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On the Developability and Distribution Parameters of the Involute Trajectory Ruled Surfaces

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Abstract. In this study, ruled surfaces formed by Frenet's trihedral of involute curve β of a given curve α are discussed. These surfaces are named as involute trajectory ruled surfaces. These type of ruled surfaces are expressed depending on the angle θ between the binormal vector b and Darboux vector D of the main curve (evolute) α. Also, some new results and theorems related to the developability of the involute trajectory ruled surfaces are obtained. Finally we illustrate these surfaces by presenting some examples.

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Keywords: Trajectory ruled surface, developable surface, distribution parameter, involute-evolute, Frenet frame, Euclidean 3-space.

1. Introduction

A ruled surface can be generated by the motion of a line in space, similar to the way that a curve can be generated by the motion of a point. So, in spatial motion, the trajectories of oriented lines embedded in a moving space or in a moving rigid body are generally called trajectory ruled surfaces (see [7]). A developable ruled surface can be defined as isometrically into the plane. The most obvious examples of developable ruled surfaces are cones and cylinders. Developable surfaces play an distinguished role since they have broad applications in many areas from engineering to manufacturing. For instance, an aircraft designer uses them to design airplane wings. In textile design one starts with a planar piece of cloth to produce garments and their quality improves if the cloth is not stretched. In naval industry one has to adapt planar sheets of steel to the form of the hull of a vessel. This can be done with a folding machine if the result is a developable surface, avoiding the application of heat and reducing the costs. They are also useful for modeling pages of a book [11] for 3D reconstruction and they can also be found in architectural constructions [12].

In the literature, there have been many studies on trajectory ruled surfaces and developable ruled surfaces. Gürsoy and Küçük gave the some new results on the geometric invariants of the closed trajectory ruled surfaces for spatial motions [5]. Also, Kucuk gave the developable of Bertrand trajectory ruled surface offsets [6]. He also showed that the striction curve of the base ruled surface is a helix if there are more than one developable Bertrand offsets within a developable ruled surface. The geometry of trajectory ruled surfaces is widely applied to the study of design problems in spatial mechanisms or space kinematics [4, 13, 14].

On the other hand, Yaylı and Saraçoglu [15] studied timelike and spacelike developable ruled surfaces in Minkowski space. Orbay and Aydemir [10] obtained the distrubition parameter, mean curvature, and Gaussian curvature, and some new results and theorems were given for developable and minimal spacelike ruled surfaces. In [1], Bayram and Bilici

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consructed a surface family possessing an involute β of a given curve α as an asymptotic curve with curvatures κ, τ. They say that if $\frac{\tau}{\kappa} \neq \mp 1$ then there exists a ruled surface possessing β as an asymptotic curve. As a result of this proposition, they concluded that the ruled surface is developable if and only if α is a unit speed helix.

In this paper, unlike existing literature, a generalization of involute trajectory ruled surfaces generated by the Frenet trihedron moving along involutes of a given curve is stated by a firmly connected angle between the binormal vector and Darboux vector of this base curve. And, some new results and theorems related to the developability of involute trajectory ruled surfaces are obtained. Additionally, we illustrate these type of ruled surfaces with three different example.

2. Preliminaries

Suppose we are given a 3-dimensional parametric curve $r(s) : I \to E^3$, $s \in I \subset \mathbb{R}$, in which s is the arch length (regular and $\|r'(s)\| = 1$) and $r(s)$ has second derivatives. We assume that $r''(s) \neq 0$, because otherwise the curve is a straight line segment or the principal normal is undefined at some point on the curve. Because $r(s)$ is a regular curve with $r''(s) \neq 0$, the Frenet frame $\{t(s), n(s), b(s)\}$ along $r(s)$ is defined, where $t(s) = r'(s)$, $n(s) = r''(s)/\|r'(s)\|$, $b(s) = t(s) \times n(s)$ are the unit tangent, principal normal and binormal vectors of the curve at the point $r(s)$, respectively. The derivative formulas of the Frenet frame are governed by the following relations [3]:

$$\frac{d}{ds}\begin{pmatrix} t(s) \\ n(s) \\ b(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & \tau(s) & 0 \end{pmatrix}\begin{pmatrix} t(s) \\ n(s) \\ b(s) \end{pmatrix}, \tag{2.1}$$

where $\kappa(s)$ and $\tau(s)$ are the curvature and torsion of the curve $r(s)$, respectively.

