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Spherical Bipolar Fuzzy Sets and Its Application in Multi Criteria Decision Making Problem

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Abstract − This paper introduces the concept of Spherical Bipolar Fuzzy sets (SBFS) as a combination of Spherical Fuzzy sets and Bipolar Valued Fuzzy Sets along with their properties. Arithmetic operations involving addition, multiplication and subtraction are presented together with their proofs. A multi criteria decision making method is established in which the evaluation values of alternatives respective to criteria are represented in SBFS. Finally, a numerical example shows the application of the proposed method.

Keywords − *Intuitionistic fuzzy set (IFS), IFS2, neutrosophic fuzzy set (NFS), bipolar valued fuzzy set, spherical fuzzy set*

1.Introduction

Researchers have introduced many extensions and generalizations of fuzzy sets in the literature. It starts from ordinary fuzzy sets and extends to recently developed types of fuzzy sets [1-11].

Atanassov's intuitionistic fuzzy sets of second type (IFS2) are characterized by a membership degree and a non- membership degree satisfying the condition that the square sum of its membership degree and non-membership degree is equal to or less than one, which is a generalization of Intuitionistic Fuzzy Sets (IFS). The motivation of introducing IFS2 is that in the real-life decision process, the sum of the support (membership) degree and the against (non- membership) degree to which an alternative satisfying a criterion provided by the decision maker may be larger than 1 but their square sum is equal to or less than 1 [12,13]. Third dimension of IFS2 is hesitancy degree, which can be calculated by $\pi_{\tilde{p}} = \left(1 - \mu_{\tilde{p}}{}^2(u) - \vartheta_{\tilde{p}}{}^2(u)\right)^{1/2}$. In the recent years IFS2 have been employed in the solutions of multi-criteria decision making problems [14,15].

Similar to IFS2, Smarandache's neutrosophic sets (NS) are represented by the following three dimensions: a truthness degree, an indeterminacy degree, and a falsity degree. NS do not deal with the hesitancy of a system but also decrease indecisiveness of inconsistent information [1,3]. Also, various authors have given their contributions towards Neutrosophic using various new methods and ideas [16 -20].

Based on the analogy with the spherical coordinates of two points, Antonov [21] introduced the new stereo metrical interpretation of the IFS elements and gave the geometrical interpretation of the distance

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between two intuitionistic fuzzy points. Yang and Chiclana [9] proposed a spherical representation, which allowed us to define a distance function between intuitionistic fuzzy sets. They have showed that the spherical distance is different from the existing distances in that it is non- linear with respect to the change of the corresponding fuzzy membership degrees, and thus it seems more appropriate than usual linear distances for non- linear contexts in 3D spaces. Their work is just on the usage of IFS on a sphere. On the surface of a sphere, the following condition is proposed:

Let $\tilde{A} = \{\langle u, \mu_{\tilde{A}}(u), \vartheta_{\tilde{A}}(u) \rangle : u \in U\}$ be an intuitionistic fuzzy set. They have,

$$\mu_{\tilde{A}} + \vartheta_{\tilde{A}} + \pi_{\tilde{A}} = 1\#(1)$$

Which can be equivalently transformed to

$$x^2 + y^2 + z^2 = 1\#(2)$$

where, $x^2 = \mu_{\tilde{A}}(u)$, $y^2 = \vartheta_{\tilde{A}}(u)$, $z^2 = \pi_{\tilde{A}}(u)$.

In the spherical representation, hesitancy can be calculated based on the given membership and non-membership values since they only consider the surface of the sphere. Besides, they measure the spherical arc distance between two IFSs. Furthermore, Gong et al. [22] introduced an approach generalizing Yang and Chiclana's work. They applied the spherical distance measure to obtain the difference between two IFSs. They first introduced an ideal opinion and each individual opinion in group decision, they briefly constructed a non- linear optimization model.

The spherical fuzzy sets introduced in this paper are based on the fact that the hesitancy of a decision maker can be defined independently from membership and non-membership degrees, satisfying the following condition:

$$0 \le {\mu_{\tilde{A}}}^2(u) + {\vartheta_{\tilde{A}}}^2(u) + {\pi_{\tilde{A}}}^2(u) \le 1\#(3)$$

On the surface of the sphere, Equation (3) becomes

$${\mu_{\tilde{A}}}^2(u) + {\vartheta_{\tilde{A}}}^2(u) + {\pi_{\tilde{A}}}^2(u) = 1, \forall\, u \in U\#(4)$$

Since Yang and Chiclana [9] and Gong et al. [22] measure the arc distance on the surface of the sphere, Euclidean distance is not measured in these works. In our spherical fuzzy sets approach, the sphere is not solid but a spherical volume. Based on this fact, Euclidean distance measurement is meaningful. This also means that any two points within the spherical volume are also on the surface of another sphere. Euclidean distance gives the shortest distance between two points in the sphere.

Bipolar-valued fuzzy sets, which was introduced by Lee [24,25] is an extension of fuzzy sets whose membership degree range is extended from the interval $[0,1]$ to $[-1,1]$. The membership degrees of the Bipolar valued fuzzy sets signify the degree of satisfaction to the property analogous to a fuzzy set and its counter-property in a bipolar valued fuzzy set, if the membership degree is 0 it means that the elements are unrelated to the corresponding property. Furthermore, if the membership degree is on $(0,1]$ it indicates that the elements somewhat fulfil the property, and if the membership degree is on $[-1,0)$ it indicates that elements somewhat satisfy the entire counter property.

In this paper, we introduce the SFS with its basic operations such as addition, subtraction, multiplication and aggregation. The idea behind SFS is to let decision makers to generalize other extensions of fuzzy sets by defining a membership function on a spherical surface and independently assign the parameters of that membership function with a larger domain.

2. Preliminaries

Definition 2.1. [12,13] Let a set U be a universe of discourse. An IFS $\tilde{A}$ is an object having the form, $\tilde{A} = \{\langle u, \mu_{\tilde{A}}(u), \vartheta_{\tilde{A}}(u) \rangle \mid u \in U\}$ where the function $\mu_{\tilde{A}}: U \to [0,1], \vartheta_{\tilde{A}}: U \to [0,1]$ and $0 \le \mu_{\tilde{A}}(u) + \vartheta_{\tilde{A}}(u) \le 1$ are degree of membership, non-membership of u to $\tilde{A}$, respectively. For any IFS $\tilde{A}$ and $u \in U$, $\pi_{\tilde{A}} = 1 - \mu_{\tilde{A}}(u) - \vartheta_{\tilde{A}}(u)$ is called degree of hesitancy of u to $\tilde{A}$.

However, in the real life, the decision makers might express their preferences about membership degrees and non-membership degrees of an alternative with respect to a criterion dissatisfies the condition that the sum of the membership and non-membership degrees should be less than or equal to1.Instead of requiring the decision makers to alter their preference information, Atanassov [12] proposed a novel concept of IFS2 (intuitionistic fuzzy sets of second type) to model this situation. This concept provides a larger preference area for decision makers.

Definition 2.2. [12,13] Let a set U be a universe of discourse. An IFS2 $\tilde{P}$ is an object having the form, $\tilde{P} = \{\langle u, \mu_{\tilde{P}}(u), \vartheta_{\tilde{P}}(u) \rangle \mid u \in U\}$ where the function $\mu_{\tilde{P}}: U \to [0,1], \vartheta_{\tilde{P}}: U \to [0,1]$ and $0 \le \mu_{\tilde{P}}(u) + \vartheta_{\tilde{P}}(u) \le 1$ are degree of membership, non-membership of u to $\tilde{P}$, respectively.

For any IFS2 $\tilde{A}$ and $u \in U, \pi_{\tilde{P}} = 1 - \mu_{\tilde{P}}(u) - \vartheta_{\tilde{P}}(u)$ is called degree of hesitancy of u to $\tilde{P}$.

The novel concept of SFS (Spherical Fuzzy Sets) provides a larger preference domain for decision makers and DM can define their hesitancy information about of an alternative with respect to a criterion.

Intuitionistic and IFS2 fuzzy membership functions are composed of membership, non-membership and hesitancy parameters, which can be calculated by $\pi_{\tilde{I}} = 1 - \mu - \vartheta$ or $\pi_{\tilde{P}} = \left(1 - \mu_{\tilde{P}}{}^2(u) - \vartheta_{\tilde{P}}{}^2(u)\right)^{1/2}$, respectively. Neutrosophic membership functions are also defined by three parameters truthiness, falsity and indeterminacy, whose sum can be between 0 and 3, and the value of each is between 0 and 1 independently. In spherical fuzzy sets, while the squared sum of membership, non-membership and hesitancy parameters can be between 0 and, each of them can be defined between 0 and 1 independently.

Definition 2.3. [26] A Spherical Fuzzy Set (SFS) $\tilde{A}_s$ of the universe of discourse U is given by,

$$\tilde{A}_s = \{\langle \mu_{\tilde{A}_s}(u), \vartheta_{\tilde{A}_s}(u), \pi_{\tilde{A}_s}(u) | u \in U \rangle\} \#(5)$$

where, $\mu_{\tilde{A}_s}: U \to [0,1]$, $\vartheta_{\tilde{A}_s}: U \to [0,1]$, $\pi_{\tilde{A}_s}: U \to [0,1]$ and

$$0 \le \mu_{\tilde{A}_s}{}^2(u) + \vartheta_{\tilde{A}_s}{}^2(u) + \pi_{\tilde{A}_s}{}^2(u) \le 1, \forall\, u \in U \#(6)$$

For each u, the numbers $\mu_{\tilde{A}_s}(u), \vartheta_{\tilde{A}_s}(u)\ and\ \pi_{\tilde{A}_s}(u)$ are the degree of membership, non-membership and hesitancy of u to $\tilde{A}_s$, respectively.

