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Cohomology of semi-invariant submanifolds of cosymplectic manifolds

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ABSTRACT

In this paper, we study de Rham cohomology class for semi-invariant submanifolds of a cosymplectic manifold. We show that there are de Rham cohomolgy class on semi-invariant submanifold of a cosymplectic manifold. Firstly, we define semi-invariant submanifolds of a cosyplectic manifold. We present an example for semi-invariant submanifold of a cosymplectic manifold.Later, We obtain characterizations, investigate the geometry of distributions which arise from the definition of semi-invariant submanifold. We obtain that invariant distribtion is always integrable and minimal. Moreover, necessary and sufficient conditions investigate for the anti-invariant distribution to be integrable and minimal. Finally, we prove that semi-invariant submanifold of a cosymplectic manifold has nontrivial de Rham cohomology class. Further, the theoretical methodology of mathematics are used to obtain results.

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1 Introduction

Cohomology groups have an important studying area for a topological manifold. If a topological space M is a manifold, we may define the dual of the cohomology groups out of differential forms defined on M. The dual groups are called the de Rham cohomology groups. Besides physicists' familiarity with differential forms, cohomology groups have several advantages over homology groups [12].

Contact geometry has a very important place in physical and other mathematical structure. Really, this structures studied thermodynamics, geometric optics and in Hamiltonian dynamics. In these days, contact structures have obtain low dimensional topology. Contact structures first appeared on partial differential equations. Gray defined an almost contact manifold by the condition that the structural group of the tangent bundle is reducible to $U(n) \times 1$. After Sasaki studied an almost contact manifold with tensor fields and Riemannian metric [14]. Later many author studied different contact structures [9, 19]. Goldberg and Yano defined and studied cosymplectic manifolds [8]. A cosymplectic manifold can be considered as an odd-dimensional analogue of a Kaehler manifold.

Bejancu defines and study CR-submanifold which generalized invariant manifold and anti invariant manifold [1]. Later, this submanifolds have been developed different type structure [10, 13, 15]. Tripathi investigated semi-invariant submanifolds of LP-cosymplectic manifold [18]. In [5], Dirik studied contact CR- submanifolds of cosymplectic manifold.

Tanno investigated topology of contact Riemannian manifold [17]. He studied the basic topological properties of contact manifolds. Fernandez and Ibanez studied de Rham cohomologies on almost contact manifolds [6]. They investigated the relation of the coeffective cohomology of some classes of almost contact manifolds with the topology of the manifold. Chinea et al. introduced topology of cosymlectic manifold [3]. Montano et al. introduced topology of 3-cosymplectic manifolds [11]. They showed that there is an action of the Lie algebra on the basic cohomology spaces of a compact 3-cosymplectic manifold with respect to the Reeb foliation.

Chen introduced cohomology of CR-submanifold [2]. He proved that there are de Rham cohomolgy class on CR-submanifold of a Kaehler manifold. Moreover, he show that this class nontrivial such that invariant distribution and anti-invariant distribution are integrable and minimal, respectively. Later, Deshmukh and Ghazal studied cohomolgy of CR-submanifold nearly Kaehler and quasi Kaehler, respectively [4,7]. In [16], Şahin obtained cohomolgy of hemi-slant submanifoldof a Kaehler manifold.
In this paper, we study de Rham cohomology of semi-invariant submanifold of cosymlectic manifold. We obtain

that there are de Rham cohomolgy class on a semi-invariat submanifold under certain conditions.

2 Semi-ınvariant submanfolds of cosymplectic manifold

Let M be an $n-$dimensional real differentiable manifolds of differentiability class C^∞ endowed with a C^∞ vector valued linear function φ, a C^∞ vector field ξ, 1-form η and Riemannian metric g , which satisfies

$$\varphi^2 = -I + \eta \otimes \xi \quad \text{and } \eta(\xi) = 1 \tag{1}$$
$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2}$$

for all $X, Y \in \Gamma(TM)$. Then, M said to be contact manifold. Also in contact manifold the following relations hold:

$$\varphi\xi = 0, \quad \eta o \varphi = 0, \quad rank(\varphi) = n-1$$

and

$$g(\varphi X, Y) = -g(X, \varphi Y).$$

A contact manifold M is called cosymplectic manifold if

$$(\nabla_X \varphi)Y = 0 \tag{3}$$

for all $X, Y \in \Gamma(TM)$.