The Frenet formulae can be interpreted kinematically as follows: If a moving point traverses the curve in such a way that s is the time parameter, then the moving frame $\{t(s), n(s), b(s)\}$ moves in accordance with Eq. (2.1). This motion contains, apart from an instantaneous translation, an instantaneous rotation with an angular velocity vector given by the Darboux vector [8].

$$D(s) = \tau(s)\, t(s) + \kappa(s)\, b(s). \tag{2.2}$$

The direction of the Darboux vector is that of the instantaneous axis of rotation and its length $\|D\| = \sqrt{\kappa^2(s) + \tau^2(s)}$ is the scalar angular velocity [8]. Let θ denote the angle from $b(s)$ to $D(s)$ measured in the sense of the shortest rotation which brings $b(s)$ into $t(s)$. Then we have

$$\kappa = \|D\| \cos\theta, \quad \tau = \|D\| \sin\theta. \tag{2.3}$$

From Eq. (2.3) we say that $\frac{\tau}{\kappa} = \tan\theta$ and if θ is a constant then $r(s)$ is a general helix.

Now consider a space curve $r(s)$ and its involutes $\gamma(s)$. If $t(s)$ denotes the unit tangent of $r(s)$ then $\gamma(s)$ has principal normal $n^*(s)$ which is the same as $t(s)$. A point on $\gamma(s)$ corresponding to a point on $r(s)$ is then given by

$$\gamma(s) = r(s) + (c - s)\, t(s), \tag{2.4}$$

where c is a constant and $t(s) = r'(s)$. On the other hand the relationship between the Frenet trihedron of $r(s)$ and that of its involutes $\gamma(s)$ can be written as [2]

$$\begin{pmatrix} t^*(s) \\ n^*(s) \\ b^*(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\cos\theta & 0 & \sin\theta \\ \sin\theta & 0 & \cos\theta \end{pmatrix}\begin{pmatrix} t(s) \\ n(s) \\ b(s) \end{pmatrix}. \tag{2.5}$$

The trace of an X oriented line along a space curve $r(s)$ is generally a trajectory ruled surface. A parametric equation of this trajectory ruled surface generated by X oriented line is given by

$$\varphi(s,k) = r(s) + k d(s), \quad s, k \in I \subset \mathbb{R},$$

where d is the unit direction vector of X oriented line. The distribution parameter (or drall) of the $\varphi(s,k)$ trajectory ruled surface is given as

$$\delta_d = \frac{\det(r', d, d')}{\|d'\|^2}. \tag{2.6}$$

A developable trajectory ruled surface is characterized by $\delta_d = 0$. If there exist a common perpendicular to two constructive rulings in the ruled surface, then the foot of the common perpendicular on the main ruling is called a central point. The locus of the central point is called striction curve. The parametrization of the sitriction curve on a trajectory ruled surface in [9] is given

$$C(s) = r(s) - \frac{\langle r', d' \rangle}{\|d'\|^2} d.$$

3. Involute Trajectory Ruled Surfaces

Let $\gamma(s)$ be the involutes of a space curve $r(s)$ and $\{t^*(s),\ n^*(s),\ b^*(s)\}$ be its Frenet frame defined as in Eq. (2.5). And also, we consider that a oriented unit line $X(s)$ in E^3 such that it is firmly connected to Frenet frame of the involutes $\gamma(s)$ is represented, uniquely with respect to this frame, in the form

$$X(s) = x_1 t^*(s) + x_2 n^*(s) + x_3\, b^*(s),\ \|X\| = 1, \tag{3.1}$$

where x_i $(i = 1,\ 2,\ 3)$ are scalar of the arc length parameter of the involutes $\gamma(s)$. The trajectory ruled surfaces generated by line $X(s)$, $t^*(s)$, $n^*(s)$ and $b^*(s)$ are

$$M : \varphi(s, v) = \gamma(s) + vX(s),$$

$$M_1 : \varphi(s, u) = \gamma(s) + ut^*(s), \tag{3.2}$$

$$M_2 : \varphi(s, z) = \gamma(s) + zn^*(s), \tag{3.3}$$

$$M_3 : \varphi(s, w) = \gamma(s) + wb^*(s), \tag{3.4}$$

respectively. We can obtained the distribution parameter of the involute trajectory ruled surface generated by $X(s)$ in E^3. Analitically, from equations (2.5), (3.1) and Frenet formulas

$$\begin{aligned} X'(s) &= (-x_1\kappa + x_2\theta' \sin\theta + x_3\ \theta' \cos\theta)\, t + (-x_2 \|D\|)\, n \\ &\quad + (x_1\tau + x_2\theta' \cos\theta - x_3\ \theta' \sin\theta)\, b. \end{aligned} \tag{3.5}$$

