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A New Paranormed Sequence Space Defined by Euler Totient Matrix

Euler Totient Matrisi Tarafından Tanımlanan Yeni Bir Paranormlu Dizi Uzayı

Merve İlkhan[1], Serkan Demiriz[2], Emrah Evren Kara[1*]

[1]Düzce University, Faculty of Arts and Sciences, Department of Mathematics, Düzce, Turkey
[2]Tokat Gaziosmanpaşa University, Faculty of Arts and Sciences, Department of Mathematics, Tokat, Turkey

Abstract

In the present paper, by using the regular matrix given by Euler Totient function, we give a new paranormed sequence space $\ell(\Phi,p)$ and prove that the spaces $\ell(\Phi,p)$ and $\ell(p)$ are linearly isomorphic. Also, we compute α-,β-,γ-duals and the Schauder basis of this space.

Keywords: α-,β-,γ-duals, Euler totient function, Möbius function, Paranormed sequence space

Öz

Bu çalışmada, Euler Totient fonksiyonu ile oluşturulan regüler bir matrisin kullanılmasıyla, yeni bir paranormlu uzay olan $\ell(\Phi,p)$ uzayını tanımladık ve bu uzayın $\ell(p)$ uzayına lineer izomorf olduğunu gösterdik. Ayrıca bu uzayın α-,β-,γ-duallerini ve Schauder bazını hesapladık.

Anahtar Kelimeler: α-,β-,γ-dualleri, Euler totient fonksiyonu, Möbius fonksiyonu, Paranormlu dizi uzayı

1. Introduction

Let ω be the space of all real valued sequences. If U is any linear subspace of ω, then U is called a sequence space. By c_0, c and ℓ_∞, we denote the spaces of all null, convergent and bounded sequences, respectively. Also, by bs, cs, ℓ_1 and $\ell_p\,(1<p<\infty)$, we denote the spaces of all bounded, convergent, absolutely and p-absolutely convergent series, respectively.

Throughout this paper, we assume that (p_n) is a bounded sequence of strictly positive real numbers with $\sup_{n\in\mathbb{N}} p_n = P$ and $S=\max\{1,P\}$. In this case, for any $\xi\in\mathbb{R}$ and all $n\in\mathbb{N}=\{1,2,3,...\}$, the inequality

$$|\xi|^{p_n}\le\max\{1,|\xi|^S\} \tag{1}$$

holds, where $\mathbb{R}$ is the real field (see [1]). Also, we write $p_n^{-1}+(p_n')^{-1}=1$ for $1<\inf_{n\in\mathbb{N}} p_n\le P$ and denote the family of all finite subsets of $\mathbb{N}$ by $\aleph$.

*Corresponding author: eevrenkara@duzce.edu.tr

Merve İlkhan orcid.org/0000-0002-0831-1474
Serkan Demiriz orcid.org/0000-0002-4662-6020
Emrah Evren Kara orcid.org/0000-0002-6398-4065

For simplicity in notation, we use the notations $\lim_n$, $\sum_n$, $\sup_n$ and $\inf_n$ instead of $\lim_{n\to\infty}$, $\sum_{n=1}^{\infty}$, $\sup_{n\in\mathbb{N}}$, $\inf_{n\in\mathbb{N}}$, respectively.

A linear topological space U over $\mathbb{R}$ is said to be a paranormed space if there is a function $g:U\to\mathbb{R}$ satisfying the following conditions:

i) g is sub-additive.

ii) $g(\theta)=0$, where θ is the zero vector of U.

iii) $g(u)=g(-u)$ for all $u\in U$.

iv) $|\xi_n-\xi|\to 0$ and $g(u_n-u)\to 0$ imply $g(\xi_n u_n-\xi u)\to 0$ for all (ξ_n) and ξ in $\mathbb{R}$ and all (u_n) and u in U.