Definition 2.1. An $(2m+1)-$dimensional Riemannian submanifold B of a cosymplectic manifold M is called a semi-invariant submanifold there exists on B two differentiable orthogonal distributions D_T and $D^\perp$ satisfying:

1. $TB = D_T \oplus D^\perp \oplus sp\{\xi\}$;

2. The distribution D_T is invariant under φ, such that $\varphi D_{T(x)} = D_{T(x)}$ for all $x \in B$;

3. The distribution $D^\perp$ is anti-invariant under φ, such that $\varphi D_x^\perp \subseteq T_x^\perp M$ for any $x \in B$, where $T_x B$ and $T_x B^\perp$ are the tangent space of B at x.

Example 2.2. In what follows, $(\mathbb{R}^{2m+1}, \varphi, \eta, \xi, g)$ will denote the manifold $\mathbb{R}^{2m+1}$ with its usual cosymplectic structure given by

$$\eta = dz, \qquad \xi = \frac{\partial}{\partial z}$$

$$\varphi\left(\sum_{i=1}^{n} (X_i \frac{\partial}{\partial x_i} + Y_i \frac{\partial}{\partial y_i}) + Z\frac{\partial}{\partial z}\right) = \sum_{i=1}^{n} (Y_i \frac{\partial}{\partial x_i} - X_i \frac{\partial}{\partial y_i}) + Y_i y_i \frac{\partial}{\partial z}$$

$$g = (\sum_{i=1}^{n} dx_i \otimes dx_i + dy_i \otimes dy_i) - \eta \otimes \eta$$

$(x_1, \dots, x_n, y_1, \dots, y_n, z)$ representing the cartesian coordinates on $\mathbb{R}^{2m+1}$. We consider a submanifold of $\mathbb{R}^7$ defined by

$$M = X(k, f, l, w, t) = (k, 0, l, f, w, 0, t).$$

Therefore a basis of TM

$$e_1 = \frac{\partial}{\partial x_1}, \qquad e_2 = \frac{\partial}{\partial y_1}, \qquad e_3 = \frac{\partial}{\partial x_3},$$
$$e_4 = \frac{\partial}{\partial y_2}, \qquad e_5 = \frac{\partial}{\partial z} = \xi$$

Moreover,

$$e_1^* = \frac{\partial}{\partial x_2}, \qquad e_2^* = \frac{\partial}{\partial y_3}$$

from a basis of $T^\perp M$.
We determine

$$D_1 = sp\{e_1, e_2\}$$

and

$$D_2 = sp\{e_3, e_4\},$$

then D_1 is invariant distribution and D_2 is anti-invariant distribution. Then

$$TM = D_1 \oplus D_2 \oplus sp\{\xi\}$$

is a semi-invariant submanifold of $\mathbb{R}^7$.
On the other hand, let $\mathcal{V}$ be a differentiable distribution on a Riemannian manifold M with Levi civita connection ∇. We determine, for all $X, Y \in \Gamma(\mathcal{V})$,

$$\sigma(X,Y) = (\nabla_X^M Y)^\perp$$

where $(\nabla_X Y)^\perp$ denotes the component of $\nabla_X Y$ in the orthogonal complementary distribution $\mathcal{V}$ in M. Let $\{E_1, E_2, \dots, E_p\}$ be an orthonormal frame of $\mathcal{V}$. We determine

$$H = \frac{1}{p}\sum_{j=1}^{p} \sigma(E_j, E_j).$$

Therefore H is well defined vector field on M. If $H = 0$ identically on M, we said to be $\mathcal{V}$ as minimal distribution.

3 Some basic result

Let B be a submanifold of a contact manifold M. Let the induced metric on M also be denoted by g. Then Gauss and Weingarten formulate are given respectively by

$$\nabla_X^M Y = \nabla_X^B Y + h(X,Y) \quad (4)$$
$$\nabla_X^M V = \nabla_X^{B\perp} V - A_V X \quad (5)$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(TM^\perp)$, where A_V is the Weingarten endomorphism associated with V, $\nabla^{B\perp}$ is the connection in the normal bundle and h is the second fundamental from of M.
The shape operator A and the second fundamental form h related by

$$g(A_V X, Y) = g(h(X,Y), V). \quad (6)$$

Let B be a submanifold of a contact manifold M with contact structure (φ, η, ξ, g). For $X \in \Gamma(TB)$ we put

$$\varphi X = TX + NX \quad (7)$$

where TX and NX denote the tangential and normal components of φX respectively.
For $V \in \Gamma(TB^\perp)$ we put

$$\varphi V = tV + nV \quad (8)$$

where tV and nV denote the tangential and normal components of φV respectively.