By diffentiating Eq. (2.4) with respect to the arc length parameter s, we have

$$\gamma'(s) = (c - s)\, \kappa n. \tag{3.6}$$

By substituting Eqs.(3.5) and (3.6) into the Eq. (2.6), the distribution parameter of this surface is

$$\begin{aligned} \delta_X &= \frac{\det(\gamma', X, X')}{\|X'\|^2}, \\ &= \frac{(c-s)\,\kappa\left[\theta'\left(x_2^2 + x_3^2\right) - x_1 x_3 \|D\|\right]}{\left(x_1^2 + x_2^2\right)\|D\|^2 + \left(x_2^2 + x_3^2\right)\theta'^2 - 2\theta' x_1 x_3 \|D\|}. \end{aligned} \tag{3.7}$$

The ruled surface developable if and only if δ_X is zero. From Eq. (3.7) we have

$$(c-s)\,\kappa\left[\theta'\left(x_2^2 + x_3^2\right) - x_1 x_3 \|D\|\right] = 0.$$

Thus we state the following theorem.

Theorem 3.1. *The involute trajectory ruled surface M is developable if and only if the angle between $b(s)$ and $D(s)$ of space curve $r(s)$ satisfies the following equality*

$$\theta = \frac{x_1 x_3}{x_2^2 + x_3^2} \int \|D\|\, ds + \lambda,$$

where λ is an arbitrary constant.

4. Special Cases

4.1. The Case $X(s) = t^*(s)$.

In this case, $x_1 = 1$, $x_2 = x_3 = 0$. Thus from Eq. (3.7)

$$\delta_{t^*} = 0.$$

We can give the following corollary

Corollary 4.1. *The involute trajectory ruled surface M_1 given by the equation (3.2) is developable.*

4.2. **The Case $X(s) = n^*(s)$.** In this case, $x_1 = x_3 = 0$, $x_2 = 1$ and from Eq. (3.7)

$$\delta_{n^*} = \frac{(c-s)\,\kappa\theta'}{\|D\|^2 + \theta'^2}. \tag{4.1}$$

So we can give following corollary.

Corollary 4.2. *If θ =constant (i.e. space curve $r(s)$ is a general helix) then the involute trajectory ruled surface M_2 given by the equation (3.3) is developable.*

4.3. **The Case $X(s) = b^*(s)$.** In this case, $x_1 = x_2 = 0$, $x_3 = 1$ and from Eq. (3.7)

$$\delta_{b^*} = \frac{(c-s)\,\kappa}{\theta'}. \tag{4.2}$$

So we can give following corollary.

Corollary 4.3. *If $c = s$ (i.e. space curve $r(s)$ is coincident with $\gamma(s)$) then the involute trajectory ruled surface M_3 given by the equation (3.4) is developable.*

From the Eqs. (4.1) and (4.2) we have

$$\frac{\delta_{n^*}}{\delta_{b^*}} = 1 + \left(\frac{\theta'}{\|D\|}\right)^2. \tag{4.3}$$

Thus, the following theorem can be given.

Theorem 4.4. *If θ denotes the angle from $b(s)$ to $D(s)$ measured in the sense of the shortest rotation wich brings $b(s)$ into $t(s)$ then there is the relationship (4.3) between Darboux vector of $r(s)$ and the distribution parameters of the involute trajectory ruled surfaces generated by $n^*(s)$ and $b^*(s)$.*

From Eq. (4.3), if θ is constant then $\frac{\delta_{n^*}}{\delta_{b^*}} = 1$. On the contrary, if $\frac{\delta_{n^*}}{\delta_{b^*}} = 1$ then θ is constant. Therefore, with respect to this condition, we can give the following theorem.

Theorem 4.5. *The space curve $r(s)$ is general helix if and only if $\frac{\delta_{n^*}}{\delta_{b^*}} = 1$.*

4.4. **The Case $X(s)$ is in the Normal Plane.** In this case x_1 is zero $\left(x_2^2 + x_3^2 = 1\right)$. From Eq. (3.7), the distribution parameters of the involute trajectory ruled surface M is

$$\delta_X = \frac{(c-s)\,\kappa\theta'}{x_2^2\,\|D\|^2 + \theta'^2}.$$

Hence the following corollary holds.