Definition 2.4. [26] Basic operators of Spherical Fuzzy Sets:

Union:

$$\tilde{A}_s \cup \tilde{B}_s = \left\{ \begin{matrix} \max\{\mu_{\tilde{A}_s}, \mu_{\tilde{B}_s}\}, \min\{\vartheta_{\tilde{A}_s}, \vartheta_{\tilde{B}_s}\}, \\ \min\left\{\left(1 - \left(\left(\max\{\mu_{\tilde{A}_s}, \mu_{\tilde{B}_s}\}\right)^2 + \left(\min\{\vartheta_{\tilde{A}_s}, \vartheta_{\tilde{B}_s}\}\right)^2\right)\right)^{1/2}, \max\{\pi_{\tilde{A}_s}, \pi_{\tilde{B}_s}\}\right\} \end{matrix} \right\}$$

Intersection:

$$\tilde{A}_s \cap \tilde{B}_s = \left\{ \begin{matrix} \min\{\mu_{\tilde{A}_s}, \mu_{\tilde{B}_s}\}, \min\{\vartheta_{\tilde{A}_s}, \vartheta_{\tilde{B}_s}\}, \\ \max\left\{\left(1 - \left(\left(min\{\mu_{\tilde{A}_s}, \mu_{\tilde{B}_s}\}\right)^2 + \left(\max\{\vartheta_{\tilde{A}_s}, \vartheta_{\tilde{B}_s}\}\right)^2\right)\right)^{1/2}, \min\{\pi_{\tilde{A}_s}, \pi_{\tilde{B}_s}\}\right\} \end{matrix} \right\}$$

Addition:

$$\tilde{A}_s \oplus \tilde{B}_s = \left\{ \begin{matrix} \left(\mu_{\tilde{A}_s}{}^2 + \mu_{\tilde{B}_s}{}^2 - \mu_{\tilde{A}_s}{}^2 \mu_{\tilde{B}_s}{}^2\right)^{1/2}, \vartheta_{\tilde{A}_s} \vartheta_{\tilde{B}_s} \\ \left((1 - \mu_{\tilde{B}_s}{}^2)\pi_{\tilde{A}_s} + (1 - \mu_{\tilde{A}_s}{}^2)\pi_{\tilde{B}_s} - \pi_{\tilde{A}_s} \pi_{\tilde{B}_s}\right)^{1/2} \end{matrix} \right\}$$

Multiplication:

$$\tilde{A}_s \otimes \tilde{B}_s = \left\{ \begin{matrix} \mu_{\tilde{A}_s}\mu_{\tilde{B}_s},\ \left(\vartheta_{\tilde{A}_s}{}^2 + \vartheta_{\tilde{B}_s}{}^2 - \vartheta_{\tilde{A}_s}{}^2\vartheta_{\tilde{B}_s}{}^2\right)^{1/2}, \\ \left(\left(1-\vartheta_{\tilde{B}_s}{}^2\right)\pi_{\tilde{A}_s} + \left(1-\vartheta_{\tilde{A}_s}{}^2\right)\pi_{\tilde{B}_s} - \pi_{\tilde{A}_s}\,\pi_{\tilde{B}_s}\right)^{1/2} \end{matrix} \right\}$$

Multiplication by a scalar; $\lambda > 0$:

$$\lambda.\tilde{A}_s = \left\{ \begin{matrix} \left(1-\left(1-\mu_{\tilde{A}_s}{}^2\right)^{\lambda}\right)^{1/2},\ \vartheta_{\tilde{A}_s}{}^{\lambda}, \\ \left(\left(1-\mu_{\tilde{A}_s}{}^2\right)^{\lambda} - \left(1-\mu_{\tilde{A}_s}{}^2 - \pi_{\tilde{A}_s}{}^2\right)^{\lambda}\right)^{1/2} \end{matrix} \right\}$$

Power of $\tilde{A}_s$; $\lambda > 0$:

$$\tilde{A}_s{}^{\lambda} = \left\{ \begin{matrix} \mu_{\tilde{A}_s}{}^{\lambda}, \left(1-\left(1-\vartheta_{\tilde{A}_s}{}^2\right)^{\lambda}\right)^{1/2} \\ \left(\left(1-\vartheta_{\tilde{A}_s}{}^2\right)^{\lambda} - \left(1-\vartheta_{\tilde{A}_s}{}^2 - \pi_{\tilde{A}_s}{}^2\right)^{\lambda}\right)^{1/2} \end{matrix} \right\}$$

Definition 2.5. Let X be the universe. Then a bipolar valued fuzzy sets A on X is defined by positive membership function $\mu_A^+ : X \to [0,1]$ and a negative membership function $\mu_A^- : X \to [-1,0]$. For sake of easiness, we shall practice the symbol $A = \{< x,\ \mu_A^+(x),\ \mu_A^-(x) > : x \in X\}$.

Definition 2.6. [24] Let A and B be two bipolar valued fuzzy sets then their union, intersection, and complement are well-defined as follows:

i. $\mu_{A\cup B}^+(x) = \max\{\mu_A^+(x), \mu_B^+(x)\}$

ii. $\mu_{A\cup B}^-(x) = \min\{\mu_A^-(x), \mu_B^-(x)\}$

iii. $\mu_{A\cap B}^+(x) = \min\{\mu_A^+(x), \mu_B^+(x)\}$

iv. $\mu_{A\cap B}^-(x) = \max\{\mu_A^-(x), \mu_B^-(x)\}$

v. $\mu_{\bar{A}}^+(x)$= 1-μ_A^+(x) and $\mu_{\bar{A}}^-(x) = -1 - \mu_A^+(x)$ for all $x \in X$.

3. Spherical Bipolar Fuzzy Sets

Definition 3.1. A spherical bipolar fuzzy set $\tilde{A}_s$ of the universe of discourse U is given by,

$$\tilde{A}_s = \left\{< u, \mu_{\tilde{A}_s}{}^{+}(\mathrm{u}), \vartheta_{\tilde{A}_s}{}^{+}(\mathrm{u}), \pi_{\tilde{A}_s}{}^{+}(\mathrm{u}), \mu_{\tilde{A}_s}{}^{-}(\mathrm{u}), \vartheta_{\tilde{A}_s}{}^{-}(\mathrm{u}), \pi_{\tilde{A}_s}{}^{-}(\mathrm{u}) | \mathrm{u} \in \mathrm{U}\right\}$$

where,

$\mu_{\tilde{A}_s}{}^{+}(u): U \to [0,1], \vartheta_{\tilde{A}_s}{}^{+}(u): U \to [0,1], \pi_{\tilde{A}_s}{}^{+}(u): U \to [0,1],$

$\mu_{\tilde{A}_s}{}^{-}(u): U \to [-1,0], \vartheta_{\tilde{A}_s}{}^{-}(u): U \to [-1,0], \pi_{\tilde{A}_s}{}^{-}(u): U \to [-1,0]$ and

$0 \le \mu_{\tilde{A}_s}{}^{2^+}(u) + \vartheta_{\tilde{A}_s}{}^{2^+}(u) + \pi_{\tilde{A}_s}{}^{2^+}(u) \le 1,$

$-1 \le -\left(\mu_{\tilde{A}_s}{}^{2^-}(u) + \vartheta_{\tilde{A}_s}{}^{2^-}(u) + \pi_{\tilde{A}_s}{}^{2^-}(u)\right) \le 0\ \forall\, u \in U.$

For each u, the numbers $\mu_{\tilde{A}_s}{}^{+}(u), \vartheta_{\tilde{A}_s}{}^{+}(u), \pi_{\tilde{A}_s}{}^{+}(u)$ are the positive membership, non-membership and the hesitancy of u to $\tilde{A}_s$ and the numbers $\mu_{\tilde{A}_s}{}^{-}(u), \vartheta_{\tilde{A}_s}{}^{-}(u), \pi_{\tilde{A}_s}{}^{-}(u)$ are the negative degree of membership, non-membership and hesitancy of u to $\tilde{A}_s$ respectively.

Definition 3.2. The basic operators of Spherical Bipolar Fuzzy sets are:

(i) Union

$$\tilde{A}_s \cup \tilde{B}_s = \left\{\begin{array}{c} <\max\left(\mu_{\tilde{A}_s}{}^{+}, \mu_{\tilde{B}_s}{}^{+}\right), \min\left(\vartheta_{\tilde{A}_s}{}^{+}, \vartheta_{\tilde{B}_s}{}^{+}\right), \\ min\left\{\left(1-\left(\left(\max\left(\mu_{\tilde{A}_s}{}^{+}, \mu_{\tilde{B}_s}{}^{+}\right)\right)^2+\left(\min\left(\vartheta_{\tilde{A}_s}{}^{+}, \vartheta_{\tilde{B}_s}{}^{+}\right)\right)^2\right)\right)^{1/2}, \max\left\{\pi_{\tilde{A}_s}{}^{+}, \pi_{\tilde{B}_s}{}^{+}\right\}\right\}, \\ \min\left(\mu_{\tilde{A}_s}{}^{-}, \mu_{\tilde{B}_s}{}^{-}\right), \max\left(\vartheta_{\tilde{A}_s}{}^{-}, \vartheta_{\tilde{B}_s}{}^{-}\right), \\ \max\left\{\left(1-\left(\left(\min\left(\mu_{\tilde{A}_s}{}^{-}, \mu_{\tilde{B}_s}{}^{-}\right)\right)^2+\left(\max\left(\vartheta_{\tilde{A}_s}{}^{-}, \vartheta_{\tilde{B}_s}{}^{-}\right)\right)^2\right)\right)^{1/2}, \min\left\{\pi_{\tilde{A}_s}{}^{-}, \pi_{\tilde{B}_s}{}^{-}\right\}\right\}> \end{array}\right\}$$

(ii) Intersection

$$\tilde{A}_s \cap \tilde{B}_s = \left\{\begin{array}{c} <\min\left(\mu_{\tilde{A}_s}{}^{+}, \mu_{\tilde{B}_s}{}^{+}\right), \max\left(\vartheta_{\tilde{A}_s}{}^{+}, \vartheta_{\tilde{B}_s}{}^{+}\right), \\ \max\left\{\left(1-\left(\left(\min\left(\mu_{\tilde{A}_s}{}^{+}, \mu_{\tilde{B}_s}{}^{+}\right)\right)^2+\left(\max\left(\vartheta_{\tilde{A}_s}{}^{+}, \vartheta_{\tilde{B}_s}{}^{+}\right)\right)^2\right)\right)^{1/2}, \min\left\{\pi_{\tilde{A}_s}{}^{+}, \pi_{\tilde{B}_s}{}^{+}\right\}\right\}, \\ \max\left(\mu_{\tilde{A}_s}{}^{-}, \mu_{\tilde{B}_s}{}^{-}\right), \min\left(\vartheta_{\tilde{A}_s}{}^{-}, \vartheta_{\tilde{B}_s}{}^{-}\right), \\ \min\left\{\left(1-\left(\left(\max\left(\mu_{\tilde{A}_s}{}^{-}, \mu_{\tilde{B}_s}{}^{-}\right)\right)^2+\left(\min\left(\vartheta_{\tilde{A}_s}{}^{-}, \vartheta_{\tilde{B}_s}{}^{-}\right)\right)^2\right)\right)^{1/2}, \max\left\{\pi_{\tilde{A}_s}{}^{-}, \pi_{\tilde{B}_s}{}^{-}\right\}\right\}> \end{array}\right\}$$

(iii) Complement

$$\tilde{A}_s{}^{c} = \left\{< u, 1-\mu_{\tilde{A}_s}{}^{+}(u), 1-\vartheta_{\tilde{A}_s}{}^{+}(u), 1-\pi_{\tilde{A}_s}{}^{+}(u), -1-\mu_{\tilde{A}_s}{}^{-}(u), -1-\vartheta_{\tilde{A}_s}{}^{-}(u), -1-\pi_{\tilde{A}_s}{}^{-}(u)\right\}, u \in U\ .$$

