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Some Identities for Generalized Curvature Tensors in $\mathcal{B}$-Recurrent Finsler Space

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Abstract − The purpose of the present paper is to consider and study a certain identities for some generalized curvature tensors in $\mathcal{B}$-recurrent Finsler space F_n in which Cartan's second curvature tensor P^i_{jkh} satisfies the generalized of recurrence condition with respect to Berwald's connection parameters G^i_{kh} which given by the condition $\mathcal{B}_m P^i_{jkh} = \lambda_m P^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh})$, where $\mathcal{B}_m$ is covariant derivative of first order (Berwald's covariant differential operator) with respect to x^m, it's called a generalized $\mathcal{B}$P-recurrent space. We shall denote it briefly by $G\mathcal{B}P\text{-}RF_n$. We have obtained Berwald's covariant derivative of first order for the h(v)-torsion tensor P^i_{kh}, the deviation tensor P^i_h and the covariant derivative of the tensor $H_{kp.h}$ (in the sense of Berwald), also we find some theorems of the R-Ricci tensor R_{jk} and the curvature vector R_j in our space. We obtained the necessary and sufficient condition for Berwald's covariant derivative of Weyl's projective curvature tensor W^i_{jkh} and its torsion tensor W^i_{kh} in our space. Also, we have proved that in $G\mathcal{B}P\text{-}RF_n$, Cartan's second curvature tensor P^i_{jkh} and the v(hv)-torsion tensor P^i_{kh} for $n = 4$.

Keywords − *Finsler space, Cartan's second curvature tensor P^i_{jkh}, Generalized $\mathcal{B}P$-recurrent space, Weyl's projective curvature tensor W^i_{jkh} , Cartan's fourth curvature tensor R^i_{jkh}*

1. Introduction

The generalized recurrent space characterized by different curvature tensors and used the sense of Berwald studied by Pandey et al. [1], and Ahsan and Ali [2], studied the properties of W-curvature tensor and its applications. The concept of the recurrent for different curvature tensors have been discussed by Qasem [3] and Matsumoto [4], they studied the generalized birecurrent of first and second kind, also studied the special birecurrent of first and second kind and W^i_{jkh} generalized birecurrent Finsler space studied by Qasem and Saleem [5]. The generalized birecurrent space was studied by Hadi [6], Qasem and Abdallah [7], Qasem and Saleem [8], Abdallah [9], Qasem and Abdallah [10-12], Qasem and Baleedi [13,14]. The generalized birecurrent Finsler space studied by Qasem [15]. F.Y.A. Qasem et al. [16] studied of GR^h-TRI affinely.

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Consider an n-dimensional Finsler space, Fig. 1., equipped with the metric function F satisfies the requisite conditions [16]. Let consider the components of the corresponding metric tensor g_{ij}, Cartan's connection parameters Γ_{jk}^{*i} and Berwald's connection parameters G_{jk}^{i} *1 .
These are symmetric in their lower indices.

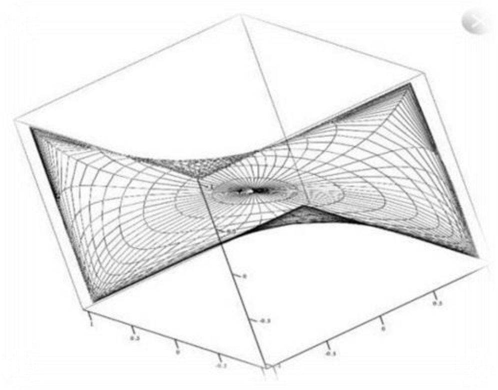

Fig.1. The figure for Finsler Space as a Locally Minkowskian Space

The vectors y_i and y^i satisfy the following relations [16]

$$\text{a) } y_i = g_{ij}\, y^j \text{ and b) } y_i\, y^i = F^2 \#(1.1)$$

The two sets of quantities g_{ij} and its associate tensor g^{ij} are related by [16]

$$g_{ij}\, g^{jk} = \delta_i^k = \begin{cases} 1 \ , & if \quad i = k \\ 0 \ , & if \quad i \neq k \end{cases} \#(1.2)$$

The tensor C_{ijk} defined by

$$C_{ijk} = \frac{1}{2}\, \dot{\partial}_i\, g_{jk} = \frac{1}{4}\, \dot{\partial}_i\, \dot{\partial}_j\, \dot{\partial}_k\, F^2 \#(1.3)$$

is known as (h) hv - torsion tensor [16].