Maddox (1967) defined the paranormed sequence space $\ell(p)$ (see also Simons (1965) and Nakano (1951)) as:

$$\ell(p)=\left\{u=(u_n)\in\omega:\sum_n|u_n|^{p_n}<\infty\right\}$$

which is the complete paranormed space by

$$g_p(u)=\left(\sum_n|u_n|^{p_n}\right)^{1/S}.$$

Let $T=(t_{mn})$ be an infinite matrix of real numbers t_{mn} for $m,n\in\mathbb{N}$ and U,V be any two sequence spaces. If $Tu=(T_m(u))\in V$ for every $u=(u_n)\in U$, then we say that T is a matrix mapping from U into V and we denote it by writing $T:U \to V$, where

$$T_m(u) = \sum_n t_{mn} u_n \tag{2}$$

for every $m \in \mathbb{N}$. Also, Tu is called as $T-$transform of $u \in U$. By $(U:V)$, we denote the class of all matrices T such that $T:U \to V$. Thus, $T \in (U:V)$ if and only if the series on the right-hand side of (2) converges for each $m \in \mathbb{N}$ and every $u \in U$, and we have $Tu \in V$ for all $u \in U$.

For the sequence spaces U and V, the set $S(U,V)$ is defined by

$$S(U,V) = \{s = (s_n) \in \omega: us = (u_n s_n) \in V \text{ for all } u = (u_n) \in U\}.$$

Then, the $\alpha-,\beta-,\gamma-$duals of a sequence space U, which are respectively denoted by U^α, U^β and U^γ are defined by

$$U^\alpha = S(U,\ell_1),\ U^\beta = S(U,cs) \text{ and } U^\gamma = S(U,bs).$$

Let (U,g) be a paranormed space. A sequence (c_n) in U is called a Schauder basis for U if and only if for each $u \in U$, there exists a unique sequence (ξ_n) of scalars such that $g(u - \sum_{n=0}^{m} \xi_n c_n) \to 0$ as $m \to \infty$. In this case, we can write $u = \sum_n \xi_n c_n$.

Let U be a sequence space and T be an infinite matrix. Then, the matrix domain U_T of T in the space U is defined by

$$U_T = \{u \in \omega: Tu \in U\}.$$

In the literature, there are many papers (Altay and Başar 2002, 2006, Aydın and Başar 2004, 2006, Kara et al. 2010, Demiriz and Çakan 2010, Karakaya and Şimşek 2012, Candan 2014a, 2015, Candan and Kılınç 2015, Candan and Güneş 2015, Ellidokuzoğlu and Demiriz 2016) about paranormed sequence spaces obtained by matrix domains of different infinite matrices. Also, for more details about matrix domains of infinite triangular matrices, one can see (Başar 2012, Et and Çolak 1995, Et and Başarır 1997, Candan 2014b, Kılınç and Candan 2017, Kama and Altay 2017, Kama et al. 2018).

Throughout the paper, φ and μ denote the Euler Totient function and Möbius function, respectively. For every $m \in \mathbb{N}$ with $m > 1, \varphi(m)$ is the number of positive integers less than m which are coprime with m and $\varphi(1) = 1$. There are some properties of function φ. For example:

i) If $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_l^{\alpha_l}$ is the prime factorization of a natural number $m > 1$, then

$$\varphi(m) = m\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\ldots\left(1 - \frac{1}{p_l}\right).$$

ii) For every $m \in \mathbb{N}$, the equality

$$m = \sum_{n|m} \varphi(n)$$

holds.

iii) If $m_1, m_2 \in \mathbb{N}$ are coprime, then the equality $\varphi(m_1 m_2) = \varphi(m_1)\varphi(m_2)$ holds (Kovac 2005).

Given any $m \in \mathbb{N}$ with $m > 1$, Möbius function μ is defined as

$$\mu(m) = \begin{cases} (-1)^l, & \text{if } m = p_1 p_2 \ldots p_l, \text{ where } p_1 p_2 \ldots p_l \text{ are} \\ & \text{non} - \text{equivalent prime numbers} \\ 0, & \text{if } p^2 \mid m \text{ for some prime numbers } p \end{cases}$$

and $\mu(1) = 1$. If $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_l^{\alpha_l}$ is the prime factorization of a natural number $m > 1$, then

$$\sum_{n \backslash m} n\mu(n) = (1 - p_1)(1 - p_2)\ldots(1 - p_l).$$

Also, the equality

$$\sum_{n \backslash m} \mu(n) = 0 \tag{3}$$

holds except for $m = 1$ and $\mu(m_1 m_2) = \mu(m_1)\mu(m_2)$, where $m_1, m_2 \in \mathbb{N}$ are coprime (Kovac 2005). One can consult to Niven et al. (1991) for more details related to these functions.