(iv) Addition

$$\tilde{A}_s \oplus \tilde{B}_s = \left\{\begin{array}{c} \left(\mu_{\tilde{A}_s}{}^{2}+\mu_{\tilde{B}_s}{}^{2}-\mu_{\tilde{A}_s}{}^{2}\mu_{\tilde{B}_s}{}^{2}\right)^{1/2}, \vartheta_{\tilde{A}_s}\vartheta_{\tilde{B}_s}, \\ \left[\left(1-\mu_{\tilde{B}_s}{}^{2}\right)\pi_{\tilde{A}_s}{}^{2}+\left(1-\mu_{\tilde{A}_s}{}^{2}\right)\pi_{\tilde{B}_s}{}^{2}-\pi_{\tilde{A}_s}{}^{2}\pi_{\tilde{B}_s}{}^{2}\right]^{1/2}, \qquad \mu_{\tilde{A}_s}{}^{-}\mu_{\tilde{B}_s}{}^{-}, \\ \left(\vartheta_{\tilde{A}_s}{}^{2^-}+\vartheta_{\tilde{B}_s}{}^{2^-}-\vartheta_{\tilde{A}_s}{}^{2^-}\vartheta_{\tilde{B}_s}{}^{2^-}\right)^{1/2}, \\ \left[\left(1-\vartheta_{\tilde{B}_s}{}^{2^-}\right)\pi_{\tilde{A}_s}{}^{2^-}+\left(1-\vartheta_{\tilde{A}_s}{}^{2^-}\right)\pi_{\tilde{B}_s}{}^{2^-}-\pi_{\tilde{A}_s}{}^{2^-}\pi_{\tilde{B}_s}{}^{2^-}\right]^{1/2} \end{array}\right\}$$

(v) Multiplication

$$\tilde{A}_s \otimes \tilde{B}_s = \left\{\begin{array}{c} \mu_{\tilde{A}_s}\mu_{\tilde{B}_s}, \left(\vartheta_{\tilde{A}_s}{}^{2}+\vartheta_{\tilde{B}_s}{}^{2}-\vartheta_{\tilde{A}_s}{}^{2}\vartheta_{\tilde{B}_s}{}^{2}\right)^{1/2}, \\ \left[\left(1-\vartheta_{\tilde{B}_s}{}^{2}\right)\pi_{\tilde{A}_s}{}^{2}+\left(1-\vartheta_{\tilde{A}_s}{}^{2}\right)\pi_{\tilde{B}_s}{}^{2}-\pi_{\tilde{A}_s}{}^{2}\pi_{\tilde{B}_s}{}^{2}\right]^{1/2}, \\ \left(\mu_{\tilde{A}_s}{}^{2^-}+\mu_{\tilde{B}_s}{}^{2^-}-\mu_{\tilde{A}_s}{}^{2^-}\mu_{\tilde{B}_s}{}^{2^-}\right)^{1/2}, \vartheta_{\tilde{A}_s}{}^{-}\vartheta_{\tilde{B}_s}{}^{-}, \\ \left[\left(1-\mu_{\tilde{B}_s}{}^{2^-}\right)\pi_{\tilde{A}_s}{}^{2^-}+\left(1-\mu_{\tilde{A}_s}{}^{2^-}\right)\pi_{\tilde{B}_s}{}^{2^-}-\pi_{\tilde{A}_s}{}^{2^-}\pi_{\tilde{B}_s}{}^{2^-}\right]^{1/2} \end{array}\right\}$$

(vi) Multiplication by a scalar; $\boldsymbol{\lambda > 0}$

$$\lambda.\tilde{A}_s = \left\{\begin{array}{c} \left(1-\left(1-\mu_{\tilde{A}_s}{}^{2}\right)^{\lambda}\right)^{1/2}, \ \vartheta_{\tilde{A}_s}{}^{\lambda}, \\ \left(\left(1-\mu_{\tilde{A}_s}{}^{2}\right)^{\lambda}-\left(1-\mu_{\tilde{A}_s}{}^{2}-\pi_{\tilde{A}_s}{}^{2}\right)^{\lambda}\right)^{1/2}, -\mu_{\tilde{A}_s}{}^{-\lambda}, \\ -\left(1-\left(1-\vartheta_{\tilde{A}_s}{}^{2^-}\right)^{\lambda}\right)^{1/2}, -\left(\left(1-\vartheta_{\tilde{A}_s}{}^{2^-}\right)^{\lambda}-\left(1-\vartheta_{\tilde{A}_s}{}^{2^-}-\pi_{\tilde{A}_s}{}^{2^-}\right)^{\lambda}\right)^{1/2} \end{array}\right\}$$

(vii) Power of $\boldsymbol{\tilde{A}_s}$; $\boldsymbol{\lambda > 0}$

$$\tilde{A}_s{}^{\lambda} = \left\{ \begin{array}{c} \mu_{\tilde{A}_s}{}^{\lambda}, \left(1-\left(1-\vartheta_{\tilde{A}_s}{}^{2}\right)^{\lambda}\right)^{1/2}, \left(\left(1-\vartheta_{\tilde{A}_s}{}^{2}\right)^{\lambda}-\left(1-\vartheta_{\tilde{A}_s}{}^{2}-\pi_{\tilde{A}_s}{}^{2}\right)^{\lambda}\right)^{1/2}, \\ -\left(1-\left(1-\mu_{\tilde{A}_s}{}^{2-}\right)^{\lambda}\right)^{1/2}, -\vartheta_{\tilde{A}_s}{}^{-\lambda}, \\ -\left(\left(1-\mu_{\tilde{A}_s}{}^{2-}\right)^{\lambda}-\left(1-\mu_{\tilde{A}_s}{}^{2-}-\pi_{\tilde{A}_s}{}^{2-}\right)^{\lambda}\right)^{1/2} \end{array} \right\}$$

Definition 3.3. For these SBFS, $\tilde{A}_s = \{< \mu_{\tilde{A}_s}{}^{+}, \vartheta_{\tilde{A}_s}{}^{+}, \pi_{\tilde{A}_s}{}^{+}, \mu_{\tilde{A}_s}{}^{-}, \vartheta_{\tilde{A}_s}{}^{-}, \pi_{\tilde{A}_s}{}^{-}\}$ and $\tilde{B}_s = \{< \mu_{\tilde{B}_s}{}^{+}, \vartheta_{\tilde{B}_s}{}^{+}, \pi_{\tilde{B}_s}{}^{+}, \mu_{\tilde{B}_s}{}^{-}, \vartheta_{\tilde{B}_s}{}^{-}, \pi_{\tilde{B}_s}{}^{-}\}$, the following are valid under the condition λ, λ_1, $\lambda_2 > 0$.

i. $\tilde{A}_s \oplus \tilde{B}_s = \tilde{B}_s \oplus \tilde{A}_s$

ii. $\tilde{A}_s \otimes \tilde{B}_s = \tilde{B}_s \otimes \tilde{A}_s$

iii. $\lambda(\tilde{A}_s \oplus \tilde{B}_s) = \lambda\tilde{A}_s \oplus \lambda\tilde{B}_s$

iv. $\lambda_1\tilde{A}_s \oplus \lambda_2\tilde{A}_s = (\lambda_1+\lambda_2)\tilde{A}_s$

v. $(\tilde{A}_s \otimes \tilde{B}_s)^{\lambda} = \tilde{A}_s{}^{\lambda} \otimes \tilde{B}_s{}^{\lambda}$

vi. $\tilde{A}_s{}^{\lambda_1} \otimes \tilde{A}_s{}^{\lambda_2} = \tilde{A}_s{}^{\lambda_1+\lambda_2}$

Proof. According to Definition 3.2, we will prove (*i*) - (*iii*) and (*v*) since (*iv*) and (*vi*) are similar to the proofs of (*iii*) and (*v*) respectively.

i.
$$\tilde{A}_s \oplus \tilde{B}_s = \left\{ \begin{array}{c} \left(\mu_{\tilde{A}_s}{}^{2}+\mu_{\tilde{B}_s}{}^{2}-\mu_{\tilde{A}_s}{}^{2}\mu_{\tilde{B}_s}{}^{2}\right)^{1/2}, \vartheta_{\tilde{A}_s}\vartheta_{\tilde{B}_s}, \\ \left[\left(1-\mu_{\tilde{B}_s}{}^{2}\right)\pi_{\tilde{A}_s}{}^{2}+\left(1-\mu_{\tilde{A}_s}{}^{2}\right)\pi_{\tilde{B}_s}{}^{2}-\pi_{\tilde{A}_s}{}^{2}\pi_{\tilde{B}_s}{}^{2}\right]^{1/2}, \ \mu_{\tilde{A}_s}{}^{-}\mu_{\tilde{B}_s}{}^{-}, \\ \left(\vartheta_{\tilde{A}_s}{}^{2-}+\vartheta_{\tilde{B}_s}{}^{2-}-\vartheta_{\tilde{A}_s}{}^{2-}\vartheta_{\tilde{B}_s}{}^{2-}\right)^{1/2}, \\ \left[\left(1-\vartheta_{\tilde{B}_s}{}^{2-}\right)\pi_{\tilde{A}_s}{}^{2-}+\left(1-\vartheta_{\tilde{A}_s}{}^{2-}\right)\pi_{\tilde{B}_s}{}^{2-}-\pi_{\tilde{A}_s}{}^{2-}\pi_{\tilde{B}_s}{}^{2-}\right]^{1/2} \end{array} \right\}$$

$$\tilde{B}_s \oplus \tilde{A}_s = \left\{ \begin{array}{c} \left(\mu_{\tilde{B}_s}{}^{2}+\mu_{\tilde{A}_s}{}^{2}-\mu_{\tilde{B}_s}{}^{2}\mu_{\tilde{A}_s}{}^{2}\right)^{1/2}, \vartheta_{\tilde{B}_s}\vartheta_{\tilde{A}_s}, \\ \left[\left(1-\mu_{\tilde{A}_s}{}^{2}\right)\pi_{\tilde{B}_s}{}^{2}+\left(1-\mu_{\tilde{B}_s}{}^{2}\right)\pi_{\tilde{A}_s}{}^{2}-\pi_{\tilde{B}_s}{}^{2}\pi_{\tilde{A}_s}{}^{2}\right]^{1/2}, \ \mu_{\tilde{B}_s}{}^{-}\mu_{\tilde{A}_s}{}^{-}, \\ \left(\vartheta_{\tilde{B}_s}{}^{2-}+\vartheta_{\tilde{A}_s}{}^{2-}-\vartheta_{\tilde{B}_s}{}^{2-}\vartheta_{\tilde{A}_s}{}^{2-}\right)^{1/2}, \\ \left[\left(1-\vartheta_{\tilde{A}_s}{}^{2-}\right)\pi_{\tilde{B}_s}{}^{2-}+\left(1-\vartheta_{\tilde{B}_s}{}^{2-}\right)\pi_{\tilde{A}_s}{}^{2-}-\pi_{\tilde{B}_s}{}^{2-}\pi_{\tilde{A}_s}{}^{2-}\right]^{1/2} \end{array} \right\}$$