The (v) hv-torsion tensor C_{ik}^h and its associate (h) hv-torsion tensor C_{ijk} are related by

$$\begin{array}{lll} \text{a) } C_{jk}^i\, y^j = C_{kj}^i\, y^j = 0, & \text{b) } y_i\, C_{jk}^i = 0, & \text{c) } C_{ijk}\, y^j = 0 \\ \text{d) } g_{rj}\, C_{ik}^r = C_{ijk}\ , & \text{e) } C_{jk}^i\ g^{jk} = C^i, & \text{f) } C_{ijk}\, g^{jk} = C_i \end{array} \quad (1.4)$$

Berwald's covariant derivative $\mathcal{B}_k T_j^i$ of an arbitrary tensor filed T_j^i with respect to x^k is given by

$$\mathcal{B}_k\, T_j^i := \partial_k\, T_j^i - \left(\dot{\partial}_r\, T_j^i\right) G_k^r + T_j^r\, G_{rk}^i - T_r^i\, G_{jk}^r \quad (1.5)$$

Berwald's covariant derivative of the metric function, the vector y^i and the unit vector l^i vanish identically [16], i.e.

$$\text{a) } \mathcal{B}_k\, y^i = 0, \text{b) } \mathcal{B}_k\, F = 0 \text{ and c) } \mathcal{B}_k\, y_i = 0 \quad (1.6)$$

But Berwald's covariant derivative of the metric tensor g_{ij} doesn't vanish, i.e. $\mathcal{B}_k\, g_{ij} \neq 0$ and given by

$$\mathcal{B}_k g_{ij} = -2\, C_{ijk|h}\, y^h = -2\, y^h\, \mathcal{B}_h\, C_{ijk} \quad (1.7)$$

Berwald's covariant differential operator with respect to x^h commutes with partial differential operator with respect to y^k, according to [16]

$$\left(\dot{\partial}_k\, \mathcal{B}_h - \mathcal{B}_h\, \dot{\partial}_k\right) T_j^i = T_j^r G_{khr}^i - T_r^i\, G_{khj}^r \quad (1.8)$$

where T_j^i is any arbitrary tensor field.

*1 The indices $i, j, k, \ldots$ assume positive integral values from 1 to n.

The hv-curvature tensor P^i_{jkh} and the v(hv)–torsion tensor P^i_{kh} satisfy [16]

$$\begin{aligned} &\text{a) } P^i_{jkh}\, y^j = P^i_{kh}, \text{ b) } g_{ir}\, P^r_{jkh} = P_{ijkh} \\ &\text{c) } g_{rp}\, P^r_{kh} = P_{kph}, \text{ d) } P^i_{jki} = P_{jk} \\ &\text{e) } P^i_{ki} = P_k, \text{ f) } P^i_{kh}\, y^k = P^i_{kh}\, y^h = 0 \end{aligned} \tag{1.9}$$

also the hv-curvature tensor P^i_{jkh} is defined by

$$\text{a) } P^i_{jkh} = \Gamma^{*i}_{jkh} + C^i_{jr}\, P^r_{kh} - C^i_{jh|k} \tag{1.10}$$

or equivalent by

$$\text{b) } P^i_{jkh} = \dot{\partial}_h\, \Gamma^{*i}_{jk} + C^i_{jr}\, C^r_{kh|s}\, y^s - C^i_{jh|k}$$

or

$$\text{c) } P^i_{jkh} = C^i_{kh|j} - C_{jkh|r}\, g^{ir} + C^r_{jk}\, P^i_{rh} - P^r_{jh}\, C^i_{rk}$$

where

$$\text{a) } \Gamma^{*i}_{jkh}\, y^j = P^i_{kh}, \qquad \text{b) } \Gamma^{*i}_{jkh}\, y^k = 0 \text{ and} \qquad \text{c) } y_i\, \Gamma^{*i}_{kjh} = -\, P_{kjh} \tag{1.11}$$

The projective curvature tensor W^i_{jkh} is known as (Wely's projective curvature tensor), the projective torsion tensor W^i_{jk} is known as (Wely's torsion tensor) and the projective deviation tensor W^i_j is known as (Wely's deviation tensor) are defined by

$$W^i_{jkh} = H^i_{jkh} + \frac{2\, \delta^i_j}{n+1} H_{[hk]} + \frac{2\, y^i}{n+1} \dot{\partial}_j H_{[kh]} + \frac{\delta^i_k}{n^2-1}\left(n\, H_{jh} + H_{hj} + y^r \dot{\partial}_j H_{hr}\right) - \frac{\delta^i_h}{n^2-1}\left(n\, H_{jk} + H_{kj} + y^r \dot{\partial}_j H_{kr}\right) \tag{1.12}$$

$$W^i_{jk} = H^i_{jk} + \frac{y^i}{n+1} H_{[jk]} + 2\, \{ \frac{\delta^i_{[j}}{n^2-1}\left(n\, H_{k]} - y^r H_{k]\,r}\right) \tag{1.13}$$

and

$$W^i_j = H^i_j - H\, \delta^i_j - \frac{1}{n+1}\left(\dot{\partial}_r H^r_j - \dot{\partial}_j H\right) y^i \tag{1.14}$$

respectively.

