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On the mixed reduction: an alternative axiomatization of the fuzzy NTU core

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Abstract

In the framework of fuzzy non-transferable-utility (NTU) games, Hwang [5] and Liao [6] introduced several extensions of reduced games and related consistency properties due to [3], Serrano and Volij [8] and Voorneveld and van den Nouweland [9] respectively. Different from the axiomatic results due to Hwang [5] and Liao [6], this paper is devoted to propose a mixed reduction and related consistency property to characterize the core on fuzzy non-transferable-utility (NTU) games.

Keywords: fuzzy NTU game; reduction; the core.
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1. Introduction

In the axiomatic characterization of solutions of standard transferable-utility (TU) games, *consistency* is a crucial property which has been applied comprehensively. If a solution is not consistent, then a subgroup of agents might not respect the original compromise but revise the payoff distribution within the subgroup. The fundamental property of solutions has been investigated in various classes of problems by applying *reduced games* always. The *core* is, perhaps, the most intuitive solution concept in game theory. Relating to the core of standard coalition games, there are several different types of imaginary reduced standard coalition games in the literature.

The theory of fuzzy games started with work of Aubin [1, 2] where the notions of a fuzzy TU game and the core of a fuzzy TU game are introduced. Hwang [5] extended the core and the reduced games proposed

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by Davis and Maschler [3] and Serrano and Volij [8] to fuzzy NTU games. Inspired by Serrano and Volij [8], Hwang [5] offered axiomatizations of the core of fuzzy NTU games. Later, Liao [6] extended the reduced game introduced by Voorneveld and van den Nouweland [9] to fuzzy NTU games and characterized the core by means of related consistency.

Different from the works of Hwang [5] and Liao [6], the reductions proposed by Hwang [5] and Liao [6] are to combine to form one alternative reduction, which we name the *mixed reduced game*. Further, we adopt related properties of consistency and its converse to characterize the core.

2. Preliminaries

Let U be the universe of players. If $N \subseteq U$ is a set of players, then a **fuzzy coalition** is a vector $\alpha \in [0,1]^N$. The i-th coordinate α_i of α is called the participation level of player i in the fuzzy coalition α. For all $T \subseteq N$, let $|T|$ be the number of elements in T. Instead of $[0,1]^T$, we will write F^T for the set of fuzzy coalitions. A player-coalition $T \subseteq N$ corresponds in a canonical way to the fuzzy coalition $e^T(N) \in F^N$, which is the vector with $e_i^T(N) = 1$ if $i \in T$, and $e_i^T(N) = 0$ if $i \in N \setminus T$. The fuzzy coalition $e^T(N)$ corresponds to the situation where the players in T fully cooperate (i.e. with participation level 1) and the players outsides T are not involved at all (i.e. they have participation level 0). Denote the zero vector in $\mathbb{R}^N$ by 0^N. The fuzzy coalition 0^N corresponds to the empty player-coalition. Note that if no confusion can arise $e^T(N)$ will be denoted by e^T.

To state the core of a fuzzy NTU game, some more notations will be needed. Let $\alpha \in F^N$, $A(\alpha, N) = \{i \in N \mid \alpha_i > 0, \alpha \in F^N\}$ is the set of players who participate in α. Let $x, y \in \mathbb{R}^N$. $x \geq y$ if $x_i \geq y_i$ for all $i \in N$; $x > y$ if $x \geq y$ and $x \neq y$; $x \gg y$ if $x_i > y_i$ for all $i \in N$. We denote $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N \mid x \geq 0^N\}$. Let $A \subseteq \mathbb{R}^N$. A is **comprehensive** if $x \in A$ and $x \geq y$ imply $y \in A$. The **boundary** of A is denoted by ∂A, and the **interior** of A is denoted by $intA$. If $x \in \mathbb{R}^N$ then $x + A = \{x + a \mid a \in A\}$.

