parallel system under the common stress. Section 4 contains the simulation study in order to compare the performance of the obtained estimators. A real life data analysis is considered for illustrative purposes of the proposed estimates.

2. Model definition

In this section, we have considered the problem of stress-strength reliability estimation of a parallel system with standby redundancy at component level. Suppose an n component parallel system with standby components that are independent but not identically distributed with original components. We assume that $X_1, \ldots, X_n$ represent the lifetimes of strength components and $Y_1, \ldots, Y_n$ are the lifetimes of independent standby strength components having exponential distributions with parameters α and β, respectively. T represents the common stress variable and follows the exponential distribution with parameter θ. This parallel system structure is shown in Figure 1a.

In the case of component level, standby redundancy is applied to the components one by one. The total lifetime of each strength component is $Z_i = X_i + Y_i$, $i = 1, \ldots, n$. Then, the cumulative distribution function (cdf) and probability density function (pdf) of Z_i, $i = 1, \ldots, n$ are given by

$$F_{Z_i}(z) = \int_0^z F_x(z-y) f_y(y) dy = \begin{cases} 1 + \frac{\alpha e^{-\beta z} - \beta e^{-\alpha z}}{\beta - \alpha} & , \alpha \neq \beta \\ 1 - e^{-\alpha z}(1 + \alpha z) & , \alpha = \beta \end{cases} \tag{2.1}$$

and

$$f_{Z_i}(z) = \begin{cases} \frac{\alpha\beta}{\beta-\alpha}(e^{-\alpha z} - e^{-\beta z}) & , \alpha \neq \beta \\ \alpha^2 z e^{-\alpha z} & , \alpha = \beta \end{cases}, \quad i = 1, \ldots, n. \tag{2.2}$$

When the active strength and standby redundancy components are identical, i.e. $\alpha = \beta$, the total lifetime distributions Z_i, $i = 1, \ldots, n$ follow the gamma distribution with shape parameter 2 and rate parameter α. In this case, the stress-strength reliability problem is reduced to the simple stress-strength reliability of the gamma components in Nojosa and Rathie [17]. Hence, we consider the case where parameters α and β are not the same in this study. Since $Z_1, \ldots, Z_n$ denote n

independent total strength random variables given in (2.1), the cdf for the lifetime of the parallel system is

$$F_{Z_{(n)}}(z) = \sum_{k=0}^{n} \binom{n}{k} \frac{(\alpha e^{-\beta z} - \beta e^{-\alpha z})^k}{(\beta-\alpha)^k}$$
$$= \sum_{k=0}^{n}\sum_{j=0}^{k} \binom{n}{k}\binom{k}{j}(-1)^j \frac{\alpha^{k-j}\beta^j}{(\beta-\alpha)^k} e^{-z[j(\alpha-\beta)+\beta k]}$$

where $Z_{(n)} = \max(Z_1, \dots, Z_n)$.

When the maximum strength component $Z_{(n)}$ is subjected to the common stress component T, the stress-strength reliability of the parallel system is obtained as

$$R = \int_0^\infty P(Z_{(n)} > T | T = t) f_T(t) dt$$
$$= 1 - \sum_{k=0}^{n}\sum_{j=0}^{k} \binom{n}{k}\binom{k}{j} \frac{(-1)^j}{(\beta-\alpha)^k} \frac{\alpha^{k-j}\beta^j\theta}{\alpha j + \beta(k-j)+\theta}. \quad (2.3)$$

Moreover, if we use n standby strength components as original components in the parallel system, we construct a new parallel system with $2n$ strength components. The stress-strength reliability of the new parallel system is derived under the common stress for comparison. Let V_i be the lifetime of i^{th} strength components $i = 1, \dots, 2n$, and follow the exponential distribution with parameter α for $1 \le i \le n$ and β for $n+1 \le i \le 2n$. In this case, the cdf for the lifetime of the $2n$ components parallel system is

$$F_{V_{(2n)}}(t) = P(\max(V_1, \dots, V_{2n}) \le t) = (1-e^{-\alpha t})^n (1-e^{-\beta t})^n.$$

Then, the stress-strength reliability of this parallel system is obtained as

$$R_V = \int_0^\infty P(V_{(2n)} > T | T = t) f_T(t) dt$$
$$= 1 - \sum_{k=0}^{n}\sum_{j=0}^{n} \binom{n}{k}\binom{n}{j} (-1)^{k+j} \frac{\theta}{\alpha k + \beta j + \theta} \quad (2.4)$$

under the common stress T.

To show the effect of the standby components at component level in the stress-strength reliability of the parallel system, we present the following graphics. In Figure 2, the comparison of two stress-strength reliabilities R in (2.3) and R_V in (2.4) are plotted with respect to the different parameters. It is seen that adding standby components increases system reliability, as expected.

3. Estimation of R

In this section, the ML and Bayes estimates of the stress-strength reliability of the aforementioned parallel system are investigated.