And so, $\tilde{A}_s \oplus \tilde{B}_s = \tilde{B}_s \oplus \tilde{A}_s$.

ii.
$$\tilde{A}_s \otimes \tilde{B}_s = \left\{ \begin{array}{c} \mu_{\tilde{A}_s}\mu_{\tilde{B}_s}, \left(\vartheta_{\tilde{A}_s}{}^{2}+\vartheta_{\tilde{B}_s}{}^{2}-\vartheta_{\tilde{A}_s}{}^{2}\vartheta_{\tilde{B}_s}{}^{2}\right)^{1/2}, \\ \left[\left(1-\vartheta_{\tilde{B}_s}{}^{2}\right)\pi_{\tilde{A}_s}{}^{2}+\left(1-\vartheta_{\tilde{A}_s}{}^{2}\right)\pi_{\tilde{B}_s}{}^{2}-\pi_{\tilde{A}_s}{}^{2}\pi_{\tilde{B}_s}{}^{2}\right]^{1/2}, \\ \left(\mu_{\tilde{A}_s}{}^{2-}+\mu_{\tilde{B}_s}{}^{2-}-\mu_{\tilde{A}_s}{}^{2-}\mu_{\tilde{B}_s}{}^{2-}\right)^{1/2}, \vartheta_{\tilde{A}_s}{}^{-}\vartheta_{\tilde{B}_s}{}^{-}, \\ \left[\left(1-\mu_{\tilde{B}_s}{}^{2-}\right)\pi_{\tilde{A}_s}{}^{2-}+\left(1-\mu_{\tilde{A}_s}{}^{2-}\right)\pi_{\tilde{B}_s}{}^{2-}-\pi_{\tilde{A}_s}{}^{2-}\pi_{\tilde{B}_s}{}^{2-}\right]^{1/2} \end{array} \right\}$$

$$\tilde{B}_s \otimes \tilde{A}_s = \left\{ \begin{array}{c} \mu_{\tilde{B}_s}\mu_{\tilde{A}_s}, \left(\vartheta_{\tilde{B}_s}{}^{2}+\vartheta_{\tilde{A}_s}{}^{2}-\vartheta_{\tilde{B}_s}{}^{2}\vartheta_{\tilde{A}_s}{}^{2}\right)^{1/2}, \\ \left[\left(1-\vartheta_{\tilde{A}_s}{}^{2}\right)\pi_{\tilde{B}_s}{}^{2}+\left(1-\vartheta_{\tilde{B}_s}{}^{2}\right)\pi_{\tilde{A}_s}{}^{2}-\pi_{\tilde{B}_s}{}^{2}\pi_{\tilde{A}_s}{}^{2}\right]^{1/2}, \\ \left(\mu_{\tilde{B}_s}{}^{2-}+\mu_{\tilde{A}_s}{}^{2-}-\mu_{\tilde{B}_s}{}^{2-}\mu_{\tilde{A}_s}{}^{2-}\right)^{1/2}, \vartheta_{\tilde{B}_s}{}^{-}\vartheta_{\tilde{A}_s}{}^{-}, \\ \left[\left(1-\mu_{\tilde{A}_s}{}^{2-}\right)\pi_{\tilde{B}_s}{}^{2-}+\left(1-\mu_{\tilde{B}_s}{}^{2-}\right)\pi_{\tilde{A}_s}{}^{2-}-\pi_{\tilde{B}_s}{}^{2-}\pi_{\tilde{A}_s}{}^{2-}\right]^{1/2} \end{array} \right\}$$

And so $\tilde{A}_s \otimes \tilde{B}_s = \tilde{B}_s \otimes \tilde{A}_s$.

iii.
$$\lambda(\tilde{A}_s \oplus \tilde{B}_s) = \lambda \left\{ \begin{array}{c} \left(\mu_{\tilde{A}_s}^{2} + \mu_{\tilde{B}_s}^{2} - \mu_{\tilde{A}_s}^{2}\mu_{\tilde{B}_s}^{2}\right)^{1/2}, \vartheta_{\tilde{A}_s}\vartheta_{\tilde{B}_s}, \\ \left[\left(1-\mu_{\tilde{B}_s}^{2}\right)\pi_{\tilde{A}_s}^{2} + \left(1-\mu_{\tilde{A}_s}^{2}\right)\pi_{\tilde{B}_s}^{2} - \pi_{\tilde{A}_s}^{2}\pi_{\tilde{B}_s}^{2}\right]^{1/2},\ \mu_{\tilde{A}_s}^{-}\mu_{\tilde{B}_s}^{-}, \\ \left(\vartheta_{\tilde{A}_s}^{2^-} + \vartheta_{\tilde{B}_s}^{2^-} - \vartheta_{\tilde{A}_s}^{2^-}\vartheta_{\tilde{B}_s}^{2^-}\right)^{1/2}, \\ \left[\left(1-\vartheta_{\tilde{B}_s}^{2^-}\right)\pi_{\tilde{A}_s}^{2^-} + \left(1-\vartheta_{\tilde{A}_s}^{2^-}\right)\pi_{\tilde{B}_s}^{2^-} - \pi_{\tilde{A}_s}^{2^-}\pi_{\tilde{B}_s}^{2^-}\right]^{1/2} \end{array} \right\}.$$

$$= \left\{ \begin{array}{c} \left(1-\left(1-\left(\mu_{\tilde{A}_s}^{2} + \mu_{\tilde{B}_s}^{2} - \mu_{\tilde{A}_s}^{2}\mu_{\tilde{B}_s}^{2}\right)\right)^{\lambda}\right)^{1/2}, \vartheta_{\tilde{A}_s}^{\lambda}\vartheta_{\tilde{B}_s}^{\lambda}, \\ \left[\begin{array}{c} \left(1-\left(\mu_{\tilde{A}_s}^{2} + \mu_{\tilde{B}_s}^{2} - \mu_{\tilde{A}_s}^{2}\mu_{\tilde{B}_s}^{2}\right)\right)^{\lambda} - \\ \left(\left(1-\left(\mu_{\tilde{A}_s}^{2} + \mu_{\tilde{B}_s}^{2} - \mu_{\tilde{A}_s}^{2}\mu_{\tilde{B}_s}^{2}\right) - \left(1-\mu_{\tilde{B}_s}^{2}\right)\pi_{\tilde{A}_s}^{2} - \left(1-\mu_{\tilde{A}_s}^{2}\right)\pi_{\tilde{B}_s}^{2} + \pi_{\tilde{A}_s}^{2}\pi_{\tilde{B}_s}^{2}\right)\right)^{\lambda} \end{array}\right]^{1/2}, \\ -\left(\mu_{\tilde{A}_s}^{-}\right)^{\lambda}\left(\mu_{\tilde{B}_s}^{-}\right)^{\lambda}, -\left(1-\left(1-\left(\vartheta_{\tilde{A}_s}^{2^-} + \vartheta_{\tilde{B}_s}^{2^-} - \vartheta_{\tilde{A}_s}^{2^-}\vartheta_{\tilde{B}_s}^{2^-}\right)\right)^{\lambda}\right)^{1/2}, \\ -\left[\begin{array}{c} \left(1-\left(\vartheta_{\tilde{A}_s}^{2^-} + \vartheta_{\tilde{B}_s}^{2^-} - \vartheta_{\tilde{A}_s}^{2^-}\vartheta_{\tilde{B}_s}^{2^-}\right)\right)^{\lambda} - \\ \left(\left(\begin{array}{c} 1-\left(\vartheta_{\tilde{A}_s}^{2^-} + \vartheta_{\tilde{B}_s}^{2^-} - \vartheta_{\tilde{A}_s}^{2^-}\vartheta_{\tilde{B}_s}^{2^-}\right) - \left(1-\vartheta_{\tilde{B}_s}^{2^-}\right)\pi_{\tilde{A}_s}^{2-} \\ -\left(1-\vartheta_{\tilde{A}_s}^{2^-}\right)\pi_{\tilde{B}_s}^{2-} + \pi_{\tilde{A}_s}^{2-}\pi_{\tilde{B}_s}^{2-} \end{array}\right)\right)^{\lambda} \end{array}\right]^{1/2} \end{array} \right\}$$

Now,

$$\lambda\tilde{A}_s \oplus \lambda\tilde{B}_s = \left\{ \begin{array}{c} \left(1-\left(1-\mu_{\tilde{A}_s}^{2}\right)^{\lambda}\right)^{1/2},\ \vartheta_{\tilde{A}_s}^{\lambda}, \\ \left(\left(1-\mu_{\tilde{A}_s}^{2}\right)^{\lambda} - \left(1-\mu_{\tilde{A}_s}^{2} - \pi_{\tilde{A}_s}^{2}\right)^{\lambda}\right)^{1/2}, -\mu_{\tilde{A}_s}^{-\lambda}, \\ -\left(1-\left(1-\vartheta_{\tilde{A}_s}^{2^-}\right)^{\lambda}\right)^{1/2}, -\left(\left(1-\vartheta_{\tilde{A}_s}^{2^-}\right)^{\lambda} - \left(1-\vartheta_{\tilde{A}_s}^{2^-} - \pi_{\tilde{A}_s}^{2^-}\right)^{\lambda}\right)^{1/2} \end{array} \right\}$$