Definition 1. *A* **fuzzy NTU game** *is a pair (N, V), where N is a non-empty and finite set of players and V is a characteristic function that assigns to each fuzzy coalition $\alpha = (\alpha_i)_{i \in N} \in F^N \setminus \{0^N\}$ a subset $V(\alpha)$ of $\mathbb{R}^{A(\alpha,N)}$, such that*

$$V(\alpha) \text{ is non-empty, closed and comprehensive,} \tag{2.1}$$

$$V(\alpha) \cap (x + \mathbb{R}_+^{A(\alpha,N)}) \text{ is bounded for every } x \in \mathbb{R}^{A(\alpha,N)}, \tag{2.2}$$

$$\text{if } x, y \in \partial V(e^N) \text{ and } x \geq y, \text{ then } x = y. (\text{ non-levelness }) \tag{2.3}$$

Denote the class of all fuzzy NTU games by Γ. Let $(N, V) \in \Gamma$. A **payoff vector** of (N, V) is a vector $x = (x_i)_{i \in N} \in \mathbb{R}^N$. Then

- A payoff vector x of $(N, V) \in \Gamma$ is **efficient (EFF)** if $x \in \partial V(e^N)$.
- A payoff vector x of $(N, V) \in \Gamma$ is **individually rational (IR)** if for all $i \in N$ and for all $j \in (0, 1]$, $jx_i \notin intV(je^{\{i\}})$.

Moreover, x is an **imputation** of (N, V) if it is EFF and IR. The set of imputations of (N, V) is denoted by $I(N, V)$.

A **solution on** Γ is a function σ which associates with each $(N, V) \in \Gamma$ a subset $\sigma(N, V)$ of $V(e^N)$.

Definition 2. (Hwang [5]) *The* **core** *$C(N, V)$ of $(N, V) \in \Gamma$ consists of all $x \in \partial V(e^N)$ that satisfy for all $\alpha \in F^N \setminus \{0^N\}$, $(\alpha_i x_i)_{i \in A(\alpha,N)} \notin intV(\alpha)$.*

3. Axioms, reductions and axiomatizations

Let σ be a solution on Γ. σ satisfies **Efficiency (EFF)** if for all $(N,V) \in \Gamma$, $\sigma(N,V) \subseteq \partial V(e^N)$. σ satisfies **One-person rationality (OPR)** if for all $(N,V) \in \Gamma$ with $|N| = 1$, $\sigma(N,V) = I(N,V)$.

Hwang [5] and Liao [6] proposed two reductions introduced by Serrano and Volij [8] and Voorneveld and van den Nouweland [9] as follows.

Definition 3. *Let $(N,V) \in \Gamma$, $x \in \mathbb{R}^N$ and $S \subseteq N$, $S \neq \emptyset$.*

- *The* **S-V reduced game with respect to** S **and** x *is the game $(S, V^{SV}_{S,x})$ defined by for all $\alpha \in F^S \setminus \{0^S\}$,*
$$V^{SV}_{S,x}(\alpha) = \bigcup_{\beta \in F^{N\setminus S}} \{y \in \mathbb{R}^{A(\alpha,S)} \mid \big(y, (\beta_i x_i)_{i \in A(\beta, N\setminus S)}\big) \in V(\alpha,\beta)\}.$$

- *The* **V-N reduced game with respect to** S **and** x *is the game $(S, V^{VN}_{S,x})$ defined by for all $\alpha \in F^S \setminus \{0^S\}$,*
$$\begin{array}{lll} V^{VN}_{S,x}(\alpha) = \{y \in \mathbb{R}^S \mid (y, x_{N\setminus S}) \in V(e^N)\} & , & \textit{if } \alpha = e^S(S), \\ V^{VN}_{S,x}(\alpha) = \bigcup\limits_{\substack{\beta \in F^{N\setminus S} \\ \beta \neq 0^{N\setminus S}}} \{y \in \mathbb{R}^{A(\alpha,S)} \mid \big(y, (\beta_i x_i)_{i \in A(\beta, N\setminus S)}\big) \in V(\alpha,\beta)\} & , & \textit{otherwise.} \end{array}$$

In this note, the S-V reduction and the V-N reduction are to combine to form one alternative reduction, which we name the *mixed reduced game*.

