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On Slant Curves in Sasakian Lorentzian 3-Manifolds

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ABSTRACT

In this paper, we study C-parallel mean curvature vector field and C-proper mean curvature vector field along a slant Frenet curve in a Sasakian Lorentzian 3-manifold. In particular, we prove that a slant Frenet curve γ in a Sasakian Lorentzian 3-manifold M satisfying $\Delta_{\dot\gamma} H = 0$ is a geodesic or pseudo-helix with $\kappa^2 = \tau^2$. For example, we find slant pseudo-helix in Lorentzian Heisenberg 3-space.

Keywords: *Slant curves; Legendre curve; Sasakian Lorentzian manifold.*

AMS Subject Classification (2020): *Primary: 58E20 ; Secondary: 53C25*

1. Introduction

In [1], Baikoussis and Blair considered submanifolds in Sasakian space forms $M(\rho)$. They defined that the mean curvature vector field H has C-parallel if $\nabla H = \lambda\xi$, where λ is a non-zero differentiable function on M and ∇ the induced Levi-Civita connection. In [8], we studied curves with C-parallel and C-proper mean curvature vector fields in the tangent and normal bundles of Sasakian 3-manifolds. So we showed that γ has a C-parallel mean curvature vector field if and only if γ is a geodesic $(\lambda = 0)$ or a helix with $\kappa = \sqrt{-\lambda\cos\theta}$ and $\tau = \lambda\sin\theta/\sqrt{-\lambda\cos\theta}$. In particular, for a Legendre curve γ in a Sasakian 3-manifold M, we proved that γ satisfies $\nabla_{\gamma'} H = \lambda\xi$ if and only if γ is a Legendre geodesic.

In this paper, we study C-parallel mean curvature vector field and C-proper mean curvature vector field in Sasakian Lorenzian 3-manifolds.

Let H be the mean curvature vector field of a curve in 3-dimensional contact Lorentzian manifolds M. The mean curvature vector field H is said to be *C-parallel* if $\nabla H = \lambda\xi$. Moreover, the vector field H is said to be *C-proper* if $\Delta H = \lambda\xi$, where ∇ denotes the operator of covariant differentiation of M. Similarly, in the normal bundle we define C-parallel and C-proper as follows: H is said to be *C-parallel in the normal bundle* if $\nabla^\perp H = \lambda\xi$, and H is said to be *C-proper in the normal bundle* if $\Delta^\perp H = \lambda\xi$, where $\nabla^\perp$ denotes the operator of covariant differentiation in the normal bundle of M.

Generalizing a Legendre curve in a 3-dimensional contact metric manifold, we consider a slant curve whose tangent vector field has constant angle with characteristic direction ξ (see [5]). For a non-geodesic slant curve in a Sasakian 3-manifold, the direction ξ becomes $\xi = \cos\alpha_0 T + \sin\alpha_0 B$, where T and B are unit tangent vector field and binormal vector field respectively, that is, the characteristic vector field ξ is orthogonal to the principal normal vector field N.

In section 3, we study a slant Frenet curve with C-parallel mean curvature vector field and C-proper mean curvature vector field in Sasakian Lorentzian 3-manifolds.
In [8], from the point of view of Riemannian structure, we found that a slant curve γ in a Sasakian 3-manifold satisfying $\Delta_{\dot\gamma} H = 0$ is a geodesic. Now, we prove that a slant Frenet curve γ in a Sasakian Lorentzian 3-manifold M satisfying $\Delta_{\dot\gamma} H = 0$ is a geodesic or pseudo-helix with $\kappa^2 = \tau^2$.
Thus, we find a necessary and sufficient condition for a slant Frenet curve with C-parallel mean curvature vector field and C-proper mean curvature vector field in Sasakian Lorenzian 3-manifolds in the normal bundle.

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2. Preliminaries

2.1. *Contact Lorentzian manifold*

Let M be a $(2n+1)$-dimensional differentiable manifold. M has an almost contact structure (φ, ξ, η) if it admits a tensor field φ of $(1,1)$, a vector field ξ and a 1-form η satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \;\; \eta(\xi) = 1.$$

Suppose M has an almost contact structure (φ, ξ, η). Then $\varphi\xi = 0$ and $\eta \circ \varphi = 0$. Moreover, the endomorphism φ has rank $2n$.