3.1. MLE of R

Let m systems be put on a test each with n original components and n cold standby components in the parallel system. The strength data is represented as $Z_{i1}, \dots, Z_{in}$, $i = 1, \dots, m$ and stress is $T_i, i = 1, \dots, m$. Then, the likelihood function of the observed sample is

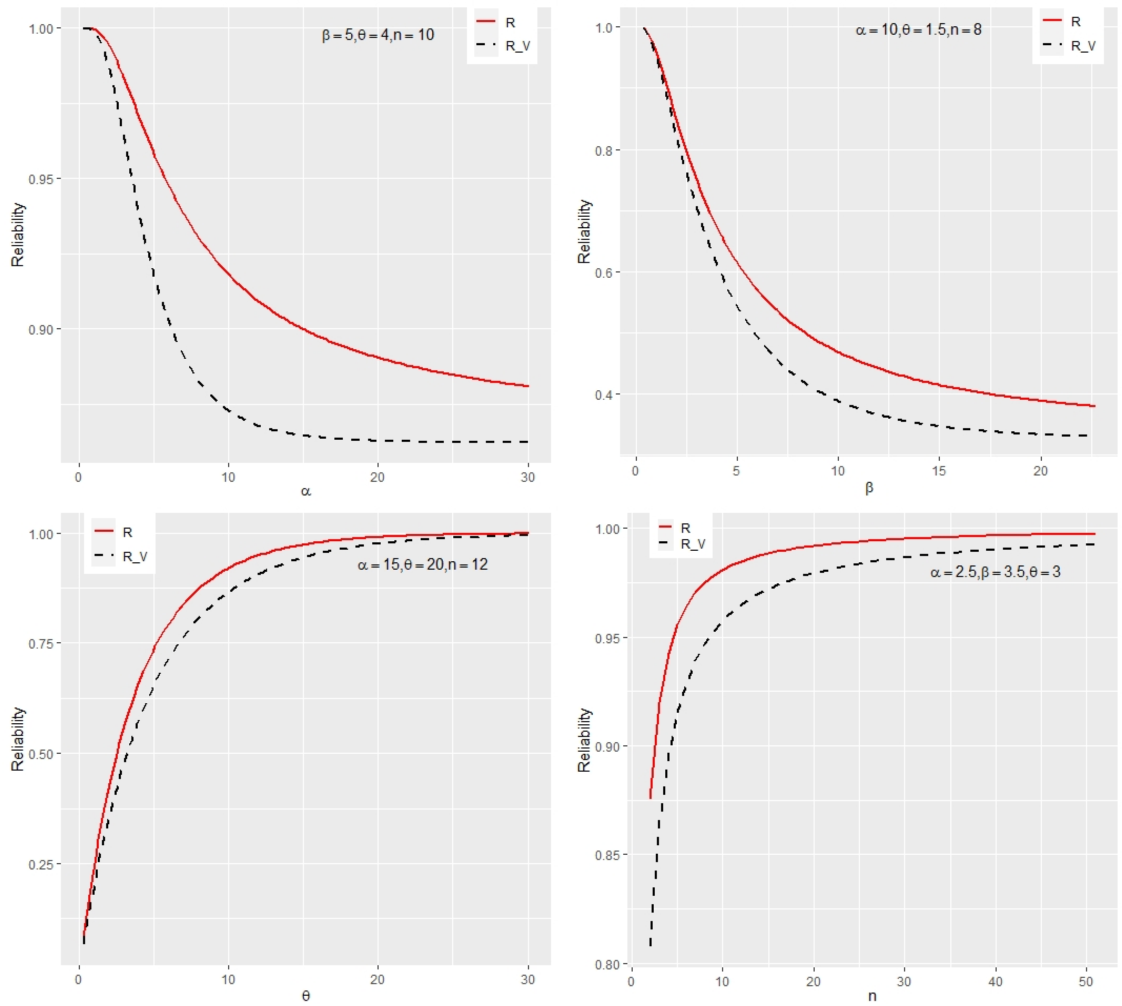


FIGURE 2. Plots for the reliabilities of R and R_V

$$L(\alpha,\beta,\theta|\mathbf{z},\mathbf{t}) = \prod_{i=1}^{m}\left(\prod_{j=1}^{n} f_Z(z_{ij})\right) f_T(t_i)$$
$$= \left(\frac{\alpha\beta}{\beta-\alpha}\right)^{nm} \exp\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\ln\left(e^{-\alpha z_{ij}} - e^{-\beta z_{ij}}\right)\right)\theta^m e^{-\theta\sum_{i=1}^{m} t_i},$$

and the log-likelihood function is given by

$$\ell(\alpha,\beta,\theta;\mathbf{z},\mathbf{t}) = nm\left(\ln(\alpha\beta) - \ln(\beta-\alpha)\right) + m\ln\theta + \sum_{i=1}^{m}\sum_{j=1}^{n}\ln\left(e^{-\alpha z_{ij}} - e^{-\beta z_{ij}}\right) - \theta\sum_{i=1}^{m} t_i.$$

The partial derivatives with respect to three parameters are obtained in the following forms:

$$\frac{\partial\ell}{\partial\alpha} = nm\left(\frac{1}{\alpha} + \frac{1}{\beta-\alpha}\right) - \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{z_{ij}e^{-\alpha z_{ij}}}{e^{-\alpha z_{ij}} - e^{-\beta z_{ij}}}, \tag{3.1}$$

$$\frac{\partial\ell}{\partial\beta} = nm\left(\frac{1}{\beta} - \frac{1}{\beta-\alpha}\right) + \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{z_{ij}e^{-\beta z_{ij}}}{e^{-\alpha z_{ij}} - e^{-\beta z_{ij}}} \tag{3.2}$$

and

$$\frac{\partial \ell}{\partial \theta} = \frac{m}{\theta} - \sum_{i=1}^{m} t_i.$$