Corollary 4.6. *If the space curve $r(s)$ is a general helix or coincident with $\gamma(s)$ then the involute trajectory ruled surface M which generated by the oriented line $X(s)$ in the normal plane is developable.*

4.5. **The Case $X(s)$ is in the Osculating Plane.** In this case x_3 is zero $\left(x_1^2 + x_2^2 = 1\right)$. From Eq. (3.7), the distribution parameters of the involute trajectory ruled surface M is

$$\delta_X = \frac{(c-s)\,\kappa\theta' x_2^2}{\|D\|^2 + x_2^2\theta'^2}.$$

Thus Corollary 4.6 can be restated for the involute trajectory ruled surface M which generated by the oriented line $X(s)$ in the osculating plane.

4.6. **The Case $X(s)$ is in the Rectifying Plane.** In this case x_2 is zero $\left(x_1^2 + x_3^2 = 1\right)$. From Eq. (3.7), the distribution parameters of the involute trajectory ruled surface M is

$$\delta_X = \frac{(c-s)\,\kappa\left[\theta' x_3^2 - x_1 x_3\,\|D\|\right]}{x_1^2\,\|D\|^2 + x_3^2\theta'^2 - 2\theta' x_1 x_3\,\|D\|}.$$

Hence the following corollary holds.

Corollary 4.7. *The involute trajectory ruled surface M which generated by the oriented line $X(s)$ in the rectifying plane is developable if and only if the angle between $b(s)$ and $D(s)$ of space curve $r(s)$ satisfies the following equality*

$$\theta = \frac{x_1}{x_3}\int \|D\|\, ds + \lambda,$$

where λ is an arbitrary constant.

4.7. **The Case the Base Curve $\gamma(s)$ is the Striction Curve $C(s)$.** From Eq. (2.6) the parametrization of the striction curve on a involute trajectory ruled surface generated by oriented line $X(s)$ is given by

$$C(s) = \gamma(s) + \frac{x_2 (c-s)\,\kappa\,\|D\|}{\|X'\|^2} X.$$

From here if the base curve $\gamma(s)$ is the striction curve $C(s)$ then we have $x_2 = 0$ or $c = s$. Hence the following theorem can be given.

Theorem 4.8. *If the base curve $\gamma(s)$ is the same as the striction curve $C(s)$ then the oriented line $X(s)$ of the involute trajectory ruled surface is in the rectifying plane or space curve $r(s)$ is coincident with $\gamma(s)$*

5. Examples of Involute Trajectory Ruled Surfaces

Example 5.1. Let $r(s) = \left(\frac{4}{5}\cos s, 1-\sin s, -\frac{3}{5}\cos s\right)$ be a unit speed curve. Then, it is easy to show that

$$\begin{aligned} t(s) &= \left(-\frac{4}{5}\sin s, -\cos s, \frac{3}{5}\sin s\right), \\ n(s) &= \left(-\frac{4}{5}\cos s, \sin s, \frac{3}{5}\cos s\right), \\ b(s) &= \left(-\frac{3}{5}, 0, -\frac{4}{5}\right), \end{aligned}$$

with curatures $\kappa = 1$ and $\tau = 0$. In this situation, the involutes $\gamma(s)$ of the curve $r(s)$ can be given by the equation

$$\gamma(s) = \left(\frac{4}{5}\cos s - \frac{4}{5}(c-s)\sin s, 1-\sin s-(c-s)\cos s, -\frac{3}{5}\cos s + \frac{3}{5}(c-s)\sin s\right),$$

where c is an arbitrary constant.

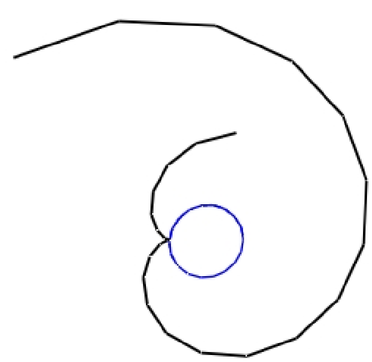

Figure 1. Cirle $r(s)$ and its involute curve $\gamma(s)$ for c=2

From Eqs. (2.2) and (2.3) we have $D(s) = b(s)$ and $\theta = 0°$, respectively. By using Eq. (2.5) we have the Frenet trihedron of the involutes $\gamma(s)$ of the curve $r(s)$

$$\begin{aligned} t^*(s) &= n(s) = \left(-\frac{4}{5}\cos s, \sin s, \frac{3}{5}\cos s\right), \\ n^*(s) &= -t(s) = \left(\frac{4}{5}\sin s, \cos s, -\frac{3}{5}\sin s\right), \\ b^*(s) &= b(s) = \left(-\frac{3}{5}, 0, -\frac{4}{5}\right). \end{aligned}$$