Proposition 3.1. For a submanifold B of a contact manifold and $X \in \Gamma(TB), V, K \in \Gamma(TB^\perp)$, we have

$$g(X, TY) = -g(TX, Y), g(X, tV) = -g(tX, V)$$

and

$$g(K, nV) = -g(nK, V).$$

Proposition 3.2. For a submanifold B of a contact manifold and $\xi \in \Gamma(TB)$, we have

$$T\xi = 0 = N\xi, \eta \circ T = 0 = \eta \circ N$$

$$T^2 + tN = I + \eta \otimes \xi, \qquad NT + nN = 0$$

$$n^2 + Nt = I, tf + Tt = 0.$$

4 Cohomology class of semi-ınvariant submanifolds

In this section, we introduce de Rham cohomology class on semi-invariant submanifold of cosymlectic manifold. Firstly, we prove the following useful lemmas.

Lemma 4.1. Let B be a semi-invariant submanifold of cosymlectic manifold M. Therefore the distribution D_T is always integrable.

Proof. For all $U, V \in \Gamma(D_T)$ and $K \in \Gamma(D^\perp)$, using (1) and (2) we have

$$g([U,V], K) = g(\nabla_U^M \varphi V, \varphi K) - g(\nabla_V^M \varphi U, \varphi K).$$

Then, by virtue of (4), we arrive,

$$g([U,V], K) = g(h(U, \varphi V) - h(V, \varphi U), \varphi K)$$

which gives our assertion.

Lemma 4.2. Let B be a semi-invariant submanifold of cosymlectic manifold M. Therefore the distribution D_T is minimal.

Proof. Firstly, for all $U \in \Gamma(D_T)$ and $K \in \Gamma(D^\perp)$, we get

$$g(U, K) = 0.$$

Then for any, $W \in \Gamma(D_T)$, we arrive,

$$g(\nabla_W^M U, K) = g(\nabla_W^M V, K). \quad (9)$$

Therefore, using (2), (3) and (9) ,we have,

$$g(\nabla_U^M U, K) = -g(\nabla_U^M \varphi K, \varphi U).$$

Hence, from (5), we get,

$$g(\nabla_U^B U, K) = g(A_{\varphi K} U, \varphi U). \quad (10)$$

Moreover, using (2) and (3), we arrive,

$$g(\nabla_{\varphi U}^M \varphi U, K) = g(\nabla_{\varphi U}^M \varphi^2 U, \varphi K).$$

Then by virtue of (1), (9) and (5) we have,

$$g(\nabla_{\varphi U}^B \varphi U, K) = -g(A_{\varphi K} U, \varphi U). \quad (11)$$

(10) and (11) we arrive,

$$g(\nabla_U^B U + \nabla_{\varphi U}^B \varphi U, K) = 0. \quad (12)$$

Let $\{E_1, \ldots, E_q, \varphi E_1, \ldots, \varphi E_q\}$ be a ortonormal base of D_T. Then,

$$H = \frac{1}{2q}\sum_{j=1}^{q}\{\sigma(E_j, E_j) + \sigma(\varphi E_j, \varphi E_j)\}$$
$$= \frac{1}{2q}\sum_{j=1}^{q}\{(\nabla^B_{E_j}E_j)^{\perp} + (\nabla^B_{\varphi E_j}\varphi E_j)^{\perp}\}$$

By virtue of $g((\nabla^B_U W)^{\perp}, K) = g(\nabla^B_U W, K)$, using (12) we have,

$$g(H, K) = 0$$

which completed that proof.

Lemma 4.3. Let B be a semi-invariant submanifold of cosymlectic manifold M. Therefore the distribution $D^{\perp}$ is integrable if and only if

$$TA_{\varphi L}K = TA_{\varphi K}L$$

for all $K, L \in \Gamma(D^{\perp})$.

Proof. For all $K, L \in \Gamma(D^{\perp})$ and $U \in \Gamma(D_T)$, using (1) and (2) we have,

$$g([K, L], U) = g(\nabla^M_K \varphi L, \varphi U) - g(\nabla^M_L \varphi K, \varphi U).$$

From (5), we get,

$$g([K, L], U) = g(-A_{\varphi L}K, \varphi U) - g(-A_{\varphi K}L, \varphi U).$$

Finally, by virtue of (7) and (8), we arrive,

$$g([K, L], U) = g(TA_{\varphi L}K - TA_{\varphi K}L, U)$$

which completes proof.

Lemma 4.4. Let B be a semi-invariant submanifold of cosymlectic manifold M. Therefore the distribution $D^{\perp}$ is minimal if and only if

$$g(h(K, TU), NK) = g(\nabla^{\perp}_K \varphi K, NU)$$

for all $K \in \Gamma(D^{\perp})$ and $U \in \Gamma(D_T)$.