Hence, the ML estimate of θ is $\widehat{\theta} = 1/\overline{T}$, and the ML estimates of α and β, that is $\widehat{\alpha}$ and $\widehat{\beta}$, are the solutions of non-linear equation system given in (3.1) and (3.2). $\widehat{\alpha}$ and $\widehat{\beta}$ can be derived with the help of numerical methods, like Newton-Raphson or Broyden's method. Therefore, $\widehat{R}$ can be obtained as

$$\widehat{R} = 1 - \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}\binom{k}{j}\frac{(-1)^j}{(\widehat{\beta}-\widehat{\alpha})^k}\frac{\widehat{\alpha}^{k-j}\widehat{\beta}^j\widehat{\theta}}{\widehat{\alpha}j + \widehat{\beta}(k-j) + \widehat{\theta}},$$

from (2.3) by using the invariance property of MLE.

3.2. Asymptotic distribution and confidence interval of R

The observed information matrix of $\tau = (\alpha, \beta, \theta)$ is given as

$$J(\tau) = -\begin{pmatrix} \frac{\partial^2 \ell}{\partial \alpha^2} & \frac{\partial^2 \ell}{\partial \alpha \partial \beta} & \frac{\partial^2 \ell}{\partial \alpha \partial \theta} \\ \frac{\partial^2 \ell}{\partial \beta \partial \alpha} & \frac{\partial^2 \ell}{\partial \beta^2} & \frac{\partial^2 \ell}{\partial \beta \partial \theta} \\ \frac{\partial^2 \ell}{\partial \theta \partial \alpha} & \frac{\partial^2 \ell}{\partial \theta \partial \beta} & \frac{\partial^2 \ell}{\partial \theta^2} \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix}.$$

Since $J_{13} = J_{31} = J_{23} = J_{32} = 0$, other entries of the matrix are

$$J_{11} = nm\left(\frac{1}{\alpha^2} - \frac{1}{(\beta-\alpha)^2}\right) + \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{z_{ij}^2 e^{-z_{ij}(\beta-\alpha)}}{(1-e^{-z_{ij}(\beta-\alpha)})^2},$$

$$J_{12} = J_{21} = \frac{nm}{(\beta-\alpha)^2} - \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{z_{ij}^2 e^{-z_{ij}(\beta-\alpha)}}{(1-e^{-z_{ij}(\beta-\alpha)})^2}$$

and

$$J_{22} = nm\left(\frac{1}{\beta^2} - \frac{1}{(\beta-\alpha)^2}\right) + \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{z_{ij}^2 e^{-z_{ij}(\beta-\alpha)}}{(1-e^{-z_{ij}(\beta-\alpha)})^2}, \qquad J_{33} = \frac{m}{\theta^2}.$$

The expectations of the entries of the observed information matrix cannot be obtained analytically. Therefore, the Fisher Information matrix $I(\tau) = E(J(\tau))$ can be obtained by using numerical methods. The MLE of R is asymptotically normal with mean R and asymptotic variance

$$\sigma_R^2 = \sum_{j=1}^{3}\sum_{i=1}^{3}\frac{\partial R}{\partial \tau_i}\frac{\partial R}{\partial \tau_j} I_{ij}^{-1},$$

where I_{ij}^{-1} is the $(i,j)^{th}$ element of the inverse of $I(\tau)$, see Rao [20]. Afterwards,

$$\sigma_R^2 = \left(\frac{\partial R}{\partial \alpha}\right)^2 I_{11}^{-1} + 2\frac{\partial R}{\partial \alpha}\frac{\partial R}{\partial \beta} I_{12}^{-1} + \left(\frac{\partial R}{\partial \beta}\right)^2 I_{22}^{-1} + \left(\frac{\partial R}{\partial \theta}\right)^2 I_{33}^{-1}. \tag{3.3}$$

Note that $I(\tau)$ can be replaced by $J(\tau)$ when $I(\tau)$ is not obtained. Therefore, an asymptotic $100(1-\gamma)\%$ confidence interval of R is given by $\left(\widehat{R}^{MLE} \pm z_{\gamma/2}\widehat{\sigma}_R\right)$ where $z_{\gamma/2}$ is the upper $\gamma/2$th quantile of the standard normal distribution and $\widehat{\sigma}_R$ is the value of σ_R at the MLE of the parameters.