The Euler totient matrix $\Phi = (\phi_{mn})$ is defined by

$$\phi_{mn} = \begin{cases} \dfrac{\varphi(n)}{m} & \text{if } n \mid m \\ 0, & \text{otherwise} \end{cases}$$

for all $m, n \in \mathbb{N}$. The inverse $\Phi^{-1} = (\phi_{mn}^{-1})$ of the matrix Φ is computed (14) as

$$\phi_{mn}^{-1} = \begin{cases} \dfrac{\mu\left(\frac{m}{n}\right)}{\varphi(m)} n & \text{if } n \mid m \\ 0, & \text{otherwise} \end{cases}$$

for all $m, n \in \mathbb{N}$. Quite recently, İlkhan and Kara (2019) have introduced new Banach sequence spaces $\ell_p(\Phi)(1 \le p < \infty)$ and $\ell_\infty(\Phi)$ derived by using a matrix operator which is comprised of Euler's totient function as

$$\ell_p(\Phi) = \left\{u = (u_n) \in \omega: \sum_m \left|\frac{1}{m}\sum_{n \backslash m} \varphi(n) u_n\right|^p < \infty\right\} (1 \le p < \infty)$$

and

$$\ell_\infty(\Phi) = \left\{u = (u_n) \in \omega: \sup_m \left|\frac{1}{m}\sum_{n \backslash m} \varphi(n) u_n\right| < \infty\right\}.$$

The paranormed spaces have more general properties than normed spaces. Thus, in this paper we generalize the normed sequence space $\ell_p(\Phi)(1 \le p < \infty)$ to paranormed space $\ell(\Phi,p)$. Also, we investigate some topological structures such as completeness, the $\alpha-,\beta-,\gamma-$duals and the basis of the space $\ell(\Phi,p)$.

2. The Paranormed Sequence Space $\ell(\Phi,p)$

In the present section, we introduce the sequence space $\ell(\Phi,p)$ by using the Euler Totient matrix Φ. Also, we prove that this space is a complete paranormed space and give the Schauder basis for this space.

Unless otherwise stated, $v=(v_n)$ will be the $\Phi-$transform of a sequence $u=(u_n)$, that is,

$$v_m = \frac{1}{m}\sum_{n\backslash m}\varphi(n)u_n$$

for all $m \in \mathbb{N}$.

Now, we introduce the sequence space $\ell(\Phi,p)$ by

$$\ell(\Phi,p) = \left\{u=(u_n)\in\omega: \sum_m \left|\frac{1}{m}\sum_{n\backslash m}\varphi(n)u_n\right|^{p_m} < \infty\right\}.$$

According to the definition of matrix domain, the sequence space $\ell(\Phi,p)$ can be represented by $\ell(\Phi,p)=(\ell(p))_\Phi$.

Let $p\in[1,+\infty)$. Then, in the case $p_m = p$ for all $m\in\mathbb{N}$, the space $\ell(\Phi,p)$ is reduced to the sequence space $\ell_p(\Phi)$.

Theorem 1. $\ell(\Phi,p)$ is complete paranormed space with the paranorm given by

$$g_\Phi(u) = \left(\sum_m \left|\frac{1}{m}\sum_{n\backslash m}\varphi(n)u_n\right|^{p_m}\right)^{1/S}$$

for all $u=(u_n)\in\ell(\Phi,p)$.

Proof. Let $u=(u_n), s=(s_n)\in\ell(\Phi,p)$. According to Maddox (1988), we can write

$$\left(\sum_m \left|\frac{1}{m}\sum_{n\backslash m}\varphi(n)(u_n+s_n)\right|^{p_m}\right)^{1/s} \le \left(\sum_m \left|\frac{1}{m}\sum_{n\backslash m}\varphi(n)u_n\right|^{p_m}\right)^{1/s} + \left(\sum_m \left|\frac{1}{m}\sum_{n\backslash m}\varphi(n)s_n\right|^{p_m}\right)^{1/s}. \tag{4}$$

The linearity of $\ell(\Phi,p)$ with respect to the co-ordinatewise addition and scalar multiplication follows from (1) and (4).

It is clear that

$$g_\Phi(\theta)=0 \text{ and } g_\Phi(u)=g_\Phi(-u)$$

for all $u\in\ell(\Phi,p)$. Also, from (1) and (4), we obtain the subadditivity of g_Φ and $g_\Phi(\xi u)\le\max\{1,|\xi|\}g_\Phi(u)$ for any $\xi\in\mathbb{R}$.

Now, let $\{u^m\}$ be any sequence in $\ell(\Phi,p)$ such that $g_\Phi(u^m-u)\to 0$ and (ξ_m) also be any sequence of scalars such that $\xi_m\to\xi$. Then, it follows from the subadditivity of g_Φ that

$$g_\Phi(u^m)\le g_\Phi(u)+g_\Phi(u^m-u).$$

Thus, $\{g_\Phi(u^m)\}$ is bounded and we have

$$g_\Phi(\xi_m u^m-\xi u) = \left(\sum_m\left|\frac{1}{m}\sum_{n\backslash m}\varphi(n)(\xi_m u_n^m-\xi u_n)\right|^{p_m}\right)^{1/S} \le |\xi_m-\xi|g_\Phi(u^m)+|\xi|g_\Phi(u^m-u)$$

which tends to zero as $m\to\infty$. This shows that scalar multiplication is continuous. Hence, g_Φ is a paranorm on $\ell(\Phi,p)$.

It remains to prove the completeness of the space $\ell(\Phi,p)$. Let $\{u^i\}$ be any Cauchy sequence in $\ell(\Phi,p)$, where $u^i=\{u_1^{(i)},u_2^{(i)},u_3^{(i)},...\}$ for each $i\in\mathbb{N}$. Then, for a give $\varepsilon>0$ there exists a positive integer $n_0(\varepsilon)$ such that

$$g_\Phi(u^i-u_j)<\varepsilon \tag{5}$$

for all $i,j\ge n_0(\varepsilon)$. Using the definition of g_Φ, we obtain for each fixed $k\in\mathbb{N}$ that

$$|\Phi_k(u^i)-\Phi_k(u^j)|\le\left[\sum_k|\Phi_k(u^i)-\Phi_k(u^j)|^{p_k}\right]^{1/S}<\varepsilon$$

for every $i,j\ge n_0(\varepsilon)$ which leads us to fact that $\{\Phi_k(u^1),\Phi_k(u^2),\Phi_k(u^3),...\}$ is a Cauchy sequence of real numbers for every fixed $k\in\mathbb{N}$. Since $\mathbb{R}$ is complete, it converges, say

$$\Phi_k(u^i)\to\Phi_k(u)$$

as $i\to\infty$ for every fixed $k\in\mathbb{N}$. With the help of these infinitely many limits $\Phi_1(u),\Phi_2(u),\Phi_3(u),...,$ we define the sequence $\{\Phi_1(u),\Phi_2(u),\Phi_3(u),...\}$. Then, from (5) we can write for each fixed $m\in\mathbb{N}$ and $i,j\ge n_0(\varepsilon)$

$$\sum_{k=1}^m|\Phi_k(u^i)-\Phi_k(u^j)|^{p_k}\le g_\Phi(u^i-u^j)^S<\varepsilon^S. \tag{6}$$

Let $i\ge n_0(\varepsilon)$. If we take $j\to\infty$ in (6) after $m\to\infty$, then we have that $g_\Phi(u^i-u)\le\varepsilon$. Finally, taking $\varepsilon=1$ in (6) and letting $i\ge n_0(1)$, it follows from Minkowski's inequality for every fixed $m\in\mathbb{N}$ that

$$\left(\sum_{k=1}^m|\Phi_k(u)|^{p_k}\right)^{1/S}\le g_\Phi(u^i-u)+g_\Phi(u^i)\le 1+g_\Phi(u^i)$$

which says that $u\in\ell(\Phi,p)$. Since $g_\Phi(u^i-u)\le\varepsilon$ for all $i\ge n_0(\varepsilon)$, we obtain that $u^i\to u$ as $i\to\infty$. As a result, it proves that $\ell(\Phi,p)$ is complete.

Note that the absolute property on the space $\ell(\Phi,p)$ does not hold, since there exists at least one sequence u in

$\ell(\Phi,p)$ such that $g_\Phi(u)\neq g_\Phi(|u|)$, where $|u|=(|u_n|)$. Thus, $\ell(\Phi,p)$ is a sequence space of non-absolute type.

Theorem 2. The sequence space $\ell(\Phi,p)$ is linearly isomorphic to the space $\ell(p)$.

Proof. To prove the theorem, the existence of a linear bijection transformation L between the spaces $\ell(\Phi,p)$ and $\ell(p)$ should be shown. For this purpose, by using the Φ-transform, we define the transformation L from $\ell(\Phi,p)$ to $\ell(p)$ by $u\to v=Lu=\Phi u$. It is trivial that L is linear. Also, L is injective since $u=\theta$ whenever $Lu=\theta$.

Let $v=(v_n)\in\ell(p)$ and define the sequence $u=(u_n)$ by

$$u_n=\sum_{k\backslash n}\frac{\mu\left(\frac{n}{k}\right)}{\varphi(n)}kv_k$$

for all $n\in\mathbb{N}$. Then, we have from (3) that

$$\begin{aligned} g_\Phi(u)&=\left(\sum_m\left|\frac{1}{m}\sum_{n\backslash m}\varphi(n)u_n\right|^{p_m}\right)^{1/S}\\ &=\left(\sum_m\left|\frac{1}{m}\sum_{n\backslash m}\varphi(n)\sum_{k\backslash n}\frac{\mu\left(\frac{n}{k}\right)}{\varphi(n)}kv_k\right|^{p_m}\right)^{1/S}\\ &=\left(\sum_m\left|\frac{1}{m}\sum_{n\backslash m}\sum_{k\backslash n}\mu\left(\frac{n}{k}\right)kv_k\right|^{p_m}\right)^{1/S}\\ &=\left(\sum_m\left|\frac{1}{m}\sum_{n\backslash m}\left(\sum_{k\backslash n}\mu(k)\right)\frac{m}{n}v_{\frac{m}{n}}\right|^{p_m}\right)^{\frac{1}{S}}\\ &=\left(\sum_m|v_m|^{p_m}\right)^{\frac{1}{S}}=g_\Phi(v)<\infty. \end{aligned}$$

This says that $u\in\ell(\Phi,p)$. Thus, L is surjective and paranorm preserving. As a result, L is a linear bijection and the spaces $\ell(\Phi,p)$ and $\ell(p)$ are linearly isomorphic.

Let $1<p_n\leq s_n$ for all $n\in\mathbb{N}$. Then, it is known that $\ell(p)\subseteq\ell(s)$ which leads us to the immediate consequence that $\ell(\Phi,p)\subseteq\ell(\Phi,s)$.

Theorem 3. Define the sequence $b^{(m)}=\{b_n^{(m)}\}_{n\in\mathbb{N}}$ of the elements of $\ell(\Phi,p)$ by

$$b_n^{(m)}=\begin{cases}\frac{\mu\left(\frac{m}{n}\right)}{\varphi(m)}n, & if\ n\mid m\\ 0, & otherwise\end{cases}$$

for every fixed $m\in\mathbb{N}$. Then, the sequence $\{b^{(m)}\}_{m\in\mathbb{N}}$ is a Schauder basis for the space $\ell(\Phi,p)$ and any $u\in\ell(\Phi,p)$ has a unique representation of the form

$$u=\sum_m\gamma_m b^{(m)},$$

where $\gamma_m=\Phi_m(u)$ for all $m\in\mathbb{N}$.

