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Research Article

Some types of f-biharmonic and bi-f-harmonic curves

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Abstract

In this paper, we determine necessary and sufficient conditions for a non-Frenet Legendre curve to be f-harmonic, f-biharmonic, bi-f-harmonic, biminimal and f-biminimal in three-dimensional normal almost paracontact metric manifold. Besides, we obtain some nonexistence theorems.

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1. Introduction

Theory of curves is one of the most important subject of differential geometry and it keeps up to date from past to present. The most popular curves in this theory are Frenet curves since these curves can be studied in many different manifolds. In particular, Legendre curves have an important role in geometry and topology of almost contact manifolds. In the literature, the most basic papers about Legendre curves studied in contact manifolds can be listed as, [2,4,5,30].

On the other hand, studies on Frenet Legendre curves are more recent in the literature. The latest studies which are a source of motivation for us can be briefly listed as, [23,31]. In the mentioned Frenet Legendre curve studies, the authors focused especially on curvature and torsion of the curve. However, in this study, different from previous studies, we handled properties of the maps, which briefly mentioned below.

Harmonic maps, which were defined by Sampson and Eells in [9], have a wide range of application areas such as physics, mathematics and engineering.

Jiang obtained biharmonic maps between Riemannian manifolds by generalizing harmonic maps, in [13]. The definition of biharmonic maps for Riemannian immersions into Euclidean space coincides with the Chen's definition of biharmonic submanifold given by using bienergy functional. In recent years, there has been a growing interest in the theory of biharmonic maps which can be divided into two main research directions. On the one side, constructing the examples and classifing results have become important from

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the differential geometric aspect. The other side, using partial differential equations from the analytic aspect (see [7, 15, 16, 24, 27–29]), because biharmonic maps are solutions of a fourth order strongly elliptic semilinear partial differential equation.

f-harmonic maps between Riemannian manifolds were introduced by Lichnerowicz in 1970 and then were examined by Eells and Lemaire, in [10]. f-harmonic maps have a physical meaning as the solution of inhomogeneous Heisenberg spin systems and continuous spin systems, [3]. For this reason, these maps are of interest not only to mathematicians but also to physicists.

On the other hand, the relations between biharmonic and f-harmonic maps were summarized by Perktaş et.al. in two ways, in [25]. The first one is to extend bienergy functional to bi-f-energy functional and obtain a new type of harmonic map called as bi-f-harmonic map. The second one is to extend the f-energy functional to the f-bienergy functional and achieve a new type harmonic map called as f-biharmonic map which are critical points of f-bienergy functional, [22, 33].

f-biharmonic maps and bi-f-harmonic maps between Riemannian Manifolds were defined by Lu, in [18, 19]. Furthermore in [21], Ou gave complete classification of f-biharmonic curves in three-dimensional Euclidean space and characterization of f-biharmonic curves in n-dimensional space forms. Moreover, bi-f-harmonic maps as a generalization of biharmonic and f-harmonic maps were introduced by Ouakkas et.al., in [22]. In addition, Roth defined a non-f-harmonic, f-biharmonic map as a proper f-biharmonic map, [26]. It should be emphasized that there is no relation between f-biharmonic and bi-f-harmonic maps.

Biminimal immersions and biminimal curves in a Riemannian manifold were defined by Loubeau and Montaldo, [17]. Finally, f-biminimal immersions were defined by Karaca and Özgür, in [11].

Motivating by these studies, in this paper, we study non-Frenet Legendre curves in three-dimensional normal almost paracontact metric manifold. First, we give the basic notions in preliminary section. Then in section 3, we give some theorems which will be needed in following subsections. In subsection 3.1, we obtain f-harmonicity conditions. In subsection 3.2, we get f-biharmonicity conditions and we determine these conditions also in α-para-Kenmotsu manifolds. In subsection 3.3, we obtain bi-f-harmonicity conditions and also discuss these conditions in various cases. Finally in subsection 3.4 and 3.5, we get biminimality and f-biminimality conditions respectively.

2. Preliminaries

This section includes some definitions and propositions that will be required throughout the paper.

Definition 2.1. For any curve γ, if $\|\gamma'\| = 1$, then curve γ is unit speed curve in Euclidean space. Let $\gamma : I \longrightarrow E^3$ be a unit speed curve in the Euclidean three-space with $\{\kappa, \tau, T, N, B\}$ Serret-Frenet apparatus. Here κ, τ denote the curvature and the torsion; $\{T, N, B\}$ denote the tangent, principal normal and binormal unit vectors of the curve γ, respectively. Then there exists an orthogonal frame $\{T, N, B\}$ which satisfies the Serret-Frenet equation

$$\begin{cases} T' = \kappa N \\ N' = -\kappa T + \tau B \\ B' = -\tau N. \end{cases}$$

Here γ is called a Frenet curve if $\kappa > 0$ and $\tau \neq 0$. Frenet frame, the best known and most frequently used frame on a curve, plays an important role in differential geometry, [12].