Definition 4. *The* **mixed reduced game with respect to** S **and** x *is the game $(S, V^{MI}_{S,x})$ defined by for all $\alpha \in F^S \setminus \{0^S\}$,*
$$V^{MI}_{S,x}(\alpha) = \bigcup_{\substack{\beta \in F^{N\setminus S} \\ \beta \neq 0^{N\setminus S}}} \{y \in \mathbb{R}^{A(\alpha,S)} \mid \big(y, (\beta_i x_i)_{i \in A(\beta, N\setminus S)}\big) \in V(\alpha,\beta)\}.$$

Consistency, originally introduced by Harsanyi [4] under the name of bilateral equilibrium, requires that if x is prescribed by σ for a game (N,V), then the projection of x to S should be prescribed by σ for the reduced game with respect to S and x for all S. Thus, the projection of x to S should be consistent with the expectations of the members of S as reflected by their reduced game.

- **S-V consistency (SVCON)**: If $(N,V) \in \Gamma$, $S \subseteq N$, $S \neq \emptyset$, and $x \in \sigma(N,V)$, then $(S, V^{SV}_{S,x}) \in \Gamma$ and $x_S \in \sigma(S, V^{SV}_{S,x})$.

Converse consistency requires that if the projection of an efficient payoff vector x to every proper S is consistent with the expectations of the members of S as reflected by their reduced game then x itself should be recommended for whole game.

- **Converse S-V consistency (CSVCON)**: If $(N,V) \in \Gamma$ with $|N| \geq 2$, $x \in \partial V(e^N)$, and for all $S \subset N$, $0 < |S| < |N|$, $(S, V^{SV}_{S,x}) \in \Gamma$ and $x_S \in \sigma(S, V^{SV}_{S,x})$, then $x \in \sigma(N,V)$.

- **Weak converse S-V consistency (WCSVCON)**: If $(N,V) \in \Gamma$ with $|N| \geq 2$, $x \in I(N,V)$, and for all $S \subset N$, $0 < |S| < |N|$, $(S, V^{SV}_{S,x}) \in \Gamma$ and $x_S \in \sigma(S, V^{SV}_{S,x})$, then $x \in \sigma(N,V)$.

From now on we restrict our attention to bounded fuzzy NTU games in this note, defined as those games (N,V) such that, there exists a real number M_v such that for all $\alpha \in F^N \setminus \{0^N\}$ and for all $x \in V(\alpha)$, $x(\alpha) \leq M_v$. We use it here in order to guarantee that all reductions are well-defined.

The following axiom is a weakening of the previous axiom, since it requires that x be individually rational as well.

"VN-reduction" instead of "SV-reduction", we introduce the **V-N consistency (VNCON)**, the **converse V-N consistency (CVNCON)** and the **weak converse V-N consistency (WCVNCON)**. Similarly, we also propose the **MI-consistency (MICON)**, the **converse MI-consistency (CMICON)** and the **weak converse MI-consistency (WCMICON)** by applying the mixed reduction. Hwang [5] and Liao [6] characterized the core by means of the SV-reduction and the VN-reduction respectively. Here, we characterize the core by means of the mixed reduction.

Lemma 1. *Let $(N, V) \in \Gamma$, $x \in V(e^N)$ and $S \subseteq N$, $S \neq \emptyset$. Then the reduced game $(S, V^{S,x}) \in \Gamma$.*

Proof. This proof can easily be deduced. Hence, we omit it. ■ □

Lemma 2. *Let $(N, V) \in \Gamma$, $x \in C(N, V)$ and $S \subseteq N$, $S \neq \emptyset$. Then for all $\alpha \in F^S \setminus \{0^S\}$,*

$$(\alpha_i x_i)_{i\in A(\alpha,S)} \notin int[\bigcup_{\substack{\beta\in F^{N\setminus S}\\ \beta\neq 0^{N\setminus S}}} \{y \in \mathbb{R}^{A(\alpha,S)} \mid \Big(y, (\beta_i x_i)_{i\in A(\beta,N\setminus S)}\Big) \in V(\alpha, \beta)\}].$$