The tensors W^i_{jkh}, W^i_{jk} and W^i_k are satisfying the following identities [16]

$$\text{a) } W^i_{jkh}\, y^j = W^i_{kh} \text{ and b) } W^i_{jk}\, y^j = W^i_k \tag{1.15}$$

The projective curvature tensor W^i_{jkh} is skew-symmetric in its indices k and h.

Cartan's third curvature tensor R^i_{jkh} , Fig.2., and the R-Ricci tensor R_{jk} in sense of Cartan, respectively, given by [16]

$$\begin{aligned} &\text{a) } R^i_{jkh} = \Gamma^{*i}_{hjk} + \left(\Gamma^{*i}_{ljk}\right) G^l_h + C^i_{jm}\left(G^m_{kh} - G^m_{kl}\, G^l_h\right) + \Gamma^{*i}_{mk}\, \Gamma^{*m}_{jh} - k/h \\ &\text{b) } R^i_{jkh}\, y^j = H^i_{kh}, \text{ c) } R_{jk}\, y^j = H_k \\ &\text{d) } R_{jk}\, y^k = R_j, \text{ e) } R^i_{jki} = R_{jk} \end{aligned} \tag{1.16}$$

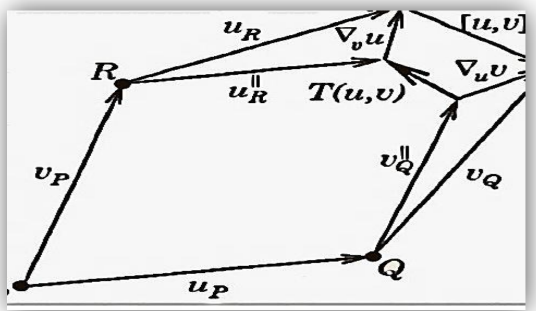

Fig.2. The figure for Covariant Derivative for Cartan's Torsion in Geometrical

Berwald curvature tensor H^i_{jkh} and h(v)–torsion tensor H^i_{kh} form the components of tensors are defined as follow [16]

$$\text{a) } H^i_{jkh} \coloneqq \partial_j G^i_{kh} + G^r_{kh} G^i_{rj} + G^i_{rhj} G^r_k - \frac{h}{k} \tag{1.17}$$

and

$$\text{b) } H^i_{kh} \coloneqq \partial_h G^i_k + G^r_k C^i_{rh} - h/k$$

They are also related by [16]

$$\text{a) } H^i_{jkh}\, y^j = H^i_{kh}, \text{ b) } H^i_{jkh} = \dot{\partial}_j H^i_{kh} \text{ and c) } H^i_{jk} = \dot{\partial}_j H^i_k \tag{1.18}$$

These tensors were constructed initially by means of the tensor H^i_h , called the deviation tensor, given by

$$\text{a) } H^i_h \coloneqq 2\, \partial_h G^i - \partial_r G^i_h\, y^r + 2\, G^i_{hs} G^s - G^i_s G^s_h \tag{1.19}$$

where

$$\text{b) } \dot{\partial}_k G^i_h = G^i_{kh}$$

In view of Euler's theorem on homogeneous functions and by contracting the indices i and h in (1.18) and (1.19), we have the following:

$$\text{a) } H^i_{jk}\, y^j = H^i_k \text{ , b) } g_{ip} H^i_{jk} = H_{jp.k} \text{ and c) } H_i\, y^i = (n-1)H \tag{1.20}$$

2. A Generalized $\mathcal{BR}$-Recurrent Space

Cartan's second curvature tensor P^i_{jkh} satisfies the condition

$$\mathcal{B}_n P^i_{jkh} = \lambda_n P^i_{jkh} \text{ , } P^i_{jkh} \neq 0 \tag{2.1}$$

is called a recurrent Finsler space, where λ_n is non-zero covariant vectors field.
A Finsler space F_n whose the curvature tensor P^i_{jkh} satisfies the condition

$$\mathcal{B}_m P^i_{jkh} = \lambda_m P^i_{jkh} + \mu_m\left(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\right), \; P^i_{jkh} \neq 0 \tag{2.2}$$

where $\mathcal{B}_m$ is covariant derivative of first order (Berwald's covariant differential operator) with respect to x^m, the quantities λ_m and μ_m are non-null covariant vectors field. It is called such space as *a generalized $\mathcal{B}P$-recurrent space*, he denoted it briefly by $G\mathcal{B}P\text{-}RF_n$.