$$\oplus$$

$$\left\{ \begin{array}{c} \left(1-\left(1-\mu_{\tilde{B}_s}^{2}\right)^{\lambda}\right)^{1/2},\ \vartheta_{\tilde{B}_s}^{\lambda}, \\ \left(\left(1-\mu_{\tilde{B}_s}^{2}\right)^{\lambda} - \left(1-\mu_{\tilde{B}_s}^{2} - \pi_{\tilde{B}_s}^{2}\right)^{\lambda}\right)^{1/2}, -\mu_{\tilde{B}_s}^{-\lambda}, \\ -\left(1-\left(1-\vartheta_{\tilde{B}_s}^{2^-}\right)^{\lambda}\right)^{1/2}, -\left(\left(1-\vartheta_{\tilde{B}_s}^{2^-}\right)^{\lambda} - \left(1-\vartheta_{\tilde{B}_s}^{2^-} - \pi_{\tilde{B}_s}^{2^-}\right)^{\lambda}\right)^{1/2} \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \left(1-\left(1-\mu_{\tilde{A}_s}^{2}\right)^{\lambda} + 1-\left(1-\mu_{\tilde{B}_s}^{2}\right)^{\lambda} - \left(1-\left(1-\mu_{\tilde{A}_s}^{2}\right)^{\lambda}\right)\left(1-\left(1-\mu_{\tilde{B}_s}^{2}\right)^{\lambda}\right)\right)^{1/2}, \vartheta_{\tilde{A}_s}^{\lambda}\vartheta_{\tilde{B}_s}^{\lambda}, \\ \left[\begin{array}{c} 1-\left(1-\left(1-\mu_{\tilde{B}_s}^{2}\right)^{\lambda}\right)\left(\left(1-\mu_{\tilde{A}_s}^{2}\right)^{\lambda} - \left(1-\mu_{\tilde{A}_s}^{2} - \pi_{\tilde{A}_s}^{2}\right)^{\lambda}\right) + \\ \left(1-\left(1-\left(1-\mu_{\tilde{A}_s}^{2}\right)^{\lambda}\right)\left(\left(1-\mu_{\tilde{B}_s}^{2}\right)^{\lambda} - \left(1-\mu_{\tilde{B}_s}^{2} - \pi_{\tilde{B}_s}^{2}\right)^{\lambda}\right)\right) - \\ \left(\left(1-\mu_{\tilde{A}_s}^{2}\right)^{\lambda} - \left(1-\mu_{\tilde{A}_s}^{2} - \pi_{\tilde{A}_s}^{2}\right)^{\lambda}\right)\left(\left(1-\mu_{\tilde{B}_s}^{2}\right)^{\lambda} - \left(1-\mu_{\tilde{B}_s}^{2} - \pi_{\tilde{B}_s}^{2}\right)^{\lambda}\right) \end{array}\right]^{1/2}, -\left(\mu_{\tilde{A}_s}^{-}\right)^{\lambda}\left(\mu_{\tilde{B}_s}^{-}\right)^{\lambda}, \\ -\left(1-\left(1-\vartheta_{\tilde{A}_s}^{2^-}\right)^{\lambda} + 1-\left(1-\vartheta_{\tilde{B}_s}^{2^-}\right)^{\lambda} - \left(1-\left(1-\vartheta_{\tilde{A}_s}^{2^-}\right)^{\lambda}\right)\left(1-\left(1-\vartheta_{\tilde{B}_s}^{2^-}\right)^{\lambda}\right)\right)^{1/2}, \\ -\left[\begin{array}{c} 1-\left(1-\left(1-\vartheta_{\tilde{B}_s}^{2^-}\right)^{\lambda}\right)\left(\left(1-\vartheta_{\tilde{A}_s}^{2^-}\right)^{\lambda} - \left(1-\vartheta_{\tilde{A}_s}^{2^-} - \pi_{\tilde{A}_s}^{2^-}\right)^{\lambda}\right) + \\ \left(1-\left(1-\left(1-\vartheta_{\tilde{A}_s}^{2^-}\right)^{\lambda}\right)\left(\left(1-\vartheta_{\tilde{B}_s}^{2^-}\right)^{\lambda} - \left(1-\vartheta_{\tilde{B}_s}^{2^-} - \pi_{\tilde{B}_s}^{2^-}\right)^{\lambda}\right)\right) - \\ \left(\left(1-\vartheta_{\tilde{A}_s}^{2^-}\right)^{\lambda} - \left(1-\vartheta_{\tilde{A}_s}^{2^-} - \pi_{\tilde{A}_s}^{2^-}\right)^{\lambda}\right)\left(\left(1-\vartheta_{\tilde{B}_s}^{2^-}\right)^{\lambda} - \left(1-\vartheta_{\tilde{B}_s}^{2^-} - \pi_{\tilde{B}_s}^{2^-}\right)^{\lambda}\right) \end{array}\right]^{1/2} \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \left(1-\left(1-\mu_{\tilde{A}_s}{}^2\right)^\lambda\left(1-\mu_{\tilde{B}_s}{}^2\right)^\lambda\right)^{1/2}, \vartheta_{\tilde{A}_s}{}^\lambda \vartheta_{\tilde{B}_s}{}^\lambda, \\ \left(\left(1-\mu_{\tilde{A}_s}{}^2\right)^\lambda\left(1-\mu_{\tilde{B}_s}{}^2\right)^\lambda - \left(1-\mu_{\tilde{A}_s}{}^2-\pi_{\tilde{A}_s}{}^2\right)^\lambda\left(1-\mu_{\tilde{B}_s}{}^2-\pi_{\tilde{B}_s}{}^2\right)^\lambda\right)^{1/2}, -\left(\mu_{\tilde{A}_s}{}^-\right)^\lambda\left(\mu_{\tilde{B}_s}{}^-\right)^\lambda, \\ -\left(1-\left(1-\vartheta_{\tilde{A}_s}{}^{2-}\right)^\lambda\left(1-\vartheta_{\tilde{B}_s}{}^{2-}\right)^\lambda\right)^{1/2}, \\ -\left(\left(1-\vartheta_{\tilde{A}_s}{}^{2-}\right)^\lambda\left(1-\vartheta_{\tilde{B}_s}{}^{2-}\right)^\lambda - \left(1-\vartheta_{\tilde{A}_s}{}^{2-}-\pi_{\tilde{A}_s}{}^{2-}\right)^\lambda\left(1-\vartheta_{\tilde{B}_s}{}^{2-}-\pi_{\tilde{B}_s}{}^{2-}\right)^\lambda\right)^{1/2} \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \left(1-\left(1-\left(\mu_{\tilde{A}_s}{}^2+\mu_{\tilde{B}_s}{}^2-\mu_{\tilde{A}_s}{}^2\mu_{\tilde{B}_s}{}^2\right)\right)^\lambda\right)^{1/2}, \vartheta_{\tilde{A}_s}{}^\lambda\vartheta_{\tilde{B}_s}{}^\lambda, \\ \left[\begin{array}{c} \left(1-\left(\mu_{\tilde{A}_s}{}^2+\mu_{\tilde{B}_s}{}^2-\mu_{\tilde{A}_s}{}^2\mu_{\tilde{B}_s}{}^2\right)\right)^\lambda - \\ \left(\left(1-\left(\mu_{\tilde{A}_s}{}^2+\mu_{\tilde{B}_s}{}^2-\mu_{\tilde{A}_s}{}^2\mu_{\tilde{B}_s}{}^2\right)-\left(1-\mu_{\tilde{B}_s}{}^2\right)\pi_{\tilde{A}_s}{}^2-\left(1-\mu_{\tilde{A}_s}{}^2\right)\pi_{\tilde{B}_s}{}^2+\pi_{\tilde{A}_s}{}^2\pi_{\tilde{B}_s}{}^2\right)\right)^\lambda \end{array}\right]^{1/2}, \\ -\left(\mu_{\tilde{A}_s}{}^-\right)^\lambda\left(\mu_{\tilde{B}_s}{}^-\right)^\lambda, -\left(1-\left(1-\left(\vartheta_{\tilde{A}_s}{}^{2-}+\vartheta_{\tilde{B}_s}{}^{2-}-\vartheta_{\tilde{A}_s}{}^{2-}\vartheta_{\tilde{B}_s}{}^{2-}\right)\right)^\lambda\right)^{1/2}, \\ -\left[\begin{array}{c} \left(1-\left(\vartheta_{\tilde{A}_s}{}^{2-}+\vartheta_{\tilde{B}_s}{}^{2-}-\vartheta_{\tilde{A}_s}{}^{2-}\vartheta_{\tilde{B}_s}{}^{2-}\right)\right)^\lambda - \\ \left(\left(\begin{array}{c} 1-\left(\vartheta_{\tilde{A}_s}{}^{2-}+\vartheta_{\tilde{B}_s}{}^{2-}-\vartheta_{\tilde{A}_s}{}^{2-}\vartheta_{\tilde{B}_s}{}^{2-}\right)-\left(1-\vartheta_{\tilde{B}_s}{}^{2-}\right)\pi_{\tilde{A}_s}{}^{2-} \\ -\left(1-\vartheta_{\tilde{A}_s}{}^{2-}\right)\pi_{\tilde{B}_s}{}^{2-}+\pi_{\tilde{A}_s}{}^{2-}\pi_{\tilde{B}_s}{}^{2-} \end{array}\right)\right)^\lambda \end{array}\right]^{1/2} \end{array} \right\}$$

And so, $\lambda(\tilde{A}_s \oplus \tilde{B}_s) = \lambda\tilde{A}_s \oplus \lambda\tilde{B}_s$.

iv. $$(\tilde{A}_s \otimes \tilde{B}_s)^\lambda = \left\{ \begin{array}{c} \mu_{\tilde{A}_s}\mu_{\tilde{B}_s}, \left(\vartheta_{\tilde{A}_s}{}^2+\vartheta_{\tilde{B}_s}{}^2-\vartheta_{\tilde{A}_s}{}^2\vartheta_{\tilde{B}_s}{}^2\right)^{1/2}, \\ \left[\left(1-\vartheta_{\tilde{B}_s}{}^2\right)\pi_{\tilde{A}_s}{}^2+\left(1-\vartheta_{\tilde{A}_s}{}^2\right)\pi_{\tilde{B}_s}{}^2-\pi_{\tilde{A}_s}{}^2\pi_{\tilde{B}_s}{}^2\right]^{1/2}, \\ \left(\mu_{\tilde{A}_s}{}^{2-}+\mu_{\tilde{B}_s}{}^{2-}-\mu_{\tilde{A}_s}{}^{2-}\mu_{\tilde{B}_s}{}^{2-}\right)^{1/2}, \vartheta_{\tilde{A}_s}{}^-\vartheta_{\tilde{B}_s}{}^-, \\ \left[\left(1-\mu_{\tilde{B}_s}{}^{2-}\right)\pi_{\tilde{A}_s}{}^{2-}+\left(1-\mu_{\tilde{A}_s}{}^{2-}\right)\pi_{\tilde{B}_s}{}^{2-}-\pi_{\tilde{A}_s}{}^{2-}\pi_{\tilde{B}_s}{}^{2-}\right]^{1/2} \end{array} \right\}^\lambda$$

$$= \left\{ \begin{array}{c} \left\{\left\{\mu_{\tilde{A}_s}{}^\lambda\mu_{\tilde{B}_s}{}^\lambda\right\}\right\}, \left(1-\left(1-\left(\vartheta_{\tilde{A}_s}{}^2+\vartheta_{\tilde{B}_s}{}^2-\vartheta_{\tilde{A}_s}{}^2\vartheta_{\tilde{B}_s}{}^2\right)\right)^\lambda\right)^{\frac{1}{2}}, \\ \left[\begin{array}{c} \left(1-\left(\vartheta_{\tilde{A}_s}{}^2+\vartheta_{\tilde{B}_s}{}^2-\vartheta_{\tilde{A}_s}{}^2\vartheta_{\tilde{B}_s}{}^2\right)\right)^\lambda - \\ \left(\left(1-\left(\vartheta_{\tilde{A}_s}{}^2+\vartheta_{\tilde{B}_s}{}^2-\vartheta_{\tilde{A}_s}{}^2\vartheta_{\tilde{B}_s}{}^2\right)-\left(1-\vartheta_{\tilde{B}_s}{}^2\right)\pi_{\tilde{A}_s}{}^2-\left(1-\vartheta_{\tilde{A}_s}{}^2\right)\pi_{\tilde{B}_s}{}^2+\pi_{\tilde{A}_s}{}^2\pi_{\tilde{B}_s}{}^2\right)\right)^\lambda \end{array}\right]^{1/2} \\ -\left(1-\left(1-\mu_{\tilde{A}_s}{}^{2-}+\mu_{\tilde{B}_s}{}^{2-}-\mu_{\tilde{A}_s}{}^{2-}\mu_{\tilde{B}_s}{}^{2-}\right)^\lambda\right)^{1/2}, -\left(\vartheta_{\tilde{A}_s}{}^-\right)^\lambda\left(\vartheta_{\tilde{B}_s}{}^-\right)^\lambda, \\ \left[\begin{array}{c} \left(1-\left(\mu_{\tilde{A}_s}{}^{2-}+\mu_{\tilde{B}_s}{}^{2-}-\mu_{\tilde{A}_s}{}^{2-}\mu_{\tilde{B}_s}{}^{2-}\right)\right)^\lambda - \\ \left(\left(\begin{array}{c} 1-\left(\mu_{\tilde{A}_s}{}^{2-}+\mu_{\tilde{B}_s}{}^{2-}-\mu_{\tilde{A}_s}{}^{2-}\mu_{\tilde{B}_s}{}^{2-}\right)-\left(1-\mu_{\tilde{B}_s}{}^{2-}\right)\pi_{\tilde{A}_s}{}^{2-}- \\ \left(1-\mu_{\tilde{A}_s}{}^{2-}\right)\pi_{\tilde{B}_s}{}^{2-}+\pi_{\tilde{A}_s}{}^{2-}\pi_{\tilde{B}_s}{}^{2-} \end{array}\right)\right)^\lambda \end{array}\right]^{1/2} \end{array} \right\}$$