3.3. Bayes estimation of R

In this section, we assume that α, β and θ are random variables that follow independent gamma prior distributions with parameters (a_i, b_i), $i = 1,2,3$, respectively. The pdf of a gamma random variable X with parameters (a_i, b_i) is given as

$$f(x) = \frac{{b_i}^{a_i}}{\Gamma(a_i)} x^{a_i-1} e^{-xb_i}, \qquad x>0, \quad a_i, b_i > 0 \quad \text{and} \quad i = 1,2,3.$$

The joint posterior density function of α, β and θ is

$$\pi(\alpha,\beta,\theta|\mathbf{z},\mathbf{t}) = I(\mathbf{z},\mathbf{t})\alpha^{nm+a_1-1}\beta^{nm+a_2-1}(\beta-\alpha)^{-nm}\theta^{a_3+m-1}e^{-\alpha b_1-\beta b_2-\theta\left(b_3+\sum_{i=1}^m t_i\right)}z_{\alpha,\beta} \tag{3.4}$$

where $I(\mathbf{z},\mathbf{t})$ is the normalizing constant and written by

$$\frac{I(\mathbf{z},\mathbf{t})^{-1}}{\Gamma(a_3+m)}\left(b_3+\sum_{i=1}^m t_i\right)^{a_3+m} = \int_0^\infty\int_0^\infty \left(\frac{\alpha\beta}{\beta-\alpha}\right)^{nm} \alpha^{a_1-1}\beta^{a_2-1}e^{-\alpha b_1-\beta b_2} z_{\alpha,\beta}\, d\alpha d\beta$$

where

$$z_{\alpha,\beta} = \exp\left(\sum_{i=1}^m\sum_{j=1}^n \ln\left(e^{-\alpha z_{ij}} - e^{-\beta z_{ij}}\right)\right).$$

The Bayes estimator of R under the SE loss function is

$$\widehat{R}_{Bayes} = \int_0^\infty\int_0^\infty\int_0^\infty R\ \pi(\alpha,\beta,\theta|\mathbf{z},\mathbf{t})\, d\alpha d\beta d\theta.$$

This integral cannot be easily computed analytically; thus, some approximation methods are needed. In order to obtain the Bayes estimate of R, we use Lindley's approximation method.

3.3.1. Lindley's approximation

Lindley [14] proposed an approximate method in order to obtain a numerical result for the computation of the ratio of two integrals. This procedure, applied to the posterior expectation of the function $u(\theta)$ for a given $\mathbf{x}$, is

$$E(u(\theta)\,|\mathbf{x}) = \frac{\int u(\theta)e^{Q(\theta)}d\lambda}{\int e^{Q(\theta)}d\lambda},$$

where $Q(\theta) = l(\theta) + \rho(\theta)$, $l(\theta)$ is the logarithm of the likelihood function and $\rho(\theta)$ is the logarithm of the prior density of θ. Using Lindley's approximation, $E(u(\theta)\,|\mathbf{x})$ is approximately estimated by

$$E(u(\theta)\,|\mathbf{x}) = \left[u + \frac{1}{2}\sum_i\sum_j (u_{ij} + 2u_i\rho_j)\sigma_{ij} + \frac{1}{2}\sum_i\sum_j\sum_k\sum_l L_{ijk}\sigma_{ij}\sigma_{kl}u_l\right]_{\widehat{\lambda}}$$
$$+\text{terms of order } n^{-2} \text{ or smaller},$$

where $\theta = (\theta_1, \theta_2, \ldots, \theta_m)$, $i,j,k,l = 1,\ldots,m$, $\widehat{\theta}$ is the MLE of θ, $u = u(\theta)$, $u_i = \partial u/\partial\theta_i$, $u_{ij} = \partial^2 u/\partial\theta_i\partial\theta_j$, $L_{ijk} = \partial^3 l/\partial\theta_i\partial\theta_j\partial\theta_k$, $\rho_j = \partial\rho/\partial\theta_j$, and $\sigma_{ij} = (i,j)$th element in the inverse of the matrix $\{-L_{ij}\}$ all evaluated at the MLE of the parameters.

For three parameter case $\theta = (\theta_1, \theta_2, \theta_3)$, Lindley's approximation gives the approximate Bayes estimate as

$$\widehat{u}_B = E(u(\theta)\,|\mathbf{x}) = u + (u_1a_1 + u_2a_2 + u_3a_3 + a_4 + a_5) + \frac{1}{2}\left[A\left(u_1\sigma_{11} + u_2\sigma_{12}\right.\right.$$
$$\left.\left. + u_3\sigma_{13}\right) + B(u_1\sigma_{21} + u_2\sigma_{22} + u_3\sigma_{23}) + C(u_1\sigma_{31} + u_2\sigma_{32} + u_3\sigma_{33})\right],$$

evaluated at $\widehat{\theta}=(\widehat{\theta}_1,\widehat{\theta}_2,\widehat{\theta}_3)$, where

$$a_i=\rho_1\sigma_{i1}+\rho_2\sigma_{i2}+\rho_3\sigma_{i3},\ i=1,2,3,$$

$$\begin{aligned}a_4&=u_{12}\sigma_{12}+u_{13}\sigma_{13}+u_{23}\sigma_{23},\ a_5=\frac{1}{2}(u_{11}\sigma_{11}+u_{22}\sigma_{22}+u_{33}\sigma_{33}),\\ A&=\sigma_{11}L_{111}+2\sigma_{12}L_{121}+2\sigma_{13}L_{131}+2\sigma_{23}L_{231}+\sigma_{22}L_{221}+\sigma_{33}L_{331},\\ B&=\sigma_{11}L_{112}+2\sigma_{12}L_{122}+2\sigma_{13}L_{132}+2\sigma_{23}L_{232}+\sigma_{22}L_{222}+\sigma_{33}L_{332},\\ C&=\sigma_{11}L_{113}+2\sigma_{12}L_{123}+2\sigma_{13}L_{133}+2\sigma_{23}L_{233}+\sigma_{22}L_{223}+\sigma_{33}L_{333}.\end{aligned}$$