Definition 2.1. A Finsler space F_n whose Cartan's second curvature tensor P^i_{jkh} satisfies the condition (2.2), where λ_m and μ_m are non-null covariant vectors field, it's called *a generalized $\mathcal{B}P$-recurrent space*. We shall denote it briefly by $G\mathcal{B}P\text{-}RF_n$.

Transvecting the condition (2.2) by y^j, using (1.6a), (1.9a), (1.1a) and (1.4c), we get

$$\mathcal{B}_m P^i_{kh} = \lambda_m P^i_{kh} + \mu_m\left(\delta^i_h\, y_k - \delta^i_k\, y_h\right) \tag{2.3}$$

Contracting the indices i and h in (2.2) and (2.3), using (1.9d), (1.9e) and in view of (1.2), we get

$$\mathcal{B}_m P_{jk} = \lambda_m P_{jk} + \mu_m (n-1)\, g_{jk} \tag{2.4}$$

$$\mathcal{B}_m P_k = \lambda_m P_k + \mu_m (n-1)\, y_k \tag{2.5}$$

Therefore, using the above assumptions and mathematical analysis results the following theorem have been derived.

Theorem 2.1. In G$\mathcal{B}$P-RF$_n$, v(hv)-torsion tensor P^i_{kh} , P-Ricci tensor P_{jk} and the curvature vector P_k (for Cartan's second curvature tensor P^i_{jkh}) are given by (2.3),(2.4) and (2.5), respectively.

Trausvecting (2.2) and (2.3) by g_{ir} , using (1.9b), (1.9c), (1.7) and in view of (1.2), we get

$$\mathcal{B}_m P_{jkrh} = \lambda_m P_{jkrh} + \mu_m \left(g_{rh}\, g_{jk} - g_{rk}\, g_{jh} \right) - 2\, y^n \mathcal{B}_n C_{irm}\, P^i_{jkh} \tag{2.6}$$

$$\mathcal{B}_m P_{krh} = \lambda_m P_{krh} + \mu_m (g_{hr}\, y_k - g_{kr}\, y_h) - 2\, y^n \mathcal{B}_n C_{irm}\, P^i_{kh} \tag{2.7}$$

Therefore, we have

Theorem 2.2. In G$\mathcal{B}$P-RF$_n$, the associate curvature tensor P_{ijkh} of the (hv)-curvature tensor P^i_{jkh} and the associate tensor P_{jkh} of v(hv)-torsion tensor P^i_{kh} (for Cartan's second curvature tensor P^i_{jkh}) is given by the equations (2.6) and (2.7), respectively.

Taking covariant derivative (Berwald's covariant differential operator) of the equation (1.10a) with respect to x^m and using condition (2.2), yields

$$\lambda_m P^i_{jkh} + \mu_m (\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}) = \mathcal{B}_m \left(\Gamma^{*i}_{jkh} + C^i_{jr}\, P^r_{kh} - C^i_{jh|k} \right)$$

By using the equation (1.10a), the above equation can be written as

$$\mathcal{B}_m \left(\Gamma^{*i}_{jkh} + C^i_{jr}\, P^r_{kh} - C^i_{jh|k}\right) = \lambda_m \left(\Gamma^{*i}_{jkh} + C^i_{jr}\, P^r_{kh} - C^i_{jh|k}\right) + \mu_m (\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}) \tag{2.8}$$

Equation (2.8) shows that the tensor $\left(\Gamma^{*i}_{jkh} + C^i_{jr}\, P^r_{kh} - C^i_{jh|k} \right)$ can't vanish, because the vanishing of it would implies the vanishing of the covariant vector field μ_m , i.e. $\mu_m = 0$, a contradiction.
Thus, it is concluded the following.

Theorem 2.3. In G$\mathcal{B}$P-RF$_n$, the tensor $\left(\Gamma^{*i}_{jkh} + C^i_{jr}\, P^r_{kh} - C^i_{jh|k} \right)$ is non-vanishing and this tensor is generalized recurrent.