Thus we obtain the involute trajectory ruled surfaces generated by $t^*(s)$, $n^*(s)$ and $b^*(s)$ as

$$\begin{aligned} \varphi_{t^*}(s,v) &= (\frac{4}{5}\cos s - \frac{4}{5}(c-s)\sin s - \frac{4}{5}v\cos s, 1 - \sin s - (c-s)\cos s + v\sin s, \\ &\quad -\frac{3}{5}\cos s + \frac{3}{5}(c-s)\sin s + \frac{3}{5}v\sin s), \\ \varphi_{n^*}(s,v) &= (\frac{4}{5}\cos s - \frac{4}{5}(c-s)\sin s - \frac{4}{5}v\cos s, 1 - \sin s - (c-s)\cos s + v\sin s, \\ &\quad -\frac{3}{5}\cos s + \frac{3}{5}(c-s)\sin s + \frac{3}{5}v\sin s), \\ \varphi_{b^*}(s,v) &= (\frac{4}{5}\cos s - \frac{4}{5}(c-s)\sin s - \frac{4}{5}v\cos s, 1 - \sin s - (c-s)\cos s + v\sin s, \\ &\quad -\frac{3}{5}\cos s + \frac{3}{5}(c-s)\sin s + \frac{3}{5}v\sin s), \end{aligned}$$

respectively, where $-5 \le v \le 5$ and $c = 2$ (Figs. 2-4).

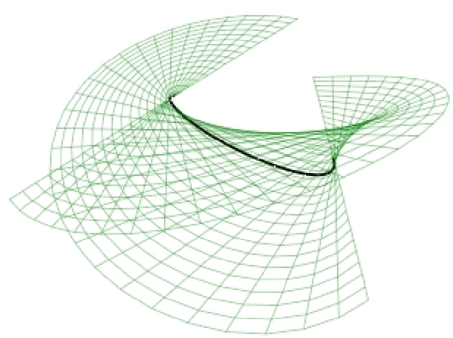

FIGURE 2. The involute trajectory ruled surface $\varphi_{t^*}(s,v)$ generated by $t^*(s)$ for $s\epsilon[-2,2]$

Example 5.2. Let $\alpha(s) = \left(\frac{\sqrt{3}}{2}\sin s, \frac{s}{2}, \frac{\sqrt{3}}{2}\cos s\right)$ be a unit speed helix. Then, it is easy to show that

$$\begin{aligned} t(s) &= \left(\frac{\sqrt{3}}{2}\cos s, \frac{1}{2}, -\frac{\sqrt{3}}{2}\sin s\right), \\ n(s) &= (-\sin s, 0, -\cos s), \\ b(s) &= \left(-\frac{1}{2}\cos s, \frac{\sqrt{3}}{2}, \frac{1}{2}\sin s\right), \end{aligned}$$

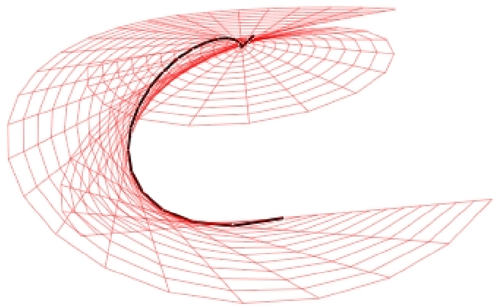

FIGURE 3. The involute trajectory ruled surface $\varphi_{n^*}(s,v)$ generated by $n^*(s)$ for $s\epsilon[-3,3]$