If a $(2n+1)$-dimensional smooth manifold M with almost contact structure (φ, ξ, η) admits a compatible Lorentzian metric such that

$$g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.1}$$

then we say M has an almost contact Lorentzian structure (η, ξ, φ, g). Setting $Y = \xi$ we have

$$\eta(X) = -g(X, \xi). \tag{2.2}$$

Next, if the compatible Lorentzian metric g satisfies

$$d\eta(X, Y) = g(X, \varphi Y),$$

then η is a contact form on M, ξ the associated Reeb vector field, g an associated metric and $(M, \varphi, \xi, \eta, g)$ is called a *contact Lorentzian manifold*.

For a contact Lorentzian manifold M, one may define naturally an almost complex structure J on $M \times \mathbb{R}$ by

$$J(X, f\frac{\mathrm{d}}{\mathrm{d}t}) = (\varphi X - f\xi, \eta(X)\frac{\mathrm{d}}{\mathrm{d}t}),$$

where X is a vector field tangent to M, t the coordinate of $\mathbb{R}$ and f a function on $M \times \mathbb{R}$. If the almost complex structure J is integrable, then the contact Lorentzian manifold M is said to be *normal* or *Sasakian*. It is known that a contact Lorentzian manifold M is normal if and only if M satisfies

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0,$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ.

Proposition 2.1 ([3, 4])**.** *An almost contact Lorentzian manifold* $(M^{2n+1}, \eta, \xi, \varphi, g)$ *is Sasakian if and only if*

$$(\nabla_X \varphi)Y = g(X, Y)\xi + \eta(Y)X.$$

Using the similar arguments and computations in [2] we obtain

Proposition 2.2 ([3, 4])**.** *Let* $(M^{2n+1}, \eta, \xi, \varphi, g)$ *be a contact Lorentzian manifold. Then*

$$\nabla_X \xi = \varphi X - \varphi h X.$$

If ξ is a killing vector field with respect to the Lorentzian metric g. Then we have

$$\nabla_X \xi = \varphi X. \tag{2.3}$$

2.2. *Frenet-Serret equations*

Let $\gamma : I \to M^3$ be a unit speed curve in Lorentzian 3-manifolds M^3 such that γ' satisfies $g(\gamma', \gamma') = \varepsilon_1 = \pm 1$. The constant ε_1 is called the *causal character* of γ. A unit speed curve γ is said to be a spacelike or timelike if its causal character is 1 or -1, respectively. A unit speed curve γ is said to be a *Frenet curve* if $g(\gamma'', \gamma'') \neq 0$. A Frenet curve γ admits a orthonormal frame field $\{T = \gamma', N, B\}$ along γ. Then the *Frenet-Serret* equations are following ([6, 7]):

$$\begin{cases} \nabla_{\gamma'} T = \quad\quad \varepsilon_2 \kappa N, \\ \nabla_{\gamma'} N = -\varepsilon_1 \kappa T \quad - \varepsilon_3 \tau B, \\ \nabla_{\gamma'} B = \quad\quad \varepsilon_2 \tau N, \end{cases} \tag{2.4}$$

where $\kappa = |\nabla_{\gamma'}\gamma'|$ is the *geodesic curvature* of γ and τ its *geodesic torsion*. The vector fields T, N and B are called tangent vector field, principal normal vector field, and binormal vector field of γ, respectively.

The constant ε_2 and ε_3 defined by $g(N,N) = \varepsilon_2$ and $g(B,B) = \varepsilon_3$, and called *second causal character* and *third causal character* of γ, respectively. Thus it satisfied $\varepsilon_1\varepsilon_2 = -\varepsilon_3$.

A Frenet curve γ is a *geodesic* if and only if $\kappa = 0$. A Frenet curve γ with constant geodesic curvature and zero geodesic torsion is called a *pseudo-circle*. A *pseudo-helix* is a Frenet curve γ whose geodesic curvature and torsion are constant.

Proposition 2.3 ([11])**.** *Let $\{E_1, E_2, E_3\}$ are orthonomal Frame field in a Lorentzian 3-manifold. Then*

$$E_1 \wedge_L E_2 = \varepsilon_3 E_3, \quad E_2 \wedge_L E_3 = \varepsilon_1 E_1, \quad E_3 \wedge_L E_1 = \varepsilon_2 E_2.$$

2.3. Slant curves

A one-dimensional integral submanifold of D in 3-dimensional contact manifold is called a *Legendre curve*, especially to avoid confusion with an integral curve of the vector field ξ. As a generalization of Legendre curve, the notion of slant curves was introduced in [5] for a contact Riemannian 3-manifold, that is, a curve in a contact 3-manifold is said to be *slant* if its tangent vector field has constant angle with the Reeb vector field.