Definition 2.2. Let (N,g) and $(\bar{N},\bar{g})$ be Riemannian manifolds. If a map $\psi:(N,g)\to(\bar{N},\bar{g})$ is a critical point of the energy functional

$$E(\psi)=\frac{1}{2}\int_N |d\psi|^2 dv_g,$$

then it is defined as a harmonic map, where v_g is the volume element of (N,g). Besides, a map called as harmonic if

$$\tau(\psi):=trace\nabla d\psi=0. \tag{2.1}$$

Here $\tau(\psi)$ is the Euler-Lagrange equation of the energy functional $E(\psi)$, where it is the tension field of map ψ and ∇ is the connection induced from the Levi-Civita connection $\nabla^{\bar{N}}$ of $\bar{N}$ and the pull-back connection ∇^{ψ}, [9,11].

Biharmonic maps are defined as below:

Definition 2.3. If $\psi:(N,g)\to(\bar{N},\bar{g})$ map is a critical point, for all variations, of the bienergy functional

$$E_2(\psi)=\frac{1}{2}\int_N |\tau(\psi)|^2 dv_g,$$

then it is called as a biharmonic map.
Bitension field $\tau_2(\psi)$, namely the Euler-Lagrange equation of a biharmonic map is given as:

$$\tau_2(\psi)=trace(\nabla^{\psi}\nabla^{\psi}-\nabla^{\psi}_{\nabla})\tau(\psi)-trace(R^{\bar{N}}(d\psi,\tau(\psi))d\psi)=0, \tag{2.2}$$

where $R^{\bar{N}}$ is the curvature tensor field of $\bar{N}$. $R^{\bar{N}}$ is defined as follows

$$R^{\bar{N}}(X_1,X_2)X_3=\nabla^{\bar{N}}_{X_1}\nabla^{\bar{N}}_{X_2}X_3-\nabla^{\bar{N}}_{X_2}\nabla^{\bar{N}}_{X_1}X_3-\nabla^{\bar{N}}_{[X_1,X_2]}X_3,$$

for any $X_1,X_2,X_3\in\Gamma(T\bar{N})$ and ∇^{ψ} is the pull-back connection, [11].

Note that harmonic maps are always biharmonic. On the other hand, non-harmonic biharmonic maps are called as proper biharmonic maps.

Definition 2.4. If $\psi:(N,g)\to(\bar{N},\bar{g})$ map is a critical point of the f-energy functional

$$E_f(\psi)=\frac{1}{2}\int_N f|d\psi|^2 dv_g,$$

where $f\in C^{\infty}(N,\mathbb{R})$ is a positive smooth function then it is defined as an f-harmonic map.
f-Tension field $\tau_f(\psi)$, namely the Euler-Lagrange equation for the f-harmonic map, is defined as:

$$\tau_f(\psi)=f\tau(\psi)+d\psi(gradf)=0. \tag{2.3}$$

f-Harmonic maps are generalizations of harmonic maps, [1,8].

Definition 2.5. If $\psi:(N,g)\to(\bar{N},\bar{g})$ map is a critical point of the f-bienergy functional

$$E_{2,f}(\psi)=\frac{1}{2}\int_N f|\tau(\psi)|^2 dv_g,$$

then it is defined as an f-biharmonic map. For an f-biharmonic map, the Euler-Lagrange equation is given by

$$\tau_{2,f}(\psi)=f\tau_2(\psi)+\Delta f\tau(\psi)+2\nabla^{\psi}_{gradf}\tau(\psi)=0, \tag{2.4}$$

where $\tau_{2,f}(\psi)$ is the f-bitension field of the map ψ. If f is a constant, the f-biharmonic map is called as a biharmonic map, [19].