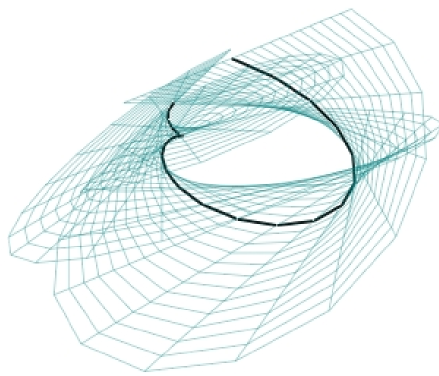

FIGURE 4. The involute trajectory ruled surface $\varphi_{n^*}(s,v)$ generated by $n^*(s)$ for $s\epsilon[-3,3]$

with curatures $\kappa = \frac{\sqrt{3}}{2}$ and $\tau = \frac{1}{2}$. In this situation, the involutes $\beta(s)$ of the curve $\alpha(s)$ can be given by the equation

$$\beta(s) = (\frac{\sqrt{3}}{2}\sin s + \frac{\sqrt{3}}{2}(c-s)\cos s, \frac{s}{2} + \frac{1}{2}(c-s), \frac{\sqrt{3}}{2}\cos s - \frac{\sqrt{3}}{2}(c-s)\sin s)$$

where c is an arbitrary constant.

From Eqs. (2.2) and (2.3) we have $D(s) = \frac{1}{2}t(s) + \frac{\sqrt{3}}{2}b(s)$ and $\theta = 30°$, respectively. By using Eq. (2.5) we have the Frenet trihedron of the involutes $\beta(s)$ of the curve $\alpha(s)$

$$\begin{aligned} t^*(s) &= n(s) = (-\sin s, 0, -\cos s), \\ n^*(s) &= -\frac{\sqrt{3}}{2}t(s) + \frac{1}{2}b(s) = (-\cos s, 0, \sin s), \\ b^*(s) &= \frac{1}{2}t(s) + \frac{\sqrt{3}}{2}b(s) = (0,1,0). \end{aligned}$$

Thus we obtain the involute trajectory ruled surfaces generated by $t^*(s)$, $n^*(s)$ and $b^*(s)$ as

$$\begin{aligned} P_{t^*}(s,v) = &(\frac{\sqrt{3}}{2}\sin s + \frac{\sqrt{3}}{2}(c-s)\cos s - v\sin s, \frac{s}{2} + \frac{1}{2}(c-s), \\ &\frac{\sqrt{3}}{2}\cos s - \frac{\sqrt{3}}{2}(c-s)\sin s - v\cos s), \end{aligned}$$

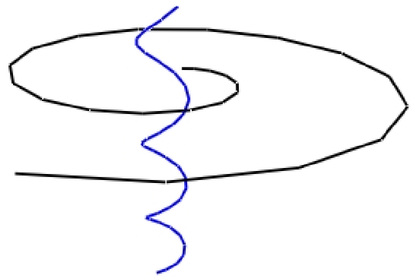

FIGURE 5. Helix $\alpha(s)$ and its involute curve $\beta(s)$ for c=6

$$
\begin{aligned}
P_{n^*}(s,v) \quad = \quad & (\frac{\sqrt{3}}{2}\sin s + \frac{\sqrt{3}}{2}(c-s)\cos s - v\cos s, \frac{s}{2} + \frac{1}{2}(c-s), \\
& \frac{\sqrt{3}}{2}\cos s - \frac{\sqrt{3}}{2}(c-s)\sin s + v\sin s),
\end{aligned}
$$

$$
\begin{aligned}
P_{b^*}(s,v) \quad = \quad & (\frac{\sqrt{3}}{2}\sin s + \frac{\sqrt{3}}{2}(c-s)\cos s, \frac{s}{2} + \frac{1}{2}(c-s) + v, \\
& \frac{\sqrt{3}}{2}\cos s - \frac{\sqrt{3}}{2}(c-s)\sin s),
\end{aligned}
$$

respectively, where $-5 \leq s, v \leq 5$ and $c = 6$ (Figs. 6-8).

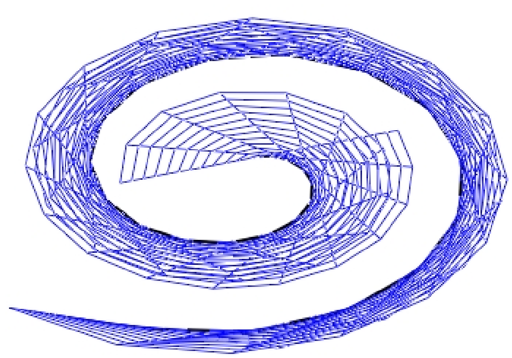

FIGURE 6. The involute trajectory ruled surface $P_{t^*}(s,v)$ generated by $t^*(s)$