Similarly a curve in a contact Lorentzian 3-manifold is said to be *slant* if its tangent vector field has constant angle with the Reeb vector field ,that is, $g(\gamma', \xi)$ is a constant. In particular, if $g(\gamma', \xi) = 0$ then γ is a Legendre curve.

Differentiating $g(T, \xi) = a$ along γ in Lorentzian Sasakian manifold, then

$$a' = g(\varepsilon_2\kappa N, \xi) + g(\gamma', \varphi\gamma') = -\varepsilon_2\kappa\eta(N).$$

This equation implies

Proposition 2.4 ([10])**.** *A non-geodesic Frenet curve γ in a Sasakian Lorentzian 3-manifold M^3 is a slant curve if and only if $\eta(N) = 0$.*

Differentiating $\eta(N) = 0$, using (2.3) and the Frenet-Serret equation (2.4) we have

Theorem 2.1 ([10])**.** *A non-geodesic slant Frenet curve in a Sasakian Lorentzian 3-manifold M is the ratio of $\tau - 1$ and κ is constant.*

In particular, let γ be a non-geodesic Frenet curve in a Sasakian Lorentzian 3-manifold M. If γ is a Legendre curve then $\tau = 1$.

Moreover, we have

Lemma 2.1 ([10])**.** *Let γ be a slant Frenet curve in 3-dimensioal almost contact Lorentzian manifold M. Then we find an orthonormal frame field in M as following:*

$$T = \gamma', \quad N = \frac{\varphi T}{\sqrt{\varepsilon_1 + a^2}}, \quad B = \frac{\xi + \varepsilon_1 a T}{\sqrt{\varepsilon_1 + a^2}},$$

also $\xi = -\varepsilon_1 a T + \sqrt{\varepsilon_1 + a^2} B$.
Thus γ is a spacelike curve with spacelike normal vector field or timelike curve.

3. Main results

Let (M, g) be a semi-Riemannian manifold and $\gamma = \gamma(s) : I \to M$ a unit speed curve in M. Then the induced (or pull-back) vector bundle γ^*TM is defined by

$$\gamma^*TM := \bigcup_{s \in I} T_{\gamma(s)}M.$$

The Levi-Civita connection ∇ of M induces a connection ∇^γ on γ^*TM as follows:

$$\nabla^\gamma_{\frac{d}{ds}} V = \nabla_{\dot\gamma} V, \ \ V \in \Gamma(\gamma^*TM).$$

The *Laplacian operator* $\Delta = \Delta^{\gamma}$ of $\gamma^* TM$ is given explicitly by

$$\Delta = -\varepsilon_1 \nabla_{\dot\gamma} \nabla_{\dot\gamma}.$$

The mean curvature vector field H of a curve γ in 3-dimensional contact Lorentzian manifolds is defined by

$$H = \varepsilon_1 \nabla_{\dot\gamma}\dot\gamma = -\varepsilon_3 \kappa N.$$

In particular, for a Legendre curve γ we get

$$H = \varepsilon_1 \nabla_{\dot\gamma}\dot\gamma = -\varepsilon_3 \kappa \varphi\dot\gamma.$$

Differentiating $\varphi\dot\gamma$ along γ, we get $\tau = 1$.

Using (2.4), we have

Lemma 3.1. *Let γ be a Frenet curve in a Sasakian Lorentzian 3-manifold M. Then*

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma = \varepsilon_3 \kappa^2 T + \varepsilon_2 \kappa' N + \varepsilon_1 \kappa\tau B, \tag{3.1}$$

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma = 3\varepsilon_3 \kappa\kappa' T + \{\varepsilon_2 \kappa'' - \kappa(\varepsilon_1\kappa^2 + \varepsilon_3\tau^2)\}N + \varepsilon_1(2\kappa'\tau + \kappa\tau')B. \tag{3.2}$$

Let H be the mean curvature vector field of a curve in 3-dimensional contact Lorentzian manifolds M. The mean curvature vector field H is said to be *C-parallel* if $\nabla H = \lambda\xi$. Moreover, the vector field H is said to be *C-proper* if $\Delta H = \lambda\xi$, where ∇ denotes the operator of covariant differentiation of M. Similarly, H is said to be *C-parallel in the normal bundle* if $\nabla^{\perp} H = \lambda\xi$, and H is said to be *C-proper in the normal bundle* if $\Delta^{\perp} H = \lambda\xi$, where $\nabla^{\perp}$ denotes the operator of covariant differentiation in the normal bundle of M.