In our system, for $(\theta_1,\theta_2,\theta_3)\equiv(\alpha,\beta,\theta)$ and $u\equiv u(\alpha,\beta,\theta)=R$ in (2.3). We compute σ_{ij}, $i,j=1,2,3$ by using the following partial derivatives $L_{11}=-J_{11}, L_{12}=L_{21}=-J_{12}$ and $L_{22}=-J_{22}$. Using logarithm of the prior density, we have

$$\rho_1=\frac{(a_1-1)}{\alpha}-b_1,\quad \rho_2=\frac{(a_2-1)}{\beta}-b_2,\quad \rho_3=\frac{(a_3-1)}{\theta}-b_3.$$

Additionally, we obtain $L_{333}=\frac{2m}{\theta^3}$ and

$$\begin{aligned}L_{111}&=2nm\left(\frac{1}{\alpha^3}+\frac{1}{(\beta-\alpha)^3}\right)-\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{z_{ij}^3e^{-z_{ij}(\beta-\alpha)}\left(1+e^{-z_{ij}(\beta-\alpha)}\right)}{\left(1-e^{-z_{ij}(\beta-\alpha)}\right)^3},\\ L_{121}=L_{112}&=-2nm\frac{1}{(\beta-\alpha)^3}+\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{z_{ij}^3e^{-z_{ij}(\beta-\alpha)}\left(1+e^{-z_{ij}(\beta-\alpha)}\right)}{\left(1-e^{-z_{ij}(\beta-\alpha)}\right)^3},\\ L_{122}=L_{221}&=2nm\frac{1}{(\beta-\alpha)^3}-\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{z_{ij}^3e^{-z_{ij}(\beta-\alpha)}\left(1+e^{-z_{ij}(\beta-\alpha)}\right)}{\left(1-e^{-z_{ij}(\beta-\alpha)}\right)^3},\\ L_{222}&=2nm\left(\frac{1}{\beta^3}-\frac{1}{(\beta-\alpha)^3}\right)+\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{z_{ij}^3e^{-z_{ij}(\beta-\alpha)}\left(1+e^{-z_{ij}(\beta-\alpha)}\right)}{\left(1-e^{-z_{ij}(\beta-\alpha)}\right)^3}.\end{aligned}$$

Let

$$S=\binom{n}{k}(-1)^k\frac{\alpha^k\beta^{n-k}\theta}{(\beta-\alpha)^n\left[\alpha\left(n-k\right)+\beta k+\theta\right]}\tag{3.5}$$

denote the common term in the first and second order partial derivatives of the parallel system reliability. We have the derivatives of reliability as given below

$$u_1=\sum_{k=0}^{n}S\frac{\{[n\alpha+k(\beta-\alpha)][\alpha(n-k)+\beta k+\theta]-\alpha(\beta-\alpha)(n-k)\}}{\alpha(\beta-\alpha)\left[\alpha\left(n-k\right)+\beta k+\theta\right]},$$

$$u_2=\sum_{k=0}^{n}S\frac{(-k)[\alpha(n-k)+\beta(k+1)+\theta]}{\beta\left[\alpha\left(n-k\right)+\beta k+\theta\right]},$$

$$u_3=\sum_{k=0}^{n}S\frac{[\alpha(n-k)+\beta k]}{\theta\left[\alpha\left(n-k\right)+\beta k+\theta\right]},$$

$$\begin{aligned}u_{11}=\sum_{k=0}^{n}S\Bigg[&\frac{n(n+1)}{(\beta-\alpha)^2}+\frac{2(n-k)}{\alpha(n-k)+\beta k+\theta}\left(-\frac{n}{\beta-\alpha}-\frac{k}{\alpha}+\frac{n-k}{\alpha(n-k)+\beta k+\theta}\right)\\ &+\frac{2kn}{\alpha(\beta-\alpha)}+\frac{k(k-1)}{\alpha^2}(k-1)\Bigg],\end{aligned}$$

$$u_{12} = \sum_{k=0}^{n} S\left[\frac{n(n+1)}{(\beta-\alpha)^2} - \frac{2(n-k)}{\alpha(n-k)+\beta k+\theta}\left(\frac{n}{\beta-\alpha}+\frac{k}{\alpha}-\frac{n-k}{\alpha(n-k)+\beta k+\theta}\right)+\frac{2kn}{\alpha(\beta-\alpha)}+\frac{k(k-1)}{\alpha^2}\right],$$

$$u_{22} = \sum_{k=0}^{n} S\left[\frac{n(n+1)}{(\beta-\alpha)^2} + \frac{2k}{\alpha(n-k)+\beta k+\theta}\left(\frac{n}{\beta-\alpha}-\frac{n-k}{\beta}+\frac{k}{\alpha(n-k)+\beta k+\theta}\right) - \frac{2n(n-k)}{\beta(\beta-\alpha)}+\frac{(k-n)(k-n+1)}{\beta^2}\right],$$