Transvecting equation (2.8) by y^j, using equations (1.6a), (1.9f), (1.1a) and (1.4c), we get the same equation (2.3).

Further, transvecting (2.8) by y_i , using equations (1.6c), (1.9g), (1.1a) and (1.4c), we get the same equation (2.6).

Now, transvecting equation (2.8) by y^k, using equations (1.6a), (1.11b), (1.9f), (1.1a), (1.4c) and in view of (1.2), we get

$$\mathcal{B}_m \left(C^i_{jh|k}\, y^k \right) = \lambda_m \left(C^i_{jh|k}\, y^k \right) + \mu_m (\delta^i_h\, y_j - g_{jh}\, y^i) \tag{2.9}$$

Transvecting (2.9) by g_{ir} , using (1.4d), (1.1a), (1.7) and in view of (1.2), we get

$$\mathcal{B}_m \left(C_{jrh|k}\, y^k\right) = \lambda_m \left(C_{jrh|k}\, y^k\right) + \mu_m (g_{rh}\, y_j - g_{jh}\, y_r) - 2\, y^n \mathcal{B}_n C_{irm} \left(C^i_{jh|k} y^k\right) \tag{2.10}$$

Therefore, it is concluded the following.

Theorem 2.4. In G$\mathcal{B}$P-RF$_n$, we have the identities (2.9) and (2.10).

Trausvecting (2.9) and (2.10) by g^{jh}, using (1.4e), (1.4f), (1.1a) and in view of (1.2), we get

$$\mathcal{B}_m \left(C^i_{|k}\, y^k\right) = \lambda_m \left(C^i_{|k}\, y^k\right) \# (2.11)$$

$$\mathcal{B}_m \left(C_{r|k}\, y^k\right) = \lambda_m \left(C_{r|k}\, y^k\right) - 2\, y^n \mathcal{B}_n C_{irm} \left(C^i_{|k}\, y^k\right), \text{where } \mathcal{B}_m\, g^{jh} = 0 \tag{2.12}$$

Therefore, we have

Theorem 2.5. In G$\mathcal{B}$P-RF$_n$, the tensor $\left(C^i_{|k}\, y^k\right)$ is recurrent and the tensor $\left(C_{r|k}\, y^k\right)$ is given by the equation (2.11).

3. The Certain Identities for Curvature Tensor P^i_{jkh}

In this section we shall obtain certain identities for some tensors to be generalized recurrent in our space of $G\mathcal{B}R\text{-}TRF_n$.

For a Riemannian space V$_4$, the projective curvature tensor P^i_{jkh} (Cartan's second curvature tensor) and the divergence of W-tensor in terms of the divergence of projective curvature tensor can be expressed as [9]

$$W^i_{jkh} = P^i_{jkh} + \frac{1}{3}\left(\delta^i_k R_{jh} - R^i_h g_{jk}\right) \tag{3.1}$$

Taking covariant derivative of first order (Berwald's covariant differential operator) of (3.1) with respect to x^m, we get

$$\mathcal{B}_m W^i_{jkh} = \mathcal{B}_m P^i_{jkh} + \frac{1}{3}\mathcal{B}_m\left(\delta^i_k R_{jh} - R^i_h g_{jk}\right) \tag{3.2}$$

Using the condition (2.2) in (3.2), we get

$$\mathcal{B}_m W^i_{jkh} = \lambda_m P^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \frac{1}{3}\mathcal{B}_m\left(\delta^i_k R_{jh} - R^i_h g_{jk}\right)$$

In view of equation (3.1) and by using (1.7), the above equation can be written as

$$\begin{aligned}\mathcal{B}_m W^i_{jkh} = \; & \lambda_m W^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) - \frac{1}{3}\lambda_m\left(\delta^i_k R_{jh} - R^i_h g_{jk}\right) \\ & + \frac{1}{3}\delta^i_k \mathcal{B}_m R_{jh} - \frac{1}{3}\left(\mathcal{B}_m R^i_h\right) g_{jk} + \frac{2}{3} R^i_h\, y^n \mathcal{B}_n C_{jkm}\end{aligned} \tag{3.3}$$

This, shows that

$$\mathcal{B}_m W^i_{jkh} = \lambda_m W^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right)$$

if and only if

$$\delta^i_k \mathcal{B}_m R_{jh} - \lambda_m\left(\delta^i_k R_{jh} - R^i_h g_{jk}\right) - \left(\mathcal{B}_m R^i_h\right) g_{jk} + 2 R^i_h\, y^n \mathcal{B}_n C_{jkm} = 0 \tag{3.4}$$

Therefore, using the above assumptions and mathematical analysis results the following theorem have been derived.