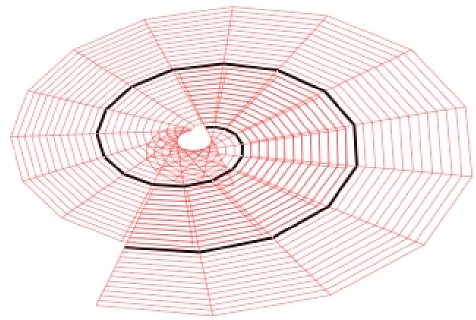

FIGURE 7. The involute trajectory ruled surface $P_{n^*}(s, v)$ generated by $n^*(s)$

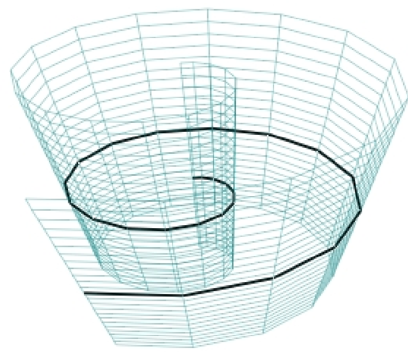

FIGURE 8. The involute trajectory ruled surface $P_{b^*}(s, v)$ generated by $b^*(s)$

Example 5.3. Let Let $\eta(s) = \left(\cos\left(\frac{\sqrt{2}}{2}s\right), \sin\left(\frac{\sqrt{2}}{2}s\right), \frac{\sqrt{2}}{2}s\right)$ be a unit speed helix. Then, it is easy to show that

$$\begin{aligned} t(s) &= \frac{\sqrt{2}}{2}\left(-\sin\left(\frac{\sqrt{2}}{2}s\right), \cos\left(\frac{\sqrt{2}}{2}s\right), 1\right), \\ n(s) &= \left(-\cos\left(\frac{\sqrt{2}}{2}s\right), -\sin\left(\frac{\sqrt{2}}{2}s\right), 0\right), \\ b(s) &= \frac{\sqrt{2}}{2}\left(\sin\left(\frac{\sqrt{2}}{2}s\right), -\cos\left(\frac{\sqrt{2}}{2}s\right), 1\right), \end{aligned}$$

with curatures $\kappa = \tau = \frac{1}{2}$. In this situation, the involutes $\zeta(s)$ of the curve $\eta(s)$ can be given by the equation

$$\zeta(s) = (\cos\left(\frac{\sqrt{2}}{2}s\right) - \frac{\sqrt{2}}{2}(c - s)\sin\left(\frac{\sqrt{2}}{2}s\right), \sin\left(\frac{\sqrt{2}}{2}s\right) + \frac{\sqrt{2}}{2}(c - s)\cos\left(\frac{\sqrt{2}}{2}s\right), \left(\frac{\sqrt{2}}{2} - 1\right)s + c),$$

where c is an arbitrary constant.

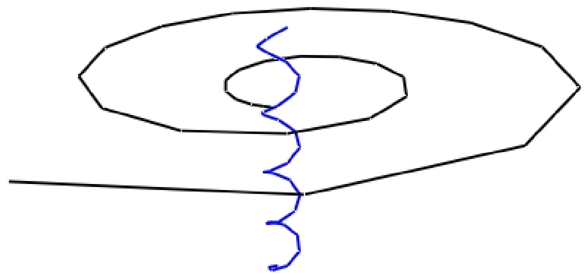

FIGURE 9. Helix $\eta(s)$ and its involute curve $\zeta(s)$ for c=10

From Eqs. (2.2) and (2.3) we have $D(s) = \frac{1}{2}[t(s) + b(s)]$ and $\theta = 45°$, respectively. By using Eq. (2.5) we have the Frenet trihedron of the involutes $\zeta(s)$ of the curve $\eta(s)$

$$\begin{aligned} t^*(s) &= n(s) = \left(-\cos\left(\frac{\sqrt{2}}{2}s\right), -\sin\left(\frac{\sqrt{2}}{2}s\right), 0\right), \\ n^*(s) &= -\frac{\sqrt{2}}{2}[t(s) - b(s)] = \left(\sin\left(\frac{\sqrt{2}}{2}s\right), -\cos\left(\frac{\sqrt{2}}{2}s\right), 0\right), \\ b^*(s) &= \frac{\sqrt{2}}{2}[t(s) + b(s)] = \left(\sin\left(\frac{\sqrt{2}}{2}s\right), 0, 1\right). \end{aligned}$$