$$u_{13} = \sum_{k=0}^{n} \frac{S}{\theta}\left[\frac{1}{\alpha(n-k)+\beta k+\theta}\left(-\theta\left(\frac{n}{\beta-\alpha}+\frac{k}{\alpha}\right)+(k-n)\left(1-\frac{2\theta}{\alpha(n-k)+\beta k+\theta}\right)\right)+\frac{n}{(\beta-\alpha)}+\frac{k}{\alpha}\right],$$

$$u_{23} = \sum_{k=0}^{n} \frac{S}{\theta}\left[\frac{1}{\alpha(n-k)+\beta k+\theta}\left(\theta\left(\frac{n}{\beta-\alpha}-\frac{n-k}{\beta}\right)+k\left(\frac{2\theta}{\alpha(n-k)+\beta k+\theta}-1\right)\right)+\frac{n-k}{\beta}\right],$$

$$u_{33} = \sum_{k=0}^{n} \frac{S}{\theta}\left(\frac{\theta}{\alpha(n-k)+\beta k+\theta}-1\right).$$

Hence, we obtain $A = \sigma_{11}L_{111} + 2\sigma_{12}L_{121} + \sigma_{22}L_{221}$, $B = \sigma_{11}L_{112} + 2\sigma_{12}L_{122} + \sigma_{22}L_{222}$ and $C = \sigma_{33}L_{333}$. Then, Bayes estimator of R, i.e. $\widehat{R}_{Lindley}$, is given as

$$\widehat{R}_{Lindley} = R + [u_1a_1 + u_2a_2 + u_3a_3 + a_4 + a_5] + \frac{1}{2}[A(u_1\sigma_{11} + u_2\sigma_{12} + u_3\sigma_{13}) + B(u_1\sigma_{21} + u_2\sigma_{22} + u_3\sigma_{23}) + C(u_1\sigma_{31} + u_2\sigma_{32} + u_3\sigma_{33})] \tag{3.6}$$

where all the parameters are evaluated at MLEs $(\widehat{\alpha}, \widehat{\beta}, \widehat{\theta})$.

4. Numerical results

In this section, a simulation study and a real-life example are presented to illustrate the obtained estimates.

4.1. Simulation study

In this subsection, we perform a simulation study to compare the performance of two different ML estimates and Bayesian estimates under informative and non-informative priors with respect to mean squared error (MSE) and estimated risk (ER). The ER of θ for the estimate $\widehat{\theta}$, is computed as

$$ER(\theta) = \frac{1}{N}\sum_{i=1}^{N}(\widehat{\theta}_i - \theta_i)^2,$$

under the SE loss function. We have used statistical software R [19] for all computations. The nleqslv package [11] in software R is used to solve the non-linear equations. The point and interval estimates are compared with respect to average lengths (AL) and coverage probabilities (CP) of 95% confidence intervals. All results are obtained based on 2500 replications.

In simulation, we have considered different sample sizes $n = 5(5)15$, $m = 25(25)100$ for $(\alpha, \beta, \theta) = (8, 2, 1.5)$ and $n = 12(4)20$, $m = 10, 20, 40, 60$ for $(\alpha, \beta, \theta) = (12, 7, 3)$. The biases and MSEs for two different ML estimates, biases and ERs of Bayes estimates under the informative and non-informative priors are presented in Table 1. The first MLE of R is computed by using (2.3) based on the solution of the non-linear equation system given in (3.1) and (3.2). ML estimates of unknown parameters α, β and θ are also obtained from the random sample of exponential distributions as $\widetilde{\alpha}=1/\overline{X}$, $\widetilde{\beta}=1/\overline{Y}$ and $\widetilde{\theta}=1/\overline{T}$. The second MLE of R, called MLE2, is computed by using $\widetilde{\alpha},\widetilde{\beta}$ and $\widetilde{\theta}$ in (2.3). In Bayesian case, the hyperparameters are selected as $a_1=2\alpha$, $b_1=2$, $a_2=2\beta$, $b_2=2$, $a_3=2\theta$, $b_3=2$ for the informative prior and zero for the non-informative case. The AL for the %95 asymptotic confidence interval of R and corresponding CP value are reported in Table 2.

From Table 1, MSE and ERs of the estimates decrease as the sample size increases as expected. We observe that Bayes estimates of R based on the informative prior perform better than that of other estimates except for $n=12$, $m=10$ case. The two ML estimates give nearly same error values but the MLE2 mostly gives better results with a small difference. For this reason, MLE2 can be preferable as an alternative method. It is observed that the AL for each interval decreases as the sample size increases and all the CPs are satisfactory.

4.2. Real data analysis

In this subsection, a real-life data set analysis is presented to illustrate the proposed methods. Nelson [16] considered the graphical methods for analyzing accelerated life test data about times to breakdown of an insulating fluid subjected to various constant elevated test voltages. The number of times to breakdown were observed and saved for each test voltage, and all the data was given in Nelson [16] (Table 1).