Theorem 3.1. In G$\mathcal{B}$P-RF$_n$ (for $n = 4$), Berwald's covariant derivative of the first order for Weyl's projective curvature tensor W^i_{jkh} is generalized recurrent if and only if (3.4) holds.

Transvecting (3.3) by y^j , using (1.6a), (1.15a), (1.1a), (1.16c) and (1.4c), yields

$$\mathcal{B}_m W^i_{kh} = \lambda_m W^i_{kh} + \mu_m\left(\delta^i_h y_k - \delta^i_k y_h\right) - \frac{1}{3}\lambda_m\left(\delta^i_k H_h - R^i_h H_k\right) + \frac{1}{3}\delta^i_k \mathcal{B}_m H_h - \frac{1}{3}\left(\mathcal{B}_m R^i_h\right) y_k \tag{3.5}$$

This, shows that

$$\mathcal{B}_m W^i_{kh} = \lambda_m W^i_{kh} + \mu_m\left(\delta^i_h y_k - \delta^i_k y_h\right) \tag{3.6}$$

if and only if

$$\delta^i_k \mathcal{B}_m H_h - \lambda_m\left(\delta^i_k H_h - R^i_h H_k\right) - \left(\mathcal{B}_m R^i_h\right) y_k = 0 \tag{3.7}$$

Therefore, it is concluded the following theorem

Theorem 3.2. In G$\mathcal{B}$P-RF$_n$ (for $n = 4$), Berwald's covariant derivative of the first order for Weyl's projective torsion tensor W^i_{kh} is given by the equation (3.6) if and only if (3.7) holds.

Transvecting (3.5) by y^k, using (1.6a), (1.15b), (1.1b), (1.2) and (1.20c), we get

$$\mathcal{B}_m W^i_h = \lambda_m W^i_h + \mu_m\left(\delta^i_h F^2 - y_h y^i\right) - \frac{1}{3}\lambda_m\left(H_h y^i - (n-1) R^i_h H\right) + \frac{1}{3} y^i \mathcal{B}_m H_h - \frac{1}{3}\left(\mathcal{B}_m R^i_h\right) F^2$$

This, shows that

$$\mathcal{B}_m W^i_h = \lambda_m W^i_h + \mu_m\left(\delta^i_h F^2 - y_h y^i\right) \tag{3.8}$$

if and only if

$$y^i \mathcal{B}_m H_h - \lambda_m\left(H_h y^i - (n-1) R^i_h H\right) - \left(\mathcal{B}_m R^i_h\right) F^2 = 0 \tag{3.9}$$

Thus, the following is derived.

Theorem 3.3. In G$\mathcal{B}$P-RF$_n$ (for $n = 4$), Berwald's covariant derivative of the first order for Weyl's projective deviation tensor W^i_h is given by the equation (3.8) if and only if (3.9) holds.

Also, the projective curvature tensor P^i_{jkh} (for a Riemannian space V$_4$) is defined by [9]

$$P^i_{jkh} = R^i_{jkh} - \frac{1}{3}\left(\delta^i_h R_{jk} - \delta^i_k R_{jh}\right) \tag{3.10}$$

Taking covariant derivative of third order (Berwald's covariant differential operator) of (3.10) with respect to x^m, we get

$$\mathcal{B}_m P^i_{jkh} = \mathcal{B}_m R^i_{jkh} - \frac{1}{3}\left(\delta^i_h \mathcal{B}_m R_{jk} - \delta^i_k \mathcal{B}_m R_{jh}\right) \tag{3.11}$$

Using the condition (2.2) in (3.11), we get

$$\mathcal{B}_m R^i_{jkh} = \lambda_m P^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \frac{1}{3}\left(\delta^i_h \mathcal{B}_m R_{jk} - \delta^i_k \mathcal{B}_m R_{jh}\right)$$

By using (3.10), the above equation can be written as

$$\mathcal{B}_m R^i_{jkh} = \lambda_m R^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \frac{1}{3}\left(\delta^i_h \mathcal{B}_m R_{jk} - \delta^i_k \mathcal{B}_m R_{jh}\right) - \frac{1}{3}\lambda_m\left(\delta^i_h R_{jk} - \delta^i_k R_{jh}\right) \tag{3.12}$$

This, shows that

$$\mathcal{B}_m R^i_{jkh} = \lambda_m R^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right)$$

if and only if

$$\left(\delta^i_h \mathcal{B}_m R_{jk} - \delta^i_k \mathcal{B}_m R_{jh}\right) - \lambda_m\left(\delta^i_h R_{jk} - \delta^i_k R_{jh}\right) = 0 \tag{3.13}$$

Thus, it is concluded the following theorem

Theorem 3.4. In G$\mathcal{B}$P-RF$_n$ (for $n = 4$), Berwald's covariant derivative of the first order for Cartan's third curvature tensor R^i_{jkh} is generalized recurrent if and only if (3.13) holds.