Now,

$$\tilde{A}_s{}^\lambda \otimes \tilde{B}_s{}^\lambda = \left\{ \begin{array}{c} \mu_{\tilde{A}_s}{}^\lambda, \left(1-\left(1-\vartheta_{\tilde{A}_s}{}^2\right)^\lambda\right)^{1/2}, \left(\left(1-\vartheta_{\tilde{A}_s}{}^2\right)^\lambda-\left(1-\vartheta_{\tilde{A}_s}{}^2-\pi_{\tilde{A}_s}{}^2\right)^\lambda\right)^{1/2}, \\ -\left(1-\left(1-\mu_{\tilde{A}_s}{}^{2-}\right)^\lambda\right)^{1/2}, -\vartheta_{\tilde{A}_s}{}^{-\lambda}, \\ -\left(\left(1-\mu_{\tilde{A}_s}{}^{2-}\right)^\lambda-\left(1-\mu_{\tilde{A}_s}{}^{2-}-\pi_{\tilde{A}_s}{}^{2-}\right)^\lambda\right)^{1/2} \end{array} \right\}$$

$$\otimes$$

$$\left\{\begin{array}{c} \mu_{\tilde{B}_s}{}^{\lambda}, \left(1-\left(1-\vartheta_{\tilde{B}_s}{}^{2}\right)^{\lambda}\right)^{1/2}, \left(\left(1-\vartheta_{\tilde{B}_s}{}^{2}\right)^{\lambda}-\left(1-\vartheta_{\tilde{B}_s}{}^{2}-\pi_{\tilde{B}_s}{}^{2}\right)^{\lambda}\right)^{1/2}, \\ -\left(1-\left(1-\mu_{\tilde{B}_s}{}^{2-}\right)^{\lambda}\right)^{1/2}, -(\vartheta_{\tilde{B}_s}{}^{-})^{\lambda}, \\ -\left(\left(1-\mu_{\tilde{B}_s}{}^{2-}\right)^{\lambda}-\left(1-\mu_{\tilde{B}_s}{}^{2-}-\pi_{\tilde{B}_s}{}^{2-}\right)^{\lambda}\right)^{1/2} \end{array}\right\}$$

$$= \left\{\begin{array}{c} \mu_{\tilde{A}_s}{}^{\lambda}\mu_{\tilde{B}_s}{}^{\lambda}, \left(\begin{array}{c} 1-\left(1-\vartheta_{\tilde{A}_s}{}^{2}\right)^{\lambda}+1-\left(1-\vartheta_{\tilde{B}_s}{}^{2}\right)^{\lambda}- \\ \left(1-\left(1-\vartheta_{\tilde{A}_s}{}^{2}\right)^{\lambda}\right)\left(1-\left(1-\vartheta_{\tilde{B}_s}{}^{2}\right)^{\lambda}\right) \end{array}\right)^{1/2}, \\ \left[\begin{array}{c} 1-\left(1-\left(1-\vartheta_{\tilde{B}_s}{}^{2}\right)^{\lambda}\right)\left(\left(1-\vartheta_{\tilde{A}_s}{}^{2}\right)^{\lambda}-\left(1-\vartheta_{\tilde{A}_s}{}^{2}-\pi_{\tilde{A}_s}{}^{2}\right)^{\lambda}\right)+ \\ 1-\left(1-\left(1-\vartheta_{\tilde{A}_s}{}^{2}\right)^{\lambda}\right)\left(\left(1-\vartheta_{\tilde{B}_s}{}^{2}\right)^{\lambda}-\left(1-\vartheta_{\tilde{B}_s}{}^{2}-\pi_{\tilde{B}_s}{}^{2}\right)^{\lambda}\right)- \\ \left(\left(1-\vartheta_{\tilde{A}_s}{}^{2}\right)^{\lambda}-\left(1-\vartheta_{\tilde{A}_s}{}^{2}-\pi_{\tilde{A}_s}{}^{2}\right)^{\lambda}\right)\left(\left(1-\vartheta_{\tilde{B}_s}{}^{2}\right)^{\lambda}-\left(1-\vartheta_{\tilde{B}_s}{}^{2}-\pi_{\tilde{B}_s}{}^{2}\right)^{\lambda}\right) \end{array}\right]^{1/2}, \\ -\left(1-\left(1-\mu_{\tilde{A}_s}{}^{2-}\right)^{\lambda}+1-\left(1-\mu_{\tilde{B}_s}{}^{2-}\right)^{\lambda}-\left(1-\left(1-\mu_{\tilde{A}_s}{}^{2-}\right)^{\lambda}\right)\left(1-\left(1-\mu_{\tilde{B}_s}{}^{2-}\right)^{\lambda}\right)\right)^{1/2}, \\ -\vartheta_{\tilde{A}_s}{}^{-\lambda}\vartheta_{\tilde{B}_s}{}^{-\lambda}, \\ \left[\begin{array}{c} 1-\left(1-\left(1-\mu_{\tilde{B}_s}{}^{2-}\right)^{\lambda}\right)\left(\left(1-\mu_{\tilde{A}_s}{}^{2-}\right)^{\lambda}-\left(1-\mu_{\tilde{A}_s}{}^{2-}-\pi_{\tilde{A}_s}{}^{2-}\right)^{\lambda}\right)+ \\ 1-\left(1-\left(1-\mu_{\tilde{A}_s}{}^{2-}\right)^{\lambda}\right)\left(\left(1-\mu_{\tilde{B}_s}{}^{2-}\right)^{\lambda}-\left(1-\mu_{\tilde{B}_s}{}^{2-}-\pi_{\tilde{B}_s}{}^{2-}\right)^{\lambda}\right)- \\ \left(\left(1-\mu_{\tilde{A}_s}{}^{2-}\right)^{\lambda}-\left(1-\mu_{\tilde{A}_s}{}^{2-}-\pi_{\tilde{A}_s}{}^{2-}\right)^{\lambda}\right)\left(\left(1-\mu_{\tilde{B}_s}{}^{2-}\right)^{\lambda}-\left(1-\mu_{\tilde{B}_s}{}^{2-}-\pi_{\tilde{B}_s}{}^{2-}\right)^{\lambda}\right) \end{array}\right]^{1/2} \end{array}\right\}$$

$$= \left\{\begin{array}{c} \mu_{\tilde{A}_s}{}^{\lambda}\mu_{\tilde{B}_s}{}^{\lambda}, \left(1-\left(1-\vartheta_{\tilde{A}_s}{}^{2}\right)^{\lambda}\left(1-\vartheta_{\tilde{B}_s}{}^{2}\right)^{\lambda}\right)^{1/2}, \\ \left(1-\vartheta_{\tilde{A}_s}{}^{2}\right)^{\lambda}\left(1-\vartheta_{\tilde{B}_s}{}^{2}\right)^{\lambda}-\left(\left(1-\vartheta_{\tilde{A}_s}{}^{2}-\pi_{\tilde{A}_s}{}^{2}\right)^{\lambda}\left(1-\vartheta_{\tilde{B}_s}{}^{2}-\pi_{\tilde{B}_s}{}^{2}\right)^{\lambda}\right)^{1/2}, \\ -\left(1-\left(1-\mu_{\tilde{A}_s}{}^{2-}\right)^{\lambda}\left(1-\mu_{\tilde{B}_s}{}^{2-}\right)^{\lambda}\right)^{1/2}, -\vartheta_{\tilde{A}_s}{}^{-\lambda}\vartheta_{\tilde{B}_s}{}^{-\lambda}, \\ -\left(1-\mu_{\tilde{A}_s}{}^{2-}\right)^{\lambda}\left(1-\mu_{\tilde{B}_s}{}^{2-}\right)^{\lambda}-\left(\left(1-\mu_{\tilde{A}_s}{}^{2-}-\pi_{\tilde{A}_s}{}^{2-}\right)^{\lambda}\left(1-\mu_{B_s}{}^{2-}-\pi_{\tilde{B}_s}{}^{2-}\right)^{\lambda}\right)^{1/2} \end{array}\right\}$$

$$= \left\{\begin{array}{c} \left\{\left\{\mu_{\tilde{A}_s}{}^{\lambda}\mu_{\tilde{B}_s}{}^{\lambda}\right\}\right\}, \left(1-\left(1-\left(\vartheta_{\tilde{A}_s}{}^{2}+\vartheta_{\tilde{B}_s}{}^{2}-\vartheta_{\tilde{A}_s}{}^{2}\vartheta_{\tilde{B}_s}{}^{2}\right)\right)^{\lambda}\right)^{\frac{1}{2}}, \\ \left[\begin{array}{c} \left(1-\left(\vartheta_{\tilde{A}_s}{}^{2}+\vartheta_{\tilde{B}_s}{}^{2}-\vartheta_{\tilde{A}_s}{}^{2}\vartheta_{\tilde{B}_s}{}^{2}\right)\right)^{\lambda}- \\ \left(\left(1-\left(\vartheta_{\tilde{A}_s}{}^{2}+\vartheta_{\tilde{B}_s}{}^{2}-\vartheta_{\tilde{A}_s}{}^{2}\vartheta_{\tilde{B}_s}{}^{2}\right)-\left(1-\vartheta_{\tilde{B}_s}{}^{2}\right)\pi_{\tilde{A}_s}{}^{2}-\left(1-\vartheta_{\tilde{A}_s}{}^{2}\right)\pi_{\tilde{B}_s}{}^{2}+\pi_{\tilde{A}_s}{}^{2}\pi_{\tilde{B}_s}{}^{2}\right)\right)^{\lambda} \end{array}\right]^{1/2} \\ -\left(1-\left(1-\mu_{\tilde{A}_s}{}^{2-}+\mu_{\tilde{B}_s}{}^{2-}-\mu_{\tilde{A}_s}{}^{2-}\mu_{\tilde{B}_s}{}^{2-}\right)^{\lambda}\right)^{1/2}, -(\vartheta_{\tilde{A}_s}{}^{-})^{\lambda}(\vartheta_{\tilde{B}_s}{}^{-})^{\lambda}, \\ \left[\begin{array}{c} \left(1-\left(\mu_{\tilde{A}_s}{}^{2-}+\mu_{\tilde{B}_s}{}^{2-}-\mu_{\tilde{A}_s}{}^{2-}\mu_{\tilde{B}_s}{}^{2-}\right)\right)^{\lambda}- \\ \left(\left(\begin{array}{c} 1-\left(\mu_{\tilde{A}_s}{}^{2-}+\mu_{\tilde{B}_s}{}^{2-}-\mu_{\tilde{A}_s}{}^{2-}\mu_{\tilde{B}_s}{}^{2-}\right)-\left(1-\mu_{\tilde{B}_s}{}^{2-}\right)\pi_{\tilde{A}_s}{}^{2-}- \\ \left(1-\mu_{\tilde{A}_s}{}^{2-}\right)\pi_{\tilde{B}_s}{}^{2-}+\pi_{\tilde{A}_s}{}^{2-}\pi_{\tilde{B}_s}{}^{2-} \end{array}\right)\right)^{\lambda} \end{array}\right]^{1/2} \end{array}\right\}$$