An engineer has a parallel system with three components and each component is isolated by using this insulating fluid. This engineer wants to decide what voltage values should be used for these components to minimize the breakdown times. For this purpose, he/she wants to compare the low and mid voltage levels (32 Kv and 34 Kv) against the high voltage level (36 Kv). Then, three components parallel system with standby components is constructed by using 32 Kv and 34 Kv data sets. It is assumed that 32 Kv, 34 Kv and 36 Kv data sets represent the strength component (**X**), standby strength component (**Y**) and stress component (**T**) observations, respectively. For the strength data sets, **X** and **Y** are used for three components of the parallel system. Based on the previous information, if the reliability value of this system exceeds 0.90, he/she prefers the parallel system with standby components.

We use 15 observations for strength components and 5 observations for stress component. Hence, a random sample of the size 15 is taken from 34 Kv data for **Y** and a random sample of the size 5 is taken from 36 Kv data for **T**. Every column of **X** and **Y** data sets represents the observations of the three components for the parallel system. Then, the observed data **X**, **Y** and $(\mathbf{Z},\mathbf{T})$ for $n=3$, $m=5$ are given as

$$\mathbf{X}=\begin{bmatrix}0.27 & 3.91 & 53.24\\ 0.40 & 9.88 & 82.85\\ 0.69 & 13.95 & 89.29\\ 0.79 & 15.93 & 100.58\\ 2.75 & 27.80 & 215.10\end{bmatrix},\ \mathbf{Y}=\begin{bmatrix}4.15 & 33.91 & 8.27\\ 4.67 & 7.35 & 36.71\\ 72.89 & 0.78 & 2.78\\ 31.75 & 4.85 & 0.19\\ 0.96 & 3.16 & 8.01\end{bmatrix},\ \mathbf{Z}=\begin{bmatrix}4.42 & 37.82 & 61.51\\ 5.07 & 17.23 & 119.56\\ 73.58 & 14.73 & 92.07\\ 32.54 & 20.78 & 100.77\\ 3.71 & 30.96 & 223.11\end{bmatrix},\ \mathbf{T}=\begin{bmatrix}1.97\\ 3.99\\ 0.99\\ 2.58\\ 25.50\end{bmatrix}.$$

We check whether data sets **X**, **Y** and **T** come from the exponential distribution or not. Kolmogorov-Smirnov (K-S), Anderson-Darling (A-D) and Cramer-von Mises (C-VM) tests are carried out for the goodness-of-fit test. Test results are listed in Table 3. It is observed that the exponential distribution provides a good fit to these data sets.

TABLE 1. Estimates of R for $(\alpha,\beta,\theta)=(8,2,1.5)$ and $(12,7,3)$

			MLE			*MLE2*			*Bayes* (Inf. prior)			*Bayes* (Non-inf. prior)		
n	m	R	$\widehat{R}$	Bias	MSE	$\widehat{R}$	Bias	MSE	$\widehat{R}$	Bias	ER	$\widehat{R}$	Bias	ER
						$(\alpha,\beta,\theta)=(8,2,1.5)$								
5	25	0.80313	0.80398	0.00084	0.00344	0.80391	0.00078	0.00342	0.79163	-0.011507	0.00318	0.79527	-0.00787	0.00364
	50		0.80270	-0.00043	0.00170	0.80287	-0.00026	0.00169	0.79804	-0.00510	0.00155	0.79866	-0.00448	0.00173
	75		0.80264	-0.00049	0.00114	0.80274	-0.00040	0.00114	0.79972	-0.00342	0.00107	0.80000	-0.00313	0.00115
	100		0.80410	0.00097	0.00088	0.80423	0.00109	0.00088	0.80197	-0.00117	0.00083	0.80213	-0.00100	0.00088
10	25	0.87624	0.87350	-0.00273	0.00227	0.87382	-0.00242	0.00221	0.86402	-0.01222	0.00201	0.86437	-0.01187	0.00252
	50		0.87483	-0.00141	0.00114	0.87523	-0.00101	0.00111	0.87036	-0.00588	0.00105	0.87032	-0.00591	0.00118
	75		0.87564	-0.00060	0.00074	0.87582	-0.00041	0.00073	0.87261	-0.00363	0.00070	0.87263	-0.00361	0.00076
	100		0.87605	-0.00019	0.00056	0.87617	-0.00007	0.00056	0.87376	-0.00248	0.00054	0.87379	-0.00245	0.00057
15	25	0.90685	0.90550	-0.00134	0.00159	0.90606	-0.00079	0.00154	0.89698	-0.00987	0.00141	0.89670	-0.01015	0.00177
	50		0.90531	-0.00153	0.00080	0.90555	-0.00130	0.00079	0.90093	-0.00591	0.00076	0.90084	-0.00600	0.00085
	75		0.90531	-0.00154	0.00055	0.90549	-0.00135	0.00055	0.90237	-0.00448	0.00053	0.90231	-0.00454	0.00058
	100		0.90677	-0.00008	0.00041	0.90677	-0.00007	0.00040	0.90450	-0.00235	0.00039	0.90452	-0.00233	0.00042
					$(\alpha,\beta,\theta)=(12,7,3)$									
12	10	0.78932	0.80325	0.01392	0.00841	0.79725	0.00792	0.00856	0.73402	-0.05530	0.00897	0.76772	-0.02161	0.01178
	20		0.79812	0.00880	0.00427	0.79439	0.00506	0.00429	0.76829	-0.02104	0.00409	0.77702	-0.01231	0.00586
	40		0.79243	0.00310	0.00239	0.79050	0.00117	0.00238	0.77949	-0.00984	0.00238	0.78039	-0.00894	0.00315
	60		0.79170	0.00237	0.00158	0.79061	0.00128	0.00157	0.78344	-0.00589	0.00150	0.78288	-0.00645	0.00201
16	10	0.81363	0.82180	0.00817	0.00801	0.81633	0.00270	0.00821	0.76609	-0.04753	0.00735	0.78599	-0.02763	0.01130
	20		0.81862	0.00499	0.00420	0.81552	0.00189	0.00425	0.79410	-0.01954	0.00349	0.79753	-0.01610	0.00577
	40		0.81374	0.00011	0.00223	0.81211	-0.00152	0.00225	0.80219	-0.01144	0.00215	0.80139	-0.01224	0.00314
	60		0.81514	0.00150	0.00150	0.81416	0.00053	0.00147	0.80773	-0.00590	0.00139	0.80720	-0.00643	0.00182
20	10	0.83059	0.83843	0.00784	0.00666	0.83377	0.00318	0.00677	0.79058	-0.04001	0.00551	0.80129	-0.02930	0.01063
	20		0.83485	0.00426	0.00375	0.83201	0.00142	0.00377	0.81131	-0.01927	0.00340	0.81240	-0.01818	0.00616
	40		0.83167	0.00109	0.00197	0.83044	-0.00014	0.00196	0.82120	-0.00939	0.00173	0.82040	-0.01019	0.00248
	60	0.83059	0.83121	0.00618	0.00138	0.83055	-0.00003	0.00137	0.82407	-0.00651	0.00135	0.82320	-0.00739	0.00177