Transvecting (3.12) by y^j, using (1.6a), (1.16b), (1.1a) and (1.16c), yields

$$\mathcal{B}_m H^i_{kh} = \lambda_m H^i_{kh} + \mu_m\left(\delta^i_h y_k - \delta^i_k y_h\right) + \frac{1}{3}\left(\delta^i_h \mathcal{B}_m y_k - \delta^i_k \mathcal{B}_m y_h\right) - \frac{1}{3}\lambda_m\left(\delta^i_h y_k - \delta^i_k y_h\right)$$

This, shows that

$$\mathcal{B}_m H^i_{kh} = \lambda_m H^i_{kh} + \mu_m\left(\delta^i_h y_k - \delta^i_k y_h\right) \tag{3.14}$$

if and only if

$$\left(\delta^i_h \mathcal{B}_m y_k - \delta^i_k \mathcal{B}_m y_h\right) - \lambda_m\left(\delta^i_h y_k - \delta^i_k y_h\right) = 0 \tag{3.15}$$

Further, transvecting (3.13) by y^k, using (1.6a), (1.20a), (1.1a), (1.1b) and (1.2), we get

$$\mathcal{B}_m H^i_h = \lambda_m H^i_h + \mu_m\left(\delta^i_h F^2 - y_h y^i\right) + \frac{1}{3}\left(\delta^i_h \mathcal{B}_m F^2 - y^i \mathcal{B}_m y_h\right) - \frac{1}{3}\lambda_m\left(\delta^i_h F^2 - y_h y^i\right) \tag{3.16}$$

This, shows that

$$\mathcal{B}_m H_h^i = \lambda_m H_h^i + \mu_m(\, \delta_h^i \, F^2 - \, y_h \, y^i \,) \tag{3.17}$$

if and only if

$$(\, \delta_h^i \, \mathcal{B}_m \, F^2 - y^i \, \mathcal{B}_m \, y_h \,) - \lambda_m(\, \delta_h^i \, F^2 - \, y_h \, y^i \,) = 0 \tag{3.18}$$

Transvecting (3.13) by g_{ip} , using (1.7), (1.6a), (1.20b) and (1.2), yields

$$\begin{aligned} \mathcal{B}_m H_{kp.h} \;=\; & \lambda_m \, H_{kp.h} \; + \mu_m(g_{hp} \, y_k - g_{kp} \, y_h) - 2 \, H_{kh}^i \;\, y^n \, \mathcal{B}_n \, C_{ipm} \\ & + \frac{1}{3}(\, g_{hp} \, \mathcal{B}_m y_k - g_{kp} \, \mathcal{B}_m \, y_h \,) - \frac{1}{3} \, \lambda_m(\, g_{hp} \, y_k - g_{kp} \, y_h \,) \end{aligned} \tag{3.19}$$

This, shows that

$$\mathcal{B}_m H_{kp.h} \; = \lambda_m \, H_{kp.h} \; + \mu_m(g_{hp} \, y_k - g_{kp} \, y_h) \tag{3.20}$$

if and only if

$$(\, g_{hp} \, \mathcal{B}_m y_k - g_{kp} \, \mathcal{B}_m \, y_h \,) - \lambda_m(\, g_{hp} \, y_k - g_{kp} \, y_h \,) - 6 \, H_{kh}^i \;\, y^n \, \mathcal{B}_n \, C_{ipm} = 0 \tag{3.21}$$

Therefore, using the above assumptions and mathematical analysis results the following theorem have been derived.

Theorem 3.5. In G$\mathcal{B}$P-RF$_\mathrm{n}$ (for $n = 4$), Berwald's covariant derivative of the first order for the h(v)-torsion tensor H_{kh}^i , the deviation tensor H_h^i and the tensor $H_{kp.h}$ are given by the equations (3.14), (3.17) and (3.20), respectively, if and only if (3.15) (3.18) and (3.21)holds.