Thus, $(\tilde{A}_s \otimes \tilde{B}_s)^{\lambda} = \tilde{A}_s{}^{\lambda} \otimes \tilde{B}_s{}^{\lambda}$.

Definition 3.4. Spherical Bipolar Weighted Arithmetic Mean (SBWAM) with respect to, $w = w_1, w_2, \ldots, w_n;\ w_i \in [0,1];\ \sum_{i=1}^{n} w_i = 1$, SBWAM is defined as,

$$SBWAM_w\left(A_{s_1}, A_{s_2}, \dots, A_{s_n}\right) = w_1 A_{s_1} + w_2 A_{s_2} + \dots + w_n A_{s_n}$$

$$= \left\{ \begin{array}{c} \left[1 - \prod_{i=1}^{n}\left(1 - \mu_{A_{s_i}}{}^{2}\right)^{w_i}\right]^{1/2}, \prod_{i=1}^{n} \vartheta_{A_{s_i}}{}^{w_i}, \\ \left[\prod_{i=1}^{n}\left(1 - \mu_{A_{s_i}}{}^{2}\right)^{w_i} - \prod_{i=1}^{n}\left(1 - \mu_{A_{s_i}}{}^{2} - \pi_{A_{s_i}}{}^{2}\right)^{w_i}\right]^{1/2}, \\ -\prod_{i=1}^{n}\left(-\mu_{A_{s_i}}{}^{2^-}\right)^{w_i}, -\left(1 - \prod_{i=1}^{n}\left(1 - \vartheta_{A_{s_i}}{}^{2^-}\right)^{w_i}\right)^{1/2}, \\ -\left[\prod_{i=1}^{n}\left(1 - \vartheta_{A_{s_i}}{}^{2^-}\right)^{w_i} - \left(\prod_{i=1}^{n} 1 - \vartheta_{A_{s_i}}{}^{2^-} - \pi_{A_{s_i}}{}^{2^-}\right)^{w_i}\right]^{1/2} \end{array} \right\} \quad (7)$$

Definition 3.5. Spherical Bipolar Weighted Geometric Mean (SBWGM) with respect to, $w = w_1, w_2, \dots, w_n$; $w_i \in [0,1]$; $\sum_{i=1}^{n} w_i = 1$, SBWGM is defined as,

$$SBWGM_w(A_1, A_2 \dots, A_n) = A_{s_1}{}^{w_1} + A_{s_2}{}^{w_2} + \dots + A_{s_n}{}^{w_n}$$

$$= \left\{ \begin{array}{c} \prod_{i=1}^{n} \mu_{A_{s_i}}{}^{w_i}, \left[1 - \prod_{i=1}^{n}\left(1 - \vartheta_{A_{s_i}}{}^{2}\right)^{w_i}\right]^{1/2}, \\ \left[\prod_{i=1}^{n}\left(1 - \vartheta_{A_{s_i}}{}^{2}\right)^{w_i} - \prod_{i=1}^{n}\left(1 - \vartheta_{A_{s_i}}{}^{2} - \pi_{A_{s_i}}{}^{2}\right)^{w_i}\right]^{1/2}, \\ -\left(1 - \prod_{i=1}^{n}\left(1 - \mu_{A_{s_i}}{}^{2^-}\right)^{w_i}\right)^{1/2}, \prod_{i=1}^{n}\left(-\vartheta_{A_{s_i}}{}^{2^-}\right)^{w_i}, \\ -\left[\prod_{i=1}^{n}\left(1 - \mu_{A_{s_i}}{}^{2^-}\right)^{w_i} - \left(\prod_{i=1}^{n} 1 - \mu_{A_{s_i}}{}^{2^-} - \pi_{A_{s_i}}{}^{2^-}\right)^{w_i}\right]^{1/2} \end{array} \right\}$$

(8)

Definition 3.6. The score function and accuracy function of sorting Spherical Bipolar fuzzy set (SBFS) are defined by,

i. Score $\left(\tilde{A}_s\right) = \frac{1}{2}\left[\begin{array}{c}\left(\mu_{\tilde{A}_s} - \pi_{\tilde{A}_s}\right)^2 - \left(\vartheta_{\tilde{A}_s} - \pi_{\tilde{A}_s}\right)^2 + \left(\mu_{\tilde{A}_s}{}^{-} - \pi_{\tilde{A}_s}{}^{-}\right)^2 - \\ \left(\vartheta_{\tilde{A}_s}{}^{-} - \pi_{\tilde{A}_s}{}^{-}\right)^2\end{array}\right]$

ii. Accuracy $\left(\tilde{A}_s\right) = \frac{1}{2}\left[\mu_{\tilde{A}_s}{}^{2} + \vartheta_{\tilde{A}_s}{}^{2} + \pi_{\tilde{A}_s}{}^{2} + \mu_{\tilde{A}_s}{}^{2^-} + \vartheta_{\tilde{A}_s}{}^{2^-} + \pi_{\tilde{A}_s}{}^{2^-}\right]$

4. Decision Making Method Based on The Spherical Fuzzy Weighted Aggregation Operator

In this section, we present a handling method for multi criteria decision making problem by means of two aggregation operators under the spherical bipolar environment.

Let $A = \{A_1, A_2, \dots, A_m\}$ be the set of all alternatives and let $c = \{C_1, C_2, \dots, C_n\}$be the set of criteria. Assume that the weight of the criteria $C_j (j = 1,2, \dots, n)$ entered by the decision maker is w_j, where $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$. In the decision process, the evaluation information of the alternative A_i on the criteria is represented by the form of a Spherical Bipolar Fuzzy Set (SBFS):

$$\tilde{A}_i = \{\langle C_j, \mu_{\tilde{A}_i}{}^{+}(C_j), \vartheta_{\tilde{A}_i}{}^{+}(C_j), \pi_{\tilde{A}_i}{}^{+}(C_j), \mu_{\tilde{A}_i}{}^{-}(C_j), \vartheta_{\tilde{A}_i}{}^{-}(C_j), \pi_{\tilde{A}_i}{}^{-}(C_j) |\, C_j \in C\rangle\}$$

where, $0 \le \mu_{\tilde{A}_i}{}^{2^+}(C_j) + \vartheta_{\tilde{A}_i}{}^{2^+}(C_j) + \pi_{\tilde{A}_i}{}^{2^+}(C_j) \le 1$ and $-1 \le -(\mu_{\tilde{A}_i}{}^{2^-}(C_j) + \vartheta_{\tilde{A}_i}{}^{2^-}(C_j) + \pi_{\tilde{A}_i}{}^{2^-}(C_j)) \le 0$ for $j = 1,2, \dots, n$ and $i = 1,2, \dots, m$.

For convenience, the value of SBFS is denoted by,

$$\alpha_{ij} = \langle \mu_{ij}{}^{+}, \vartheta_{ij}{}^{+}, \pi_{ij}{}^{+}, \mu_{ij}{}^{-}, \vartheta_{ij}{}^{-}, \pi_{ij}{}^{-} \rangle \qquad (j = 1,2, \dots, m \; and \; i = 1,2, \dots, n)$$

Therefore, we get a SBFS decision matrix, $D = (\alpha_{ij})_{m\times n}$.

$$D = (\alpha_{ij})_{m\times n} = \begin{bmatrix} \langle {\mu_{11}}^+, {\vartheta_{11}}^+, {\pi_{11}}^+, {\mu_{11}}^-, {\vartheta_{11}}^-, {\pi_{11}}^- \rangle & \langle {\mu_{12}}^+, {\vartheta_{12}}^+, {\pi_{12}}^+, {\mu_{12}}^-, {\vartheta_{12}}^-, {\pi_{12}}^- \rangle & \cdots & \langle {\mu_{1n}}^+, {\vartheta_{1n}}^+, {\pi_{1n}}^+, {\mu_{1n}}^-, {\vartheta_{1n}}^-, {\pi_{1n}}^- \rangle \\ \langle {\mu_{21}}^+, {\vartheta_{21}}^+, {\pi_{21}}^+, {\mu_{21}}^-, {\vartheta_{21}}^-, {\pi_{21}}^- \rangle & \langle {\mu_{22}}^+, {\vartheta_{22}}^+, {\pi_{22}}^+, {\mu_{22}}^-, {\vartheta_{22}}^-, {\pi_{22}}^- \rangle & \cdots & \langle {\mu_{2n}}^+, {\vartheta_{2n}}^+, {\pi_{2n}}^+, {\mu_{2n}}^-, {\vartheta_{2n}}^-, {\pi_{2n}}^- \rangle \\ \vdots & \vdots & & \vdots \\ \langle {\mu_{m1}}^+, {\vartheta_{m1}}^+, {\pi_{m1}}^+, {\mu_{m1}}^-, {\vartheta_{m1}}^-, {\pi_{m1}}^- \rangle & \langle {\mu_{m2}}^+, {\vartheta_{m2}}^+, {\pi_{m2}}^+, {\mu_{m2}}^-, {\vartheta_{m2}}^-, {\pi_{m2}}^- \rangle & \cdots & \langle {\mu_{mn}}^+, {\vartheta_{mn}}^+, {\pi_{mn}}^+, {\mu_{mn}}^-, {\vartheta_{mn}}^-, {\pi_{mn}}^- \rangle \end{bmatrix}$$

Then, the aggregating spherical bipolar value α_i for $A_i (i = 1,2,\dots,m)$ is,

$\alpha_i = \langle {\mu_i}^+, {\vartheta_i}^+, {\pi_i}^+, {\mu_i}^-, {\vartheta_i}^-, {\pi_i}^- \rangle = SBWAM_w(\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{in})$ or

$\alpha_i = \langle {\mu_i}^+, {\vartheta_i}^+, {\pi_i}^+, {\mu_i}^-, {\vartheta_i}^-, {\pi_i}^- \rangle = SBWGM_w(\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{in})$ is obtained by equations (7) and (8).