Definition 2.6. [25] If $\psi : (N, g) \to (\bar{N}, \bar{g})$ map is a critical point of the bi-f-energy functional

$$E_{f,2}(\psi) = \frac{1}{2} \int_N |\tau_f(\psi)|^2 dv_g,$$

then it is called as a bi-f-harmonic map.
For $\tau_{f,2}(\psi)$ bi-f-tension field of the map ψ, the Euler-Lagrange equation of a bi-f-harmonic map is given as

$$\tau_{f,2}(\psi) = trace((\nabla^\psi f(\nabla^\psi \tau_f(\psi)) - f\nabla^\psi_{\nabla^N} \tau_f(\psi) + fR^{\bar{N}}(\tau_f(\psi), d\psi)d\psi)) = 0. \quad (2.5)$$

Definition 2.7. If an immersion $\psi : (N, g) \to (\bar{N}, \bar{g})$ is a critical point of the bienergy functional $E_2(\psi)$, for variations normal to the image $\psi(N) \subset \bar{N}$ with fixed energy, then it is called biminimal. Equivalently, ψ is a critical point of the λ-bienergy functional

$$E_{2,\lambda}(\psi) = E_2(\psi) + \lambda E(\psi),$$

where $\lambda \in \mathbb{R}$ is a constant. For a λ-biminimal immersion, the Euler-Lagrange equation is

$$[\tau_{2,\lambda}(\psi)]^\perp = [\tau_2(\psi)]^\perp - \lambda[\tau(\psi)]^\perp = 0, \quad (2.6)$$

where $[.]^\perp$ denotes the normal component of [.], [11, 17].

Definition 2.8. If an immersion $\psi : (N, g) \to (\bar{N}, \bar{g})$ is a critical point of the f-bienergy functional $E_{2,f}(\psi)$, for variations normal to the image $\psi(N) \subset \bar{N}$ with fixed energy, then it is called f-biminimal. Equivalently, ψ is a critical point of the λ-f-bienergy functional

$$E_{2,\lambda,f}(\psi) = E_{2,f}(\psi) + \lambda E_f(\psi),$$

where $\lambda \in \mathbb{R}$ is a constant. Then an immersion is f-biminimal if

$$[\tau_{2,\lambda,f}(\psi)]^\perp = [\tau_{2,f}(\psi)]^\perp - \lambda[\tau_f(\psi)]^\perp = 0. \quad (2.7)$$

If f is a constant then the f-biminimal map turns into a biminimal map, [11].

Definition 2.9. A differentiable manifold N^{2n+1} is called an almost paracontact metric manifold if the following conditions are satisfied:

$$\varphi^2 = I - \eta \otimes \xi, \quad \varphi\xi = 0, \quad \eta(\xi) = 1, \quad g(\varphi X_1, \varphi X_2) = -g(X_1, X_2) + \eta(X_1)\eta(X_2). \quad (2.8)$$

Here φ is a tensor field type $(1, 1)$, ξ is a vector field, η is a 1-form, X_1, $X_2 \in TN$, I is the identity endomorphism on vector fields and finally g is a pseudo-Riemannian compatible metric with signature $(n + 1, n)$. Besides this, $\eta \circ \varphi = 0$ and $rank(\varphi) = 2n$ in an almost paracontact metric manifold. From (2.8) it is easy to see that $g(X_1, \varphi X_2) = -g(\varphi X_1, X_2)$ and $g(X_1, \xi) = \eta(X_1)$ for any X_1, $X_2 \in TN$ and the fundamental 2-form of N is defined by $\Phi(X_1, X_2) = g(X_1, \varphi X_2)$.
An almost paracontact metric manifold $(N, \varphi, \xi, \eta, g)$ is said to be normal if

$$N(X_1, X_2) - 2d\eta(X_1, X_2)\xi = 0,$$

where N is the Nijenhuis torsion tensor of φ, [14, 32].

The curvature tensor field equation of a three-dimensional normal almost paracontact metric manifold with $\alpha, \beta = constant$ is as follows.

$$\begin{aligned} R(X_1, X_2)X_3 &= \left(\frac{r}{2} + 2\left(\alpha^2 + \beta^2\right)\right)(g(X_2, X_3)X_1 - g(X_1, X_3)X_2) \quad (2.9) \\ &+ g(X_1, X_3)\left(\frac{r}{2} + 3\left(\alpha^2 + \beta^2\right)\right)\eta(X_2)\xi - \left(\frac{r}{2} + 3(\alpha^2 + \beta^2)\right)\eta(X_2)\eta(X_3)X_1 \\ &- g(X_2, X_3)\left(\frac{r}{2} + 3(\alpha^2 + \beta^2)\right)\eta(X_1)\xi + \left(\frac{r}{2} + 3(\alpha^2 + \beta^2)\right)\eta(X_1)\eta(X_3)X_2, \end{aligned}$$

where $X_1, X_2, X_3 \in TN$ and r is the scalar curvature, [24].