Contracting the indices i and h in (3.12), using (1.16e) and in view of (1.2), we get

$$\mathcal{B}_m \, R_{jk} = \lambda_m R_{jk} + (n \, - 1) \, \mu_m \, g_{jk} + \frac{1}{3}(\, n \, - 1 \,) \, \mathcal{B}_m R_{jk} - \frac{1}{3}(\, n - 1) \, \lambda_m \, R_{jk} \tag{3.22}$$

This, shows that

$$\mathcal{B}_m \, R_{jk} = \lambda_m R_{jk} + (n \, - 1) \, \mu_m \, g_{jk}$$

if and only if

$$\mathcal{B}_m R_{jk} = \; \lambda_m \, R_{jk} \tag{3.23}$$

Therefore, it is concluded the following.

Theorem 3.6. In G$\mathcal{B}$P-RF$_\mathrm{n}$ (for $n = 4$), Berwald's covariant derivative of the first order for the R-Ricci tensor R_{jk} is non- vanishing if and only if R-Ricci tensor R_{jk} is recurrent.

Transvecting (3.22) by y^j, using (1.6a), (1.16c) and (1.1a), yields

$$\mathcal{B}_m \, H_k = \lambda_m H_k + (n \, - 1) \, \mu_m \, y_k + \frac{1}{3}(\, n \, - 1 \,) \, \mathcal{B}_m H_k - \frac{1}{3}(\, n - 1) \, \lambda_m \, H_k \tag{3.24}$$

This, shows that

$$\mathcal{B}_m \, H_k = \lambda_m H_k + (n \, - 1) \, \mu_m \, y_k \tag{3.25}$$

if and only if

$$\mathcal{B}_m H_k = \; H_k \tag{3.26}$$

Further, transvecting (3.22) by y^k , using (1.6a), (1.16d) and (1.1a), we get

$$\mathcal{B}_m \, R_j = \lambda_m R_j + (n \, - 1) \, \mu_m \, y_j + \frac{1}{3}(\, n \, - 1 \,) \, \mathcal{B}_m R_j - \frac{1}{3}(\, n - 1) \, \lambda_m \, R_j \tag{3.27}$$

This, shows that

$$\mathcal{B}_m \, R_j = \lambda_m R_j + (n \, - 1) \, \mu_m \, y_j \tag{3.28}$$

if and only if

$$\mathcal{B}_m R_j = \; \lambda_m \, R_j$$

Therefore, it is concluded the following.

Theorem 3.7. In G$\mathcal{B}$P-$\mathrm{RF_n}$ (for $n = 4$), Berwald's covariant derivative of the first order for the curvature vector H_k and the curvature vector R_j are non- vanishing if and only if the curvature vector H_k and the curvature vector R_j are recurrent.

4. Conclusion

A Finsler space is called generalized $\mathcal{B}$P-recurrent if it satisfies the condition (2.2).

In G$\mathcal{B}$P-$\mathrm{RF_n}$, Berwald's covariant derivative of the first order for v(hv)-torsion tensor P^i_{kh}, P-Ricci tensor P_{jk} and the curvature vector P_k (for Cartan's second curvature tensor P^i_{jkh}) are given by (2.3),(2.4) and (2.5), respectively.

In G$\mathcal{B}$P-$\mathrm{RF_n}$, the associate curvature tensor P_{ijkh} of the (hv)-curvature tensor P^i_{jkh} and the associate tensor P_{jkh} of v(hv)-torsion tensor P^i_{kh} (for Cartan's second curvature tensor P^i_{jkh}) is given by the equations (2.6) and (2.7), respectively also we have the identities (2.9) and (2.10).
In G$\mathcal{B}$P-$\mathrm{RF_n}$ (for $n = 4$), the necessary and sufficient condition of Weyl's projective curvature tensor W^i_{jkh} to be generalized recurrent are given by the equation (3.4).

In G$\mathcal{B}$R-$\mathrm{TRF_n}$ (for $n = 4$), the necessary and sufficient conditions of Berwald's covariant derivative of the first order for the torsion tensor W^i_{kh} , the deviation tensor W^i_h , the h(v)-torsion tensor H^i_{kh} , the deviation tensor H^i_h and the tensor $H_{kp.h}$ are given by equations (3.6), (3.8), (3.14), (3.17) and (3.20), respectively.
In G$\mathcal{B}$R-$\mathrm{TRF_n}$ (for $n = 4$), the necessary and sufficient conditions of Cartan's third curvature tensor R^i_{jkh} is generalized recurrent and given by equation (3.11).
Author recommend the need for continuing research and development in generalized $\mathcal{B}$P-recurrent spaces and interlard it with the properties of special spaces for Finsler space.

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