In summary, the decision procedure for the proposed method can be summarized as follows:

Step 1. Construct the decision matrix provided by the decision maker as:

$$(\alpha_{ij})_{m\times n} = \langle {\mu_{ij}}^+, {\vartheta_{ij}}^+, {\pi_{ij}}^+, {\mu_{ij}}^-, {\vartheta_{ij}}^-, {\pi_{ij}}^- \rangle_{m\times n}.$$

Step 2. Compute α_i by calculating the weighted arithmetic average values by $SBWAM_w$ or $SBWGM_w$

Step 3. Calculate the score values of score (α_i) $i = 1,2,\dots,m$ for the collective overall SBF number of $\alpha_i (i = 1,2,\dots,m)$

Step 4. Give the ranking order of the alternative according to the score values, and then give the best choice.

5. Numerical Example

In this section, an example for a multi criteria decision making problem of engineering alternatives is used as a demonstration of the application of the proposed decision-making method in a realistic scenario, as well as the application and effectiveness of the proposed decision-making method.

Let us consider the decision-making problem.

There is an investment company which wants to invest a sum of money in the best option. There is a panel with four possible alternatives to invest the money:

1) A_1 is a car company,
2) A_2 is a food company,
3) A_3 is a computer company,
4) A_4 is a arms company.

The investment company must take a decision according to the following three criteria:

1) C_1 is the risk,
2) C_2 is the growth,
3) C_3 is the environmental impact.

The weight vector of the criteria is given by,

$$w = (0.35, 0.25, 0.4)$$

Then,

STEP 1. Construct the decision matrix provided by the decision maker:

$$\begin{bmatrix} \begin{pmatrix} 0.2, 0.3, 0.4, \\ -0.1, -0.4, -0.6 \end{pmatrix} & \begin{pmatrix} 0.4, 0.5, 0.5, \\ -0.2, -0.3, -0.4 \end{pmatrix} & \begin{pmatrix} 0.2, 0.2, 0.5, \\ -0.5, -0.6, -0.2 \end{pmatrix} \\ \begin{pmatrix} 0.1, 0.2, 0.6, \\ -0.2, -0.5, -0.6 \end{pmatrix} & \begin{pmatrix} 0.6, 0.1, 0.2, \\ -0.3, -0.4, -0.5 \end{pmatrix} & \begin{pmatrix} 0.5, 0.1, 0.7, \\ -0.1, -0.3, -0.5 \end{pmatrix} \\ \begin{pmatrix} 0.2, 0.3, 0.3, \\ -0.5, -0.5, -0.6 \end{pmatrix} & \begin{pmatrix} 0.2, 0.3, 0.5, \\ -0.1, -0.2, -0.5 \end{pmatrix} & \begin{pmatrix} 0.2, 0.3, 0.8, \\ -0.1, -0.2, -0.3 \end{pmatrix} \\ \begin{pmatrix} 0.1, 0.4, 0.7, \\ -0.3, -0.5, -0.7 \end{pmatrix} & \begin{pmatrix} 0.1, 0.4, 0.7, \\ -0.3, -0.5, -0.7 \end{pmatrix} & \begin{pmatrix} 0.2, 0.5, 0.8 \\ -0.2, -0.4, -0.5 \end{pmatrix} \end{bmatrix}$$

STEP 2. Calculate the weighted arithmetic average value α_i for $i = 1,2,3,4$.

$$\alpha_1 = [0.2675, 0.2898, 0.4709, -0.0512, -0.4809, -0.4263]$$

$$\alpha_2 = [0.3509, 0.2898, 0.6633, -0.0281, -0.4080, -0.5475]$$

$$\alpha_3 = [0.2002, 0.2999, 0.6369, -0.0308, -0.4498, -0.4163]$$

$$\alpha_4 = [0.1487, 0.4374, 0.7481, -0.0651, -0.4640, -0.6509]$$

STEP 3. Calculate the score values s(α_i) $for\ i = 1,2,3,4$ for the collective overall SBF number $\alpha_i, i = 1,2,3,4$.

$$s(\alpha_1) = 0.0731$$

$$s(\alpha_2) = 0.1042$$

$$s(\alpha_3) = 0.0961$$

$$s(\alpha_4) = 0.2856$$

STEP 4. Rank of the investment company according to the score values is:

$$A_4 > A_2 > A_3 > A_1$$

Thus A_4 is the best alternative, (i.e.,) the best option to invest money is arms company.

6. Conclusion

In this paper, we studied the concept of Spherical Bipolar Fuzzy Sets (SBFS) by using Spherical Fuzzy Sets (SFS), Bipolar fuzzy set (BFS) and some of its basic operational relations. Two aggregation operators: Spherical bipolar weighted arithmetic average operator and Spherical bipolar weighted geometric average operator were proposed and they were applied to multi criteria decision making problems. Finally, a numerical example is provided to illustrate the application of the developed approach.

References

[1] F. Smarandache. *A Unifying Field in Logics. Neutrosophy: Neutrosophic Probability, Set and Logic.Rehoboth*: American Research Press (1999) 1-141.

[2] G. İntepe, E. Bozdağ, T. Koç, *The Selection of Technology Forecasting Method Using A Multi-Criteria Interval-Valued Intuitionistic Fuzzy Group Decision Making Approach*, Computers and Industrial Engineering 65(2) (2013) 277-285.

[3] F. Smarandache, *Neutrosophic Set- A Generalization of The Intuitionistic Fuzzy Set*, International Journal of Pure and Applied Mathematics 24(3) (2004) 1-15.

[4] I. Grattan-Guinness, *Fuzzy Membership Mapped onto Interval and Many-Valued Quantities*, Mathematical Logic Quarterly 22(1) (1976) 149-160.

[5] J. M. Garibaldi, T. Özen, *Uncertain Fuzzy Reasoning: A Case Study in Modelling Expert Decision Making*. IEEE Transactions on Fuzzy Systems 15(1) (2007) 16-30.

[6] K. T. Atanassov, *Intuitionistic Fuzzy Sets*, Fuzzy sets and Systems 20(1) (1986) 87-96.

[7] L. A. Zadeh, *Fuzzy Sets*, Information and Control 8 (1965) 338-353.

[8] L. A. Zadeh, *The Concept of a Linguistic Variable and its application to Approximate Reasoning*, Information Sciences 8(3) (1975) 199-249.

[9] Y. Yang, F. Chiclana, *Intuitionistic Fuzzy Sets: Spherical Representation and Distances*, International Journal of Intelligent Systems 24(4) (2009) 399-420.

[10] R. R. Yager, *On The Theory of Bags*, International Journal of General Systems 13(1) (1986) 23–37.

[11] V. Torra, *Hesitant Fuzzy Sets*, International Journal of Intelligent Systems 25(6) (2010) 529-539.

[12] K. Atanassov, *Geometrical Interpretation of The Elements of The Intuitionistic Fuzzy Objects*, Preprint IM-MFAIS, 1-89, Sofia. Reprinted: International Journal Bioautomation 20 (S1) (2016) 27-42.

[13] K. Atanassov, *Intuitionistic Fuzzy Sets*, Springer, Heidelberg (1999).

[14] P. J. Ren, Z. Xu, X. J. Gou, *Pythagorean Fuzzy TODIM Approach to Multi-Criteria Decision Making*, Applied Soft Computing 31 (2016) 246–259.

[15] X. Gou, Z. Xu, P. Ren, *The Properties of Continuous Pythagorean Fuzzy Information*, International Journal of Intelligent Systems 31(5) (2016) 401-424.

[16] S. Broumi, M. Murugappan, M. Talea, A. Bakali, F. Smarandache, P. K. Singh, A. Dey, *Single Valued $(2n+1)$ Sided Polygonal Neutrosophic Numbers and Single Valued $(2n)$ Sided Polygonal Neutrosophic Numbers*, Neutrosophic Sets and Systems 25 (2019) 54-65.

[17] S. Broumi, M. Talea, A. Bakali, P. K. Singh, F. Smarandache, *Energy and Spectrum Analysis of Interval Valued Neutrosophic Graph using MATLAB*, Neutrosophic Sets and Systems 24 (2019) 46-60.

[18] S. Broumi, M. Talea, A. Bakali, F. Smarandache, P. K. Singh, M. Murugappan, V. V. Rao, A *Neutrosophic Technique Based Efficient Routing Protocol for MANET Based on Its Energy and Distance*, Neutrosophic Sets and Systems 24 (2019) 61-69.

[19] S. Broumi, A. Dey, M. Talea, A. Bakali, F. Smarandache, D. Nagarajan, M. Lathamaheswari, R. Kumar, *Shortest Path Problem using Bellman Algorithm under Neutrosophic Environment*, Complex and Intelligent Systems 5 (2019) 409-416. https://doi.org/10.1007/s40747-019-0101-8

[20] S. Broumi, M. Talea, A. Bakali, F. Smarandache, D. Nagarajan, M. Lathamaheswari, M. Parimala, *Shortest Path Problem in Fuzzy, Intuitionistic Fuzzy and Neutrosophic Environment: An Overview*, Complex and Intelligent Systems 5 (2019) 371-378. https://doi.org/10.1007/s40747-019-0098-z

[21] I. Antonov, *On A New Geometrical Interpretation of The Intuitionistic Fuzzy Sets*, Notes on Intuitionistic Fuzzy Sets 1(1) (1995) 29–31.

[22] Z. Gong, X. Xu, Y. Yang, Y. Zhou, H. Zhang, *The Spherical Distance for Intuitionistic Fuzzy Sets and Its Application in Decision Analysis*, Technological and Economic Development of Economy 22(3) (2016) 393-415.

[23] R. Sambuc, *Function Φ-Flous, Application a l'aide au Diagnostic en Pathologie Thyroidienne*. University of Marseille (1975).

[24] K. M. Lee, 2000, *Bipolar Valued Fuzzy Sets and Their Operations*, Proc. Int. Conf. on Intelligent Technologies, Bangkok, Thailand, (2000) 307-312.

[25] K. M. Lee, *Comparison of Interval Valued Fuzzy Sets, Intuitionistic Fuzzy Sets and Bipolar-Valued Fuzzy Sets*, Journal of Korean Institute of Intelligent Systems 14(2) (2004) 125-129.

[26] F. K. Gündoğu, C. Kahraman, *Spherical Fuzzy Sets and Spherical Fuzzy TOPSIS Method*, Journal of Intelligent & Fuzzy Systems 36(1) (2019) 337-352.