Proposition 2.10. [31]*For a three-dimensional almost paracontact metric manifold N, the following conditions are mutually equivalent:*

i- *N is normal,*

ii- *there exist α, β functions on N such that*

$$(\nabla_{X_1}\varphi)X_2 = \alpha\left(g(\varphi X_1, X_2)\xi - \eta(X_2)\varphi X_1\right) + \beta\left(g(X_1, X_2)\xi - \eta(X_2)X_1\right), \quad (2.10)$$

iii- *there exist α, β functions on N such that*

$$\nabla_{X_1}\xi = \alpha\left(X_1 - \eta(X_1)\xi\right) + \beta\varphi X_1. \quad (2.11)$$

Moreover, the functions α, β appearing in (2.10) and (2.11) are given by

$$2\alpha = trace\{X_1 \to \nabla_{X_1}\xi\}, \qquad 2\beta = trace\{X_1 \to \varphi\nabla_{X_1}\xi\} \quad (2.12)$$

where $X_1, X_2 \in TN$.

Definition 2.11. A three-dimensional normal almost paracontact metric manifold is called as α-para-Kenmotsu Manifold if $\alpha \neq 0$, $\beta = 0$ and α is constant, [31].

Definition 2.12. Let $(N, \varphi, \xi, \eta, g)$ be a three-dimensional normal almost paracontact metric manifold where $\alpha, \beta =$ constant and $\gamma : I \subseteq \mathbb{R} \to (N, g)$ be an immersed curve. The map $c_\gamma : I \to \mathbb{R}$ given with the formula

$$c_\gamma(s) = g(V(s), \xi) = \eta(V(s)),$$

is the structural function of γ where $V = \gamma'$. Then γ is called as Legendre curve if $c_\gamma = \eta(V(s)) = 0$, [6].

Note that in this paper, we study with a non-Frenet Legendre curve $\gamma : I \subseteq \mathbb{R} \longrightarrow N$, parametrized by arc length on a pseudo-Riemannian space form N (a three-dimensional normal almost paracontact metric manifold with $\alpha, \beta =$ constant).
Throughout this study, we will say Legendre curves with null normal instead of non-Frenet Legendre curves with null normal.

3. Non-Frenet Legendre curves

In this section, before presenting our main results about some types of non-Frenet Legendre curves in three-dimensional normal almost paracontact metric manifolds, we will give some theorems which will be needed throughout the paper.
$\gamma : I \subseteq \mathbb{R} \longrightarrow N$ is a non-Frenet Legendre curve in a three-dimensional pseudo-Riemannian manifold and has two types defined as follows, [31].

The first type is called a Legendre curve with null tangent (null curve) and satisfies

$$g(\gamma', \gamma') = 0,$$

and the second type is called a Legendre curve with null normal and satisfies

$$g(\gamma', \gamma') = \varepsilon_1 = 1.$$

Theorem 3.1. [31] *Let N be a three-dimensional normal almost paracontact metric manifold. If $\gamma : I \subseteq \mathbb{R} \longrightarrow N$ is a Legendre curve with null tangent, then*

$$\nabla_{\gamma'}\gamma' = \vartheta\gamma',$$

where ϑ is a function. Note that after a reparametrization γ is a geodesic.

Theorem 3.2. [31] *Let N be a three-dimensional normal almost paracontact metric manifold. $\gamma : I \subseteq \mathbb{R} \longrightarrow N$ is a Legendre curve with null normal iff $g(\gamma', \gamma') = \varepsilon_1 = 1$ and*

$$\nabla_{\gamma'}\gamma' = -\alpha(\xi \pm \varphi\gamma'),$$

where α is a function defined in (2.12), $\alpha = \pm\delta \neq 0$ along γ and $\delta = g(\nabla_{\gamma'}\gamma', \vartheta\gamma')$.

Here $\nabla_{\gamma'}$ denotes the covariant differentiation along γ.

Theorem 3.3. [31] *There is no Legendre curve with null binormals in a three-dimensional normal almost paracontact metric manifold.*

Let $\gamma : I \subseteq \mathbb{R} \longrightarrow N$ be a non-Frenet Legendre curve in a three-dimensional normal almost paracontact metric manifold and $\left\{\gamma' = V,\ \varphi\gamma' = \varphi V,\ \xi\right\}$ are orthonormal vector fields along γ. In this case, the tension fields reduce to

$$\tau(\gamma) = \nabla_V V \tag{3.1}$$

and

$$\tau_2(\gamma) = \nabla_V^3 V - R(V, \nabla_V V)V, \tag{3.2}$$

see [20].