Proof. Let $(N, V) \in \Gamma$, $x \in C(N, V)$ and $S \subseteq N$ with $S \neq \emptyset$. Let $\alpha \in F^S \setminus \{0^S\}$. Suppose that

$$(\alpha_i x_i)_{i\in A(\alpha,S)} \in int[\bigcup_{\substack{\beta\in F^{N\setminus S}\\ \beta\neq 0^{N\setminus S}}} \{y \in \mathbb{R}^{A(\alpha,S)} \mid \Big(y, (\beta_i x_i)_{i\in A(\beta,N\setminus S)}\Big) \in V(\alpha, \beta)\}].$$

Then there exists $\beta \in F^{N\setminus S}$ such that

$$\Big((\alpha_i x_i)_{i\in A(\alpha,S)}, (\beta_i x_i)_{i\in A(\beta,N\setminus S)}\Big) \in intV(\alpha, \beta).$$

Let $\overline{\alpha} = (\alpha, \beta) \in F^N \setminus \{0^N\}$. Hence $(\overline{\alpha}_i x_i)_{i\in A(\overline{\alpha},N)} \in intV(\overline{\alpha})$. This contradicts to that $x \in C(N, V)$. ■ □

Lemma 3.

1. *The core satisfies MICON.*
2. *The core satisfies CMICON.*

Proof. First, we show that the core satisfies MICON. Let $(N, V) \in \Gamma$, $x \in C(N, V)$ and $S \subseteq N$ with $S \neq \emptyset$. By Lemma 1, $(S, V^{MI}_{S,x}) \in \Gamma$. Let $\alpha \in F^S \setminus \{0^S\}$. Since

$$V^{MI}_{S,x}(\alpha) = \bigcup_{\substack{\beta\in F^{N\setminus S}\\ \beta\neq 0^{N\setminus S}}} \{y \in \mathbb{R}^{A(\alpha,S)} \mid \Big(y, (\beta_i x_i)_{i\in A(\beta,N\setminus S)}\Big) \in V(\alpha, \beta)\},$$

$(\alpha_i x_i)_{i\in A(\alpha,S)} \notin intV^{MI}_{S,x}(\alpha)$ by Lemma 2. In particular, if $\alpha = e^S(S)$ then $x_S \notin intV^{MI}_{S,x}(e^S(S))$. Since $x_S \in V^{MI}_{S,x}(e^S(S))$, $x_S \in \partial V^{MI}_{S,x}(e^S(S))$. That is, x_S is EFF in the reduced game $(S, V^{MI}_{S,x})$. So if $x \in C(N, V)$, for all $S \subseteq N$ with $S \neq \emptyset$, $x_S \in C(S, V^{MI}_{S,x})$.

Next, we show that the core satisfies CMICON. Let $(N, V) \in \Gamma$ with $|N| \geq 2$ and let $x \in \partial V(e^N)$. Suppose for all $S \subset N$ such that $0 < |S| < |N|$, $(S, V^{MI}_{S,x}) \in \Gamma$ and $x_S \in C(S, V^{MI}_{S,x})$. We will show that $x \in C(N, V)$. Let $i \in N$ and $\alpha \in F^N \setminus \{0^N\}$ with $\alpha_i \neq 0$. Consider the reduced game $(\{i\}, V^{MI}_{\{i\},x})$. Then $\alpha_i x_i \notin intV^{MI}_{\{i\},x}(\alpha_i)$ by $x_i \in C(\{i\}, V^{MI}_{\{i\},x})$. Since

$$V^{MI}_{\{i\},x}(\alpha_i) = \bigcup_{\substack{\beta\in F^{N\setminus S}\\ \beta\neq 0^{N\setminus S}}} \{y \in \mathbb{R}^{\{i\}} \mid \Big(y, (\beta_j x_j)_{j\in A(\beta,N\setminus\{i\})}\Big) \in V(\alpha_i, \beta)\},$$

$\alpha_i x_i \notin int \bigcup_{\substack{\beta\in F^{N\setminus S}\\ \beta\neq 0^{N\setminus S}}} \{y \in \mathbb{R}^{\{i\}} \mid \Big(y, (\beta_j x_j)_{j\in A(\beta,N\setminus\{i\})}\Big) \in V(\alpha_i, \beta)\}$. Hence, we have that $(\alpha_k x_k)_{k\in A(\alpha,N)} \notin intV(\alpha)$. ■ □