Proof. For all $K \in \Gamma(D^{\perp})$ and $U \in \Gamma(D_T)$, from (9), (1), (2) and (7), we have,

$$g(\nabla_K K, U) = g(\nabla_K \varphi K, TU) + g(\nabla_K \varphi K, NU).$$

By virtue of (5), we get,

$$g(\nabla_K K, U) = -g(A_{\varphi K}K, TU) + g(\nabla^{\perp}_K \varphi K, NU).$$

Then, using (6) and (7), we arrive,

$$g(\nabla_K K, U) = -g(h(K, TU), NK) + g(\nabla^{\perp}_K \varphi K, NU)$$

which gives our assertion.

Now, we denote an orthonomal frame $\{E_1, \ldots, E_q, \varphi E_1, \ldots, \varphi E_q\}$ of the distribution D_T. Let $\{w^1, \ldots, w^q, w^{q+1}, \ldots, w^{2q}\}$ be the 1-forms on B satisfiying

$$w^i(K) = 0, \qquad i \in \{1, \ldots, 2q\},$$
$$w^i(E_j) = \delta_{ij}, i, j \in \{1, \ldots, q\}, \tag{13}$$
$$w^k(\varphi E_j) = \delta_{kj}, \qquad k \in \{q+1, \ldots, 2q\}, \qquad , j \in \{1, \ldots, q\}$$

for all $K \in \Gamma(D^{\perp})$. Therefore, we arrive

$$w = w^1 \wedge \ldots \wedge w^{2q}. \tag{14}$$

Hence w defines a $2q$-form on submanifold B.

Theorem 4.4. Let B be a closed semi-invariant submanifold of a cosymlectic manifold M . Therefore the $2q$ $-$form w defines a canonical de Rham cohomology class given by

$$c(B) = [w] \in H^{2q}(B, \mathbb{R}), dimD_T = 2q.$$

Moreover $c(B)$ is non-trivial if D_T is integrable and $D^{\perp}$ is minimal.

Proof. Firstly, using (14), we arrive,

$$dw = \sum_{k=1}^{2q}(-1)^k w^1 \wedge \ldots \wedge w^{2q}.$$

By virtue of (13), for all $U_1, \ldots, U_{2q} \in \Gamma(D_T)$ and $K, L \in \Gamma(D^{\perp})$, we show that $dw = 0$ if and only if

$$dw = (K, U_1, \ldots, U_{2q}) = 0 \tag{15}$$

and

$$dw = (K, L, U_1, \ldots, U_{2q}) = 0. \tag{16}$$

Hence, $D^{\perp}$ must be integrable for (15) equality to occur and D_T must be minimal for (16) equality to occur. But two conditions always exist for semi-invariant submanifolds of cosymplectic manifold . Accordingly, w is closed form on M. Therefore, w defines a de Rham cohomolgy class $c(B)$ such that

$$c(B) = [w] \in H^{2q}(B, \mathbb{R}).$$

On the other hand, we denote $\{E_{2q+1}, \ldots, E_{2q+p}\}$ and $\{w^{2q+1}, \ldots, w^{2q+p}\}$ an orthonormal frame and dual frame of

D_T, respectively. Let $w^* = w^{2q+1} \wedge \ldots \wedge w^{2q+p}$ be p-form on M. Therefore similarly way for w, we can say that, $D^\perp$ is minimal and D_T is integrable, then w^* is closed, hence $2q$ -form w is coclosed. We know that, B is closed submanifold, then w is harmonic. Since w is nontrivial, the cohomology class $[w]$ characterize by w is nontrivial in $H^{2q}(B, \mathbb{R})$.

5 Discussion and conclusion

Contact geometry has an important application for many sciences such as physics, geometric optics, technology, thermodynamics, classical mechanics, medical sciences and classical mechanics. Researchers have increased studies on this field from different areas in recent years. The improvement of the contact geometry depends on the differential geometry of the manifolds with structures. Another subclass of contact geometry is the cosymplectic manifolds. Topology of cosymplectic manifolds is less explored, and there is a shortlist of papers in the mathematical literature on this topic. The works on this subject will be useful tools for the applications for topological of the cosymplectic manifolds. Hence, we studied de Rham cohomology class for semi-invariant submanifolds of cosymplectic manifolds. Consequently, the results obtained in this article provide contribution to investigate topological properties of different submanifolds in cosymplectic manifolds.

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