Using the information given above, we can rewrite $\nabla_{\gamma'}\gamma'$ for a non-Frenet Legendre curve with null normal as

$$\nabla_V V = -\alpha(\xi \pm \varphi V), \tag{3.3}$$

in a three-dimensional normal almost paracontact metric manifold.

3.1. f-harmonic non-Frenet Legendre curves

In this subsection, we obtained the f-harmonicity conditions for a Legendre curve with null normal in a three-dimensional normal almost paracontact metric manifold N where $\alpha, \beta =$ constant.
Let $\gamma : I \subseteq \mathbb{R} \longrightarrow N$ be a Legendre curve with null normal. With the help of Definition 2.4 and equations (3.1), (3.3), the f-harmonicity condition for a Legendre curve with null normal is given as below:

$$\begin{aligned} \tau_f(\gamma) &= f\tau(\gamma) + d\gamma(gradf) \\ &= f\nabla_V V + f'V \\ &= f(-\alpha(\xi \pm \varphi V)) + f'V \\ &= f'V \mp \alpha f\varphi V - \alpha f\xi \\ &= 0. \end{aligned} \tag{3.4}$$

With the help of equation (3.4), we can state the following nonexistence theorem.

Theorem 3.4. *There is no f-harmonic Legendre curve with null normal in a three-dimensional normal almost paracontact metric manifold where $\alpha, \beta =$ constant.*

Proof. From (3.4), it is clear that f' is zero so f is a constant function. □

3.2. f-biharmonic non-Frenet Legendre curves

Here, we get the f-biharmonicity conditions for a Legendre curve with null normal in a three-dimensional normal almost paracontact metric manifold N, where $\alpha, \beta =$ constant. Let determine the f-biharmonicity condition for $\gamma : I \subseteq \mathbb{R} \longrightarrow N$ Legendre curve with null normal. We determine $\nabla_V^2 V$, $\nabla_V^3 V$ and $R(V, \nabla_V V)V$ for a Legendre curve with null normal given as below:

$$\nabla_V^2 V = -\alpha\beta(\varphi V \pm \xi), \tag{3.5}$$

$$\nabla_V^3 V = -\alpha\beta^2(\pm\varphi V + \xi), \tag{3.6}$$

$$R(V, \nabla_V V)V = \pm\alpha(\frac{r}{2} + 2(\alpha^2 + \beta^2))\varphi V - \alpha(\alpha^2 + \beta^2)\xi. \tag{3.7}$$

Here by substituting equations (3.1), (3.2), (3.5), (3.6) and (3.7) into the f-bitension field formula, we obtain the f-biharmonicity condition for a Legendre curve with null normal as:

$$\begin{aligned}
\tau_{2,f}(\gamma) &= f\tau_2(\gamma) + \Delta f \tau(\gamma) + 2\nabla^{\gamma}_{gradf}\tau(\gamma) \\
&= f(\nabla^3_V V - R(V, \nabla_V V)V) + f^{''}\nabla_V V + 2f^{'}\nabla^2_V V \\
&= (\mp(3\beta^2 + \frac{r}{2} + 2\alpha^2)f - 2\beta f^{'} \mp f^{''})\varphi V + (\alpha^2 f \mp 2\beta f^{'} - f^{''})\xi \\
&= 0. \qquad (3.8)
\end{aligned}$$

From (3.8), we get following theorem.

Theorem 3.5. *Let N be a three-dimensional normal almost paracontact metric manifold where $\alpha, \beta =$ constant and $\gamma : I \subseteq \mathbb{R} \longrightarrow N$ be a Legendre curve with null normal. Then γ is an f-biharmonic Legendre curve with null normal iff following differential equations satisfied,*

$$\begin{cases} \mp(3\beta^2 + \frac{r}{2} + 2\alpha^2)f - 2\beta f^{'} \mp f^{''} = 0, \\ \alpha^2 f \mp 2\beta f^{'} - f^{''} = 0. \end{cases} \qquad (3.9)$$

Now we give an interpretation of Theorem 3.5.

Case I: If N is a α-para-Kenmotsu manifold then we have following equations from (3.9):

$$\begin{cases} \mp(\frac{r}{2} + 2\alpha^2)f \mp f^{''} = 0, \\ \alpha^2 f - f^{''} = 0. \end{cases}$$

From this case, we obtain the following corollary.