Thus we obtain the involute trajectory ruled surfaces generated by $t^*(s)$, $n^*(s)$ and $b^*(s)$ as ,

$$\begin{aligned} K_{t^*}(s,v) &= (\cos\left(\frac{\sqrt{2}}{2}s\right) - \frac{\sqrt{2}}{2}(c-s)\sin\left(\frac{\sqrt{2}}{2}s\right) - v\cos\left(\frac{\sqrt{2}}{2}s\right), \sin\left(\frac{\sqrt{2}}{2}s\right) + \\ &\qquad \frac{\sqrt{2}}{2}(c-s)\cos\left(\frac{\sqrt{2}}{2}s\right) - v\sin\left(\frac{\sqrt{2}}{2}s\right), \ \frac{\sqrt{2}}{2}s + (c-s)), \\ K_{n^*}(s,v) &= (\cos\left(\frac{\sqrt{2}}{2}s\right) - \frac{\sqrt{2}}{2}(c-s)\sin\left(\frac{\sqrt{2}}{2}s\right) + v\sin\left(\frac{\sqrt{2}}{2}s\right), \sin\left(\frac{\sqrt{2}}{2}s\right) + \\ &\qquad \frac{\sqrt{2}}{2}(c-s)\cos\left(\frac{\sqrt{2}}{2}s\right) - v\cos\left(\frac{\sqrt{2}}{2}s\right), \ \frac{\sqrt{2}}{2}s + (c-s)), \\ K_{b^*}(s,v) &= (\cos\left(\frac{\sqrt{2}}{2}s\right) - \frac{\sqrt{2}}{2}(c-s)\sin\left(\frac{\sqrt{2}}{2}s\right) + v\sin\left(\frac{\sqrt{2}}{2}s\right), \sin\left(\frac{\sqrt{2}}{2}s\right) + \\ &\qquad \frac{\sqrt{2}}{2}(c-s)\cos\left(\frac{\sqrt{2}}{2}s\right), \ \frac{\sqrt{2}}{2}s + (c-s) + v), \end{aligned}$$

respectively, where $-5 \le s, v \le 5$ and $c = 10$ (Figs. 10-12).

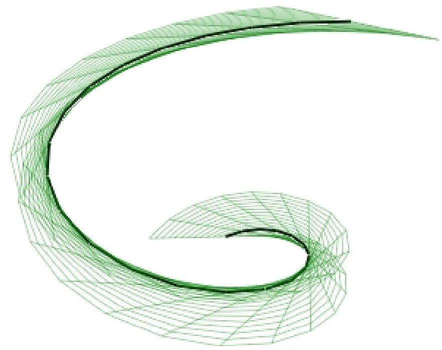

FIGURE 10. The involute trajectory ruled surface: $K_{t^*}(s,v)$ generated by $t^*(s)$

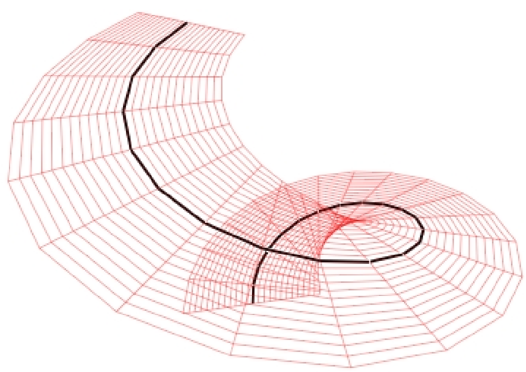

FIGURE 11. The involute trajectory ruled surface: $K_{n^*}(s,v)$ generated by $n^*(s)$

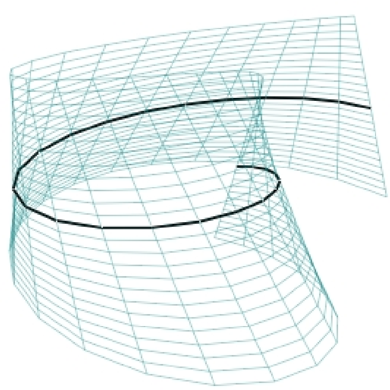

FIGURE 12. The involute trajectory ruled surface: $K_{b^*}(s,v)$ generated by $b^*(s)$

6. Conclusion

Involute trajectory ruled surfaces occurs as a result of the continuosly movement of the frenet vectors along the involute curve. In this study, Involute trajectory ruled surfaces is stated by a firmly connected angle between the binormal vector and Darboux vector of this base curve. Also, some new results and theorems related to the developability of the involute of trajectory ruled surfaces are obtained. It is hoped that this study will provide the impetus for new studies and contribute to the study of trajectory ruled surfaces.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

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