TABLE 2. Average lengths and coverage probabilities of R for $(\alpha,\beta,\theta)=(8,2,1.5)$ and $(12,7,3)$

n	m	R	AL	CP	n	m	R	AL	CP
5	25	0.80314	0.22488	0.92200	12	10	0.78933	0.35302	0.87320
	50		0.16184	0.93560		20		0.26235	0.91960
	75		0.13273	0.94360		40		0.19079	0.93040
	100		0.11477	0.93840		60		0.15704	0.93680
10	25	0.87624	0.18283	0.91240	16	10	0.81363	0.33748	0.86760
	50		0.13070	0.93320		20		0.25054	0.90360
	75		0.10691	0.93600		40		0.18285	0.92520
	100		0.09266	0.93520		60		0.14982	0.93240
15	25	0.90685	0.15545	0.90200	20	10	0.83059	0.32468	0.87360
	50		0.11211	0.93240		20		0.24059	0.90520
	75		0.09208	0.94520		40		0.17522	0.93720
	100		0.07921	0.94120		60		0.14416	0.93400

In this case, the MLE of the parameters are obtained as $(\widehat{\alpha},\widehat{\beta},\widehat{\theta})=(0.0183,0.8230,0.1427)$ for our model. The estimates of R are listed in Table 4 based on these ML estimates. Bayes estimates of R are obtained under three different priors like as Prior 1: $a_i=b_i=2$, $i=1,2,3$, Prior 2: $a_1=0.5084$, $b_1=0.0123$, $a_2=0.5539$, $b_2=0.0377$, $a_3=0.5677$, $b_3=0.0810$ and Prior 3: $a_i=b_i=0$, $i=1,2,3$. The hyperparameters in Prior 2 are obtained by using moment estimation of Gamma distribution based on the data **X**, **Y** and **T**. This method can be preferable when the prior information is not available.

TABLE 3. Goodness-of-fit test for the real data set

Data	*MLE*	$K-S$	$p-value$	$A-D$	$p-value$	$C-VM$	$p-value$
X	$\widehat{\alpha}$ =0.0243	0.3094	0.0900	3.7203	0.0123	0.4030	0.0697
Y	$\widehat{\beta}$ =0.0680	0.3030	0.1020	1.5471	0.1659	0.2900	0.1439
T	$\widehat{\theta}$ =0.1427	0.3658	0.4134	0.7282	0.5276	0.1336	0.4562

TABLE 4. Estimates of R for the real data set

(n,m)	MLE	MLE2	Lindley (Prior 1)	Lindley (Prior 2)	Lindley (Prior 3)
(3,5)	0.99459	0.99801	0.93044	0.91603	0.94869
	(0.98770,1.00)	(0.98721,1.00)			

5. Conclusion

In this paper, the estimation problem has been considered for the stress-strength reliability of the parallel system when the cold standby redundancy available. It is assumed that stress, strength and standby components come from the exponential distribution. Bayes estimate is approximated by using Lindley's approximation under two different priors, and compared with the maximum likelihood estimates. It is observed that when the prior information is available, the estimated risk of Bayes estimate is smaller than risks of the other estimates. When the prior information is not available, the estimated risks of two maximum likelihood and Bayes estimates are closing to each other as sample size increases.

It is known that the convolution of the independent and non-identical random variables generally has mixed forms. Hence, the total lifetime of the strength component and its corresponding standby component for the other lifetime distributions will be more complicated for this reliability problem. We hope to report our results in this regard in the near future.

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