Corollary 3.6. *Let N be a three-dimensional α-para-Kenmotsu manifold and $\gamma : I \subseteq \mathbb{R} \longrightarrow N$ be a Legendre curve with null normal. Then γ is an f-biharmonic Legendre curve with null normal iff the function f and the constant scalar curvature r are given by*

$$f(s) = c_1 e^{\alpha s} + c_2 e^{-\alpha s} \quad and \quad r = -6\alpha^2,$$

respectively.

3.3. Bi-f-harmonic non-Frenet Legendre curves

In this subsection, we handle bi-f-harmonic non-Frenet Legendre curves in three-dimensional normal almost paracontact metric manifold where $\alpha, \beta =$ constant and we obtain the bi-f-harmonicity condition for this type of curves. Moreover, we determine this condition for different cases of a Legendre curve with null normal in α-para-Kenmotsu manifold.

Let determine the bi-f-harmonicity condition for a Legendre curve with null normal. By using the equations (3.3), (3.5), (3.6) and the curvature tensor

$$R(\nabla_V V, V)V = \mp\alpha(\frac{r}{2} + 2(\alpha^2 + \beta^2))\varphi V + \alpha(\alpha^2 + \beta^2)\xi, \qquad (3.10)$$

we obtain the bi-f-harmonicity condition for a Legendre curve with null normal as below:

$$\begin{aligned}
\tau_{f,2}(\gamma) &= trace\big((\nabla^{\gamma} f(\nabla^{\gamma}\tau_f(\gamma)) - f\nabla^{\gamma}_{\nabla^N}\tau_f(\gamma) + fR^{\bar{N}}(\tau_f(\gamma), d\gamma)d\gamma\big) \\
&= (ff^{''})^{'}V + (3ff^{''} + 2(f^{'})^2)\nabla_V V + 4ff^{'}\nabla^2_V V + f^2\nabla^3_V V + f^2 R(\nabla_V V, V)V \\
&= \big[(ff^{''})^{'}\big]V \\
&+ \big[\mp(3ff^{''} + 2(f^{'})^2)\alpha - 4ff^{'}\alpha\beta \mp \alpha\beta^2 f^2 \mp \alpha\frac{r}{2}f^2 \mp 2\alpha(\alpha^2 + \beta^2)f^2\big]\varphi V \\
&+ \big[-(3ff^{''} + 2(f^{'})^2)\alpha \mp 4ff^{'}\alpha\beta - \alpha\beta^2 f^2 + \alpha(\alpha^2 + \beta^2)f^2\big]\xi \\
&= 0. \qquad (3.11)
\end{aligned}$$

Via (3.11), we get the following theorems.

Theorem 3.7. *Let N be a three-dimensional normal almost paracontact metric manifold where $\alpha, \beta =$ constant and $\gamma : I \subseteq \mathbb{R} \longrightarrow N$ be a Legendre curve with null normal. Then γ is a bi-f-harmonic curve iff following differential equation system satisfied,*

$$\begin{cases} (ff^{''})^{'} = 0, \\ \mp(3ff^{''} + 2(f^{'})^2) - 4ff^{'}\beta \mp \beta^2 f^2 \mp \frac{r}{2}f^2 \mp 2(\alpha^2+\beta^2)f^2 = 0, \\ -(3ff^{''} + 2(f^{'})^2) \mp 4ff^{'}\beta - \beta^2 f^2 + (\alpha^2+\beta^2)f^2 = 0. \end{cases} \tag{3.12}$$

From now, our calculations will be made according to the positive sign of the (3.12) differential equation system. We have similar results for negative sign.

Theorem 3.8. *Let N be a three-dimensional normal almost paracontact metric manifold where $\alpha, \beta =$ constant and $\gamma : I \subseteq \mathbb{R} \longrightarrow N$ be a Legendre curve with null normal. Then γ is a bi-f-harmonic Legendre curve with null normal iff the function f is the solution of following differential equation*

$$3ff^{''} + 2(f^{'})^2 - 4ff^{'}\beta - \alpha^2 f^2 = 0$$

where $(ff^{''})^{'} = 0$ and the constant scalar curvature r is given by

$$r = -6(\alpha^2 + \beta^2).$$

Now we give interpretations of Theorem 3.7.

Case I: If N is a α-para-Kenmotsu manifold then we have following differential equation system from (3.12);

$$\begin{cases} (ff^{''})^{'} = 0, \\ (3ff^{''} + 2(f^{'})^2) + \frac{r}{2}f^2 + 2\alpha^2 f^2 = 0, \\ -(3ff^{''} + 2(f^{'})^2) + \alpha^2 f^2 = 0. \end{cases}$$

So we have the following corollary.