Lemma 4. *Let $(N,V) \in \Gamma$, $x \in V(e^N)$ and $S \subseteq N$, $S \neq \emptyset$. Then x is EFF in (N,V) if and only if x_S is EFF in the reduced game $(S, V^{MI}_{S,x})$.*

Proof. Clearly, $x_S \in V^{MI}_{S,x}(e^S(S))$. If x_S is not EFF in the reduced game $(S, V^{MI}_{S,x})$, then there exists $y_S \in V^{MI}_{S,x}(e^S(S))$ such that $y_S > x_S$. Hence, $(y_S, x_{N\setminus S}) \in V(e^N)$ and $(y_S, x_{N\setminus S}) > x$. Therefore x is not EFF in (N,V). Similarly, if x is not EFF in (N,V), then there exists $y \in V(e^N)$ such that $y \gg x$. Hence, $(y_S, x_{N\setminus S}) \in V(e^N)$. Thus, $y_S \in V^{MI}_{S,x}(e^S(S))$ and $y_S \gg x_S$. Therefore, x_S is not EFF in the reduced game $(S, V^{MI}_{S,x})$. ■ □

Lemma 5. *Let σ be a solution on Γ. If σ satisfies OPR and MICON then it also satisfies EFF.*

Proof. Assume, on the contrary, that there exist $(N,V) \in \Gamma$ and $x \in \sigma(N,v)$ such that x is not EFF in (N,V). Let $i \in N$. Consider the reduced game $(\{i\}, V^{MI}_{\{i\},x})$. By MICON of σ, $x_i \in \sigma(\{i\}, V^{MI}_{\{i\},x})$. By OPR of σ, x_i is EFF in the reduced game $(\{i\}, V^{MI}_{\{i\},x})$. By Lemma 4, x is EFF in (N,V). Thus, the desired contradiction has been obtained. ■ □

Theorem 3.1. *A solution σ on Γ satisfies OPR, MICON and CMICON if and only if for all $(N,V) \in \Gamma$, $\sigma(N,V) = C(N,V)$.*

Proof. By Lemma 3, the core satisfies MICON and CMICON. Clearly, the core satisfies OPR.

To prove uniqueness, assume that a solution σ satisfies OPR, MICON and CMICON. By Lemma 5, σ satisfies EFF. Let$(N,V) \in \Gamma$. The proof proceeds by induction on the number $|N|$. If $|N| = 1$ then by OPR of σ, $\sigma(N,V) = I(N,V) = C(N,V)$. Assume that $\sigma(N,V) = C(N,V)$ if $|N| < k$, $k \geq 2$.
The case $|N| = k$:
First we show that $\sigma(N,V) \subseteq C(N,V)$. Let $x \in \sigma(N,V)$. Since σ satisfies EFF, $x \in X(N,V)$. By MICON of σ, for all $S \subset N$ with $0 < |S| < |N|$, $x_S \in \sigma(S, V^{MI}_{S,x})$. By the induction hypothesis, for all $S \subset N$ with $0 < |S| < |N|$, $x_S \in \sigma(S, V^{MI}_{S,x}) = C(S, V^{MI}_{S,x})$. By CMICON of the core, $x \in C(N,V)$. The opposite inclusion may be shown analogously by interchanging the roles of σ and C. Hence, we have that $\sigma(N,V) = C(N,V)$.
■ □

The following examples show that each of the axioms used in Theorem 1 is logically independent of the others.

Example 1. *Let $\sigma(N,V) = \emptyset$ for all $(N,V) \in \Gamma$. Then σ satisfies MICON and CMICON, but it violates OPR.*

Example 2. *Let $\sigma(N,V) = I(N,V)$ for all $(N,V) \in \Gamma$. Then σ satisfies OPR and CMICON, but it violates MICON.*

Example 3. *Define the solution σ on Γ by*

$$\sigma(N,V) = \begin{cases} I(N,V) & , \text{ if } |N| = 1 \\ \emptyset & , \text{otherwise.} \end{cases}$$

Then σ satisfies OPR and MICON, but it violates CMICON.

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In order to show the logical independence of the used axioms $|U| \geq 2$ is needed.

Conflict of interest

The authors declare that they have no conflict of interest.

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