For a slant Frenet curve γ in Sasakian Lorentzian 3-manifolds, Using the Lemma 2.1 and (3.1) we find that γ satisfies $\nabla_{\dot\gamma} H = \lambda\xi$ if and only if

$$\begin{cases} \kappa^2 = \varepsilon_1 a\lambda, \\ \kappa' = 0, \\ \kappa\tau = \lambda\sqrt{\varepsilon_1 + a^2}. \end{cases} \tag{3.3}$$

Therefore we obtain:

Theorem 3.1. *Let γ be a slant Frenet curve in a Sasakian Lorentzian 3-manifold with C-parallel mean curvature vector field. we have*
(i) If γ is a Legendre curve or $\lambda = 0$, then it is a geodesic.
(ii) If γ is not Legendre curve and $\lambda \neq 0$, then it is a pseudo-helix with $\kappa = \sqrt{\varepsilon_1 a\lambda}$, $\tau = \sqrt{\frac{(\varepsilon_1 + a^2)\lambda}{\varepsilon_1 a}}$.

Next, for a slant Frenet curve γ in Sasakian Lorentzian 3-manifolds, from the Lemma 2.1 and (3.2) we find that γ satisfies $\Delta_{\dot\gamma} H = \lambda\xi$ if and only if

$$\begin{cases} 3\kappa\kappa' = -a\lambda, \\ \kappa'' - \varepsilon_2\kappa(\varepsilon_1\kappa^2 + \varepsilon_3\tau^2) = 0, \\ \varepsilon_3(2\kappa'\tau + \kappa\tau') = \lambda\sqrt{\varepsilon_1 + a^2}. \end{cases} \tag{3.4}$$

Hence we have:

Proposition 3.1. *Let γ be a slant Frenet curve in a Sasakian Lorentzian 3-manifold. Then γ has a C-proper mean curvature vector field if and only if γ satisfies $\Delta_{\dot\gamma} H = 0$.*

Proof. We assume that $\lambda = \lambda_0 \neq 0$, where λ_0 is a constant. Then from the above first equation we get $\kappa^2 = -\frac{2}{3}(a\lambda_0)s + c$, c is a constant. Applying this result to the second equation of (3.4), it is a contradiction. □

Moreover, using (3.4) for the case of $\lambda = 0$ we have

Theorem 3.2. *Let γ be a slant Frenet curve in a Sasakian Lorentzian 3-manifold. Then γ satisfies $\Delta_{\dot\gamma} H = 0$ if and only if γ is a geodesic or pseudo-helix with $\kappa^2 = \tau^2$.*

Remark 3.1. In [8], from the point of view of Riemannian structure, we proved that a slant curve γ in a Sasakian 3-manifold satisfying $\Delta_{\dot\gamma} H = 0$ is a geodesic.

The *normal bundle* of γ in M is defined by

$$T^{\perp}\gamma = \bigcup_{s\in I} (\mathbb{R}\dot\gamma(s))^{\perp},$$

here γ is a spacelike or timelike. The connection $\nabla^{\perp}$ of the normal bundle $T^{\perp}\gamma$ is called the *normal connection*. The Laplacian operator

$$\Delta^{\perp} = -\varepsilon_1 \nabla^{\perp}_{\dot\gamma}\nabla^{\perp}_{\dot\gamma}$$

of the normal bundle $T^{\perp}\gamma$ is called the *normal Laplacian* of γ.

Then from (2.4) we have:

Lemma 3.2. *Let γ be a Frenet curve in contact Lorentzian 3-manifold M. Then*

$$\nabla^{\perp}_{\dot\gamma}\nabla^{\perp}_{\dot\gamma}\dot\gamma = \varepsilon_2\kappa' N + \varepsilon_1\kappa\tau B, \tag{3.5}$$

$$\nabla^{\perp}_{\dot\gamma}\nabla^{\perp}_{\dot\gamma}\nabla^{\perp}_{\dot\gamma}\dot\gamma = (\varepsilon_2\kappa'' - \varepsilon_3\kappa\tau^2)N + \varepsilon_1(2\kappa'\tau + \kappa\tau')B. \tag{3.6}$$

For a slant Frenet curve γ in Sasakian Lorentzian 3-manifolds, from the Lemma 2.1 and (3.5) we find that γ satisfies $\nabla^{\perp}_{\dot\gamma} H = \lambda\xi$ if and only if

$$\begin{cases} a\lambda = 0, \\ \kappa' = 0, \\ \kappa\tau = \lambda\sqrt{\varepsilon_1 + a^2}. \end{cases} \tag{3.7}$$

From which, we have

Theorem 3.3. *Let γ be a non-geodesic slant Frenet curve in a Sasakian Lorentzian 3-manifold. Then γ has a C-parallel mean curvature vector field in normal bundle if and only if γ is a pseudo-circle($\lambda = 0$) or a Legendre helix($\lambda \neq 0$) with $\kappa = \lambda$ and $\tau = 1$, λ is a non-zero constant.*

Proof. From the second equation of (3.7) we can see that κ is a constant. Using the first equation of (3.7), we get $\lambda = 0$ or γ is a Legendre curve. If $\lambda = 0$, then a slant Frenet curve γ becomes a pseudo-circle as κ is a constant and $\tau = 0$. If $\lambda \neq 0$ then a slant Frenet curve γ is a Legendre helix and $\lambda = \kappa$. □

Next, from the Lemma 2.1 and (3.6) we find that γ satisfies $\Delta^{\perp}_{\dot\gamma} H = \lambda\xi$ if and only if

$$\begin{cases} a\lambda = 0, \\ \varepsilon_2\kappa'' - \varepsilon_3\kappa\tau^2 = 0, \\ -\varepsilon_1(2\kappa'\tau + \kappa\tau') = \lambda\sqrt{\varepsilon_1 + a^2}. \end{cases} \tag{3.8}$$

From which, we have

Theorem 3.4. *Let γ be a non-geodesic slant Frenet curve in a Sasakian Lorentzian 3-manifold. Then γ has a C-proper mean curvature vector field in the normal bundle if and only if γ is a pseudo-circle($\lambda = 0$) or a Legendre curve($\lambda \neq 0$) with $\kappa = p\cos(s) + q\sin(s)$, $\tau = 1$ and $\lambda = 2\{p\sin(s) - q\cos(s)\}$ where p and q are constants.*

Proof. (I) For the case of $\lambda = 0$, we have

$$\begin{cases} \varepsilon_2\kappa'' - \varepsilon_3\kappa\tau^2 = 0, \\ 2\kappa'\tau + \kappa\tau' = 0. \end{cases} \tag{3.9}$$

Since a curve γ is a non-geodesic slant Frenet curve, by Theorem 2.1, $\tau = Q\kappa + 1$, where Q is a constant. From the second equation of (3.9), we have that $\kappa' = 0$ or $3Q\kappa + 2 = 0$.

For the case of $\kappa' = 0$, we get $\kappa = constant \neq 0$ and $\tau = 0$.

For the case of $3Q\kappa + 2 = 0$, using the first equation of (3.9) we have $\tau = 0$. However, it is contradictory to slant Frenet curve condition.

Hence, for a non-geodesic slant Frenet curve γ in a Sasakian Lorentzian 3-manifold, γ satisfies $\Delta^{\perp}_{\dot\gamma} H = 0$ if and only if γ is a pseudo-circle with $\kappa = constant \neq 0$ and $\tau = 0$.

(II) For the case of $\lambda \neq 0$, we can see that γ is a Legendre curve satisfying

$$\begin{cases} \kappa'' + \kappa = 0, \\ 2\kappa' = -\lambda. \end{cases} \tag{3.10}$$

From this, for a slant Frenet curve γ in a Sasakian Lorentzian 3-manifold, γ satisfies $\Delta^{\perp}_{\gamma} H = \lambda\xi$ if and only if γ is a Legendre curve with $\kappa = p\cos(s) + q\sin(s),\ \tau = 1$ and $\lambda = 2\{p\sin(s) - q\cos(s)\}$ where p and q are constants. □

For Riemannian structure, we found

Remark 3.2 ([8]). Let γ be a non-geodesic slant curve in a Sasakian 3-manifold. Then the slant curve γ has a C-proper mean curvature vector field in the normal bundle if and only if γ is a circle($\lambda = 0$) or a Legendre curve($\lambda \neq 0$) with $\kappa = a\exp(s) + b\exp(-s),\ \tau = 1$ and $\lambda = -2\{a\exp(s) - b\exp(-s)\}$ where a and b are constants.

4. Example

The Heisenberg group $\mathbb{H}_3$ is a Lie group which is diffeomorphic to $\mathbb{R}^3$ and the group operation is defined by

$$(x, y, z) * (\overline{x}, \overline{y}, \overline{z}) = (x + \overline{x}, y + \overline{y}, z + \overline{z} + \frac{x\overline{y}}{2} - \frac{\overline{x}y}{2}).$$

The mapping

$$\mathbb{H}_3 \to \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| \, a, b, c \in \mathbb{R} \right\} : (x, y, z) \mapsto \begin{pmatrix} 1 & x & z + \frac{xy}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

is an isomorphism between $\mathbb{H}_3$ and a subgroup of $GL(3, \mathbb{R})$.
Now, we take the contact form

$$\eta = dz + (ydx - xdy).$$

Then the characteristic vector field of η is $\xi = \frac{\partial}{\partial z}$.
Now, we equip the Lorentzian metric as following:

$$g = dx^2 + dy^2 - (dz + (ydx - xdy))^2.$$

We take a left-invariant Lorentzian orthonormal frame field (e_1, e_2, e_3) on $(\mathbb{H}_3, g)$:

$$e_1 = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z},\ e_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z},\ e_3 = \frac{\partial}{\partial z},$$

and the commutative relations are derived as follows:

$$[e_1, e_2] = 2e_3,\ [e_2, e_3] = [e_3, e_1] = 0.$$

Then the endomorphism field φ is defined by

$$\varphi e_1 = e_2,\ \varphi e_2 = -e_1,\ \varphi e_3 = 0.$$

The Levi-Civita connection ∇ of $(\mathbb{H}_3, g)$ is described as ([9])

$$\begin{aligned} \nabla_{e_1} e_1 = \nabla_{e_2} e_2 = \nabla_{e_3} e_3 = 0,\ \ \nabla_{e_1} e_2 = e_3 = -\nabla_{e_2} e_1, \\ \nabla_{e_2} e_3 = -e_1 = \nabla_{e_3} e_2,\ \ \nabla_{e_3} e_1 = e_2 = \nabla_{e_1} e_3. \end{aligned} \tag{4.1}$$

The contact form η satisfies $d\eta(X, Y) = g(X, \varphi Y)$. Moreover the structure (η, ξ, φ, g) is Sasakian. The Riemannian curvature tensor R of $(\mathbb{H}_3, g)$ is given by

$$\begin{aligned} R(e_1, e_2)e_1 &= 3e_2, & R(e_1, e_2)e_2 &= -3e_1, \\ R(e_2, e_3)e_2 &= -e_3, & R(e_2, e_3)e_3 &= -e_2, \\ R(e_3, e_1)e_3 &= e_1, & R(e_3, e_1)e_1 &= e_3, \end{aligned}$$

the others are zero.

The sectional curvature is given by ([3])

$$K(\xi, e_i) = -R(\xi, e_i, \xi, e_i) = -1, \ for\ i = 1, 2,$$

and

$$K(e_1, e_2) = R(e_1, e_2, e_1, e_2) = 3.$$

Hence Lorentzian Heisenberg space $(\mathbb{H}_3, g)$ is the Lorentzian Sasakian space forms with constant holomorphic sectional curvature $\mu = 3$.

Let γ be a slant Frenet curve in Lorentzian Heisenberg space (H_3, g) parametrized by arc-length. Then the tangent vector field has the form

$$T = \gamma' = \sqrt{\varepsilon_1 + a^2}\cos\beta e_1 + \sqrt{\varepsilon_1 + a^2}\sin\beta e_2 + ae_3, \tag{4.2}$$

where $a = constant,\ \beta = \beta(s)$. Using (4.1), we get

$$\nabla_{\gamma'}\gamma' = \sqrt{\varepsilon_1 + a^2}(\beta' + 2a)(-\sin\beta e_1 + \cos\beta e_2). \tag{4.3}$$

since γ is a non-geodesic, we may assume that $\kappa = \sqrt{\varepsilon_1 + a^2}(\beta' + 2a) > 0$ without loss of generality. Then the normal vector field

$$N = -\sin\beta e_1 + \cos\beta e_2.$$

The binormal vector field $\varepsilon_3 B = T \wedge_L N = -a\cos\beta e_1 - a\sin\beta e_2 - \sqrt{\varepsilon_1 + a^2}e_3$. From the Lemma 2.1, we see that $\varepsilon_2 = 1$, so we have $\varepsilon_3 = -\varepsilon_1$. Hence

$$B = \varepsilon_1(a\cos\beta e_1 + a\sin\beta e_2 + \sqrt{\varepsilon_1 + a^2}e_3).$$

Using the Frenet-Serret equation (2.4), we have

Lemma 4.1. *Let γ be a Frenet slant curve in Lorentzian Heisenberg space $(\mathbb{H}_3, g)$ parametrized by arc-length. Then γ admits a orthonormal frame field $\{T, N, B\}$ along γ and*

$$\begin{aligned} \kappa &= \sqrt{\varepsilon_1 + a^2}(\beta' + 2a), \\ \tau &= 1 + \varepsilon_1 a(\beta' + 2a). \end{aligned} \tag{4.4}$$

Let $\gamma(s) = (x(s), y(s), z(s))$ be a curve in Lorentzian Heisenberg space $(\mathbb{H}_3, g)$. Then the tangent vector field γ' of γ is

$$\gamma' = \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right) = \frac{dx}{ds}\frac{\partial}{\partial x} + \frac{dy}{ds}\frac{\partial}{\partial y} + \frac{dz}{ds}\frac{\partial}{\partial z}.$$

Using the relations:

$$\frac{\partial}{\partial x} = e_1 + ye_3,\ \frac{\partial}{\partial y} = e_2 - xe_3,\ \frac{\partial}{\partial z} = e_3,$$

If γ is a slant curve in $(\mathbb{H}_3, g)$, then from (4.2) the system of differential equations for γ are given by

$$\frac{dx}{ds}(s) = \sqrt{\varepsilon_1 + a^2}\cos\beta(s), \tag{4.5}$$

$$\frac{dy}{ds}(s) = \sqrt{\varepsilon_1 + a^2}\sin\beta(s), \tag{4.6}$$

$$\frac{dz}{ds}(s) = a + \sqrt{\varepsilon_1 + a^2}(x(s)\sin\beta(s) - y(s)\cos\beta(s)).$$

Now, we construct a slant Frenet curve γ with C-parallel mean curvature vector fields in the Lorentzian Heisenberg space $(\mathbb{H}_3, g)$. From the Theorem 3.1 and (4.4) we have

Proposition 4.1. *Let $\gamma : I \to (\mathbb{H}_3, g)$ be a non-geodesic slant Frenet curve parametrized by arc-length in the Lorentzian Heisenberg space $(\mathbb{H}_3, g)$. Then*
(i) γ satisfies C-parallel mean curvature vector fields if and only if γ is a slant pseudo-helix with

$$\beta'(s) = \frac{\sqrt{\varepsilon_1 a\lambda}}{\sqrt{\varepsilon_1 + a^2}} - 2a, \quad for\ a = \eta(\gamma').$$

(ii) *γ satisfies $\Delta^{\perp}_{\gamma} H = 0$ if and only if γ is a slant pseudo-helix with*

$$\beta'(s) = -a \pm \sqrt{\varepsilon_1 + a^2}.$$

Namely, β' is a constant, say A, hence $\beta(s) = As + b,\ b \in \mathbb{R}$. Thus, from (4.5) and (4.6) we have the following result :

Corollary 4.1 ([10]). *Let $\gamma : I \to (\mathbb{H}_3, g)$ be a non-geodesic slant Frenet curve parametrized by arc-length in the Lorentzian Heisenberg space $(\mathbb{H}_3, g)$. If γ is slant pseudo-helix , then the parametric equations of γ are given by*

$$\begin{cases} x(s) = \frac{1}{A}\sqrt{\varepsilon_1 + a^2}\sin(As+b) + x_0, \\ y(s) = -\frac{1}{A}\sqrt{\varepsilon_1 + a^2}\cos(As+b) + y_0, \\ z(s) = \{a + \frac{\varepsilon_1 + a^2}{A}\}s - \frac{\sqrt{\varepsilon_1 + a^2}}{A}\{x_0\cos(As+b) + y_0\sin(As+b)\} + z_0. \end{cases}$$

where b, x_0, y_0, z_0 are constants.

In particular, from the Theorem 3.1, if γ is a Legendre curve, it is a geodesic.

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