Corollary 3.9. *Let N be a three-dimensional α-para-Kenmotsu manifold and $\gamma : I \subseteq \mathbb{R} \longrightarrow N$ be a Legendre curve with null normal. Then γ is a bi-f-harmonic Legendre curve with null normal iff the function f is the solution of following differential equation,*

$$3ff^{''} + 2(f^{'})^2 - \alpha^2 f^2 = 0$$

where $(ff^{''})^{'} = 0$ and the constant scalar curvature r is given by

$$r = -6\alpha^2.$$

Case II: If N is a three-dimensional normal almost paracontact metric manifold where $\alpha, \beta =$ constant and $f.f^{''} = c = constant \neq 0$ then we have following differential equation system from (3.12);

$$\begin{cases} 3c + 2(f^{'})^2 - 4ff^{'}\beta + \beta^2 f^2 + \frac{r}{2}f^2 + 2(\alpha^2+\beta^2)f^2 = 0, \\ -3c - 2(f^{'})^2 + 4ff^{'}\beta - \beta^2 f^2 + (\alpha^2+\beta^2)f^2 = 0. \end{cases}$$

Therefore we obtain the following corollary.

Corollary 3.10. *Let N be a three-dimensional normal almost paracontact metric manifold where $\alpha, \beta =$ constant, $\gamma : I \subseteq \mathbb{R} \longrightarrow N$ be a Legendre curve with null normal and*

$f.f^{''}=c=constant\neq 0$. Then γ is a bi-f-harmonic Legendre curve with null normal iff function f is the solution of following differential equation,

$$3c+2(f^{'})^2-4\beta ff^{'}-\alpha^2f^2=0$$

and the constant scalar curvature r is given by

$$r=-6(\alpha^2+\beta^2).$$

Case III: If N is a α-para-Kenmotsu manifold and $f.f^{''}=c=constant\neq 0$ then we have following differential equation system from (3.12);

$$\begin{cases}3c+2(f^{'})^2+\frac{r}{2}f^2+2(\alpha^2)f^2=0,\\ -3c-2(f^{'})^2+(\alpha^2)f^2=0.\end{cases}$$

Then we have the following corollary.

Corollary 3.11. *Let N be a three-dimensional α-para-Kenmotsu manifold, $\gamma:I\subseteq\mathbb{R}\longrightarrow N$ be a Legendre curve with null normal and $f.f^{''}=c=constant\neq 0$. Then γ is a bi-f-harmonic Legendre curve with null normal iff the function f either*

$$f(s)=\mp\frac{\sqrt{3c}\tanh(\frac{1}{2}(2\alpha c_1-\sqrt{2}\alpha s))}{\sqrt{\alpha^2\tanh^2(\frac{1}{2}(2\alpha c_1-\sqrt{2}\alpha s))-\alpha^2}}$$

or

$$f(s)=\mp\frac{\sqrt{3c}\tanh(\frac{1}{2}(2\alpha c_1+\sqrt{2}\alpha s))}{\sqrt{\alpha^2\tanh^2(\frac{1}{2}(2\alpha c_1+\sqrt{2}\alpha s))-\alpha^2}}$$

where $c_1=constant$ and the constant scalar curvature r are given by

$$r=-6\alpha^2.$$

3.4. Biminimal non-Frenet Legendre curves

In this subsection, we handle biminimal Legendre curves with null normal in a three-dimensional normal almost paracontact metric manifold N where $\alpha,\beta=$ constant.
From Definition 2.7, we know that if a curve is biminimal then its tension and bitension fields satisfy the following formula;

$$[\tau_{2,\lambda}(\gamma)]^{\perp}=[\tau_2(\gamma)]^{\perp}-\lambda[\tau(\gamma)]^{\perp}=0. \tag{3.13}$$

Let determine the biminimality condition for a Legendre curve with null normal. For a Legendre curve with null normal, the tension and bitension fields are given by

$$\tau(\gamma)=\mp\alpha\varphi V-\alpha\xi,$$

$$\tau_2(\gamma)=\alpha^3\xi\mp\alpha(\frac{r}{2}+2\alpha^2+3\beta^2)\varphi V,$$

respectively. Then with the help of equation (3.13), we get the biminimality condition for a Legendre curve with null normal given as below:

$$\begin{aligned}[\tau_{2,\lambda}(\gamma)]^{\perp}&=[\tau_2(\gamma)]^{\perp}-\lambda[\tau(\gamma)]^{\perp}\\ &=(\alpha^3+\lambda\alpha)\xi\mp\alpha(\frac{r}{2}+2\alpha^2+3\beta^2-\lambda\alpha)\varphi V\\ &=0.\end{aligned} \tag{3.14}$$

Then from (3.14), we get following corollary.

Theorem 3.12. *Let N be a three-dimensional normal almost paracontact metric manifold and $\gamma : I \subseteq \mathbb{R} \longrightarrow N$ be a Legendre curve with null normal. Then γ is a biminimal Legendre curve with null normal iff the constant scalar curvature r is given by*

$$r = -4\alpha^2 - 6\beta^2 - 2\alpha^3,$$

for $\lambda = -\alpha^2$.

3.5. f-biminimal non-Frenet Legendre curves

In this last subsection, we give f-biminimality conditions for a non-Frenet Legendre curve in a three-dimensional normal almost paracontact metric manifold and α-para-Kenmotsu manifold.

A curve is called as f-biminimal if following equation satisfied,

$$[\tau_{2,\lambda,f}(\gamma)]^{\perp} = [\tau_{2,f}(\gamma)]^{\perp} - \lambda[\tau_f(\gamma)]^{\perp} = 0. \tag{3.15}$$

Let determine the f-biminimality condition for a Legendre curve with null normal. For a Legendre curve with null normal, the f-tension and f-bitension fields are given by,

$$\begin{aligned} \tau_f(\gamma) &= f^{'}V \mp \alpha f\varphi V - \alpha f\xi, \\ \tau_{2,f}(\gamma) &= (\mp(3\beta^2 + \frac{r}{2} + 2\alpha^2)f - 2\beta f^{'} \mp f^{''})\varphi V + (\alpha^2 f \mp 2\beta f^{'} - f^{''})\xi, \end{aligned}$$

respectively. By substituting these tension fields to the equation (3.15), we obtain the f-biminimality condition for a Legendre curve with null normal as follows:

$$\begin{aligned} [\tau_{2,\lambda,f}(\gamma)]^{\perp} &= (\alpha^2 f \mp 2\beta f^{'} - f^{''} + \alpha\lambda f)\xi + (\mp(\frac{r}{2} + 2\alpha^2 + 3\beta^2 - \alpha\lambda)f - 2\beta f^{'} \mp f^{''})\varphi V \\ &= 0. \end{aligned}$$

Hence, we obtain following theorems.

Theorem 3.13. *Let N be a three-dimensional normal almost paracontact metric manifold where $\alpha, \beta =$ constant and $\gamma : I \longrightarrow N$ be a Legendre curve with null normal. Then γ is an f-biminimal Legendre curve with null normal iff following differential equation system satisfied,*

$$\begin{cases} (\alpha^2 + \alpha\lambda)f \mp 2\beta f^{'} - f^{''} = 0, \\ \mp(\frac{r}{2} + 2\alpha^2 + 3\beta^2 - \alpha\lambda)f - 2\beta f^{'} \mp f^{''} = 0. \end{cases} \tag{3.16}$$

Theorem 3.14. *Let N be a three-dimensional normal almost paracontact metric manifold where $\alpha, \beta =$ constant and $\gamma : I \subseteq \mathbb{R} \longrightarrow N$ be a Legendre curve with null normal. Then γ is an f-biminimal Legendre curve with null normal iff the function f and the constant scalar curvature r are given by*

$$f(s) = c_1 e^{\left(\beta - \sqrt{\beta^2 + (\alpha^2 + \alpha\lambda)}\right)s} + c_2 e^{\left(\beta + \sqrt{\beta^2 + (\alpha^2 + \alpha\lambda)}\right)s},$$

$$r = -6(\alpha^2 + \beta^2).$$

Now we give an interpretation of Theorem 3.13.

Case I: If N is a α-para-Kenmotsu manifold then we get following differential equation system from (3.16):

$$\begin{cases} (\alpha^2 + \alpha\lambda)f - f^{''} = 0, \\ (\frac{r}{2} + 2\alpha^2 - \alpha\lambda)f + f^{''} = 0. \end{cases}$$

Hence, we have the following corollary.

Corollary 3.15. *Let N be a α-para-Kenmotsu manifold and $\gamma : I \subseteq \mathbb{R} \longrightarrow N$ be a Legendre curve with null normal. Then γ is an f-biminimal Legendre curve with null normal iff the function f is given by*

$$f(s) = c_1 e^{(-\sqrt{\alpha^2+\alpha\lambda})s} + c_2 e^{(\sqrt{\alpha^2+\alpha\lambda})s}$$

and the constant scalar curvature r is given by

$$r = -6\alpha^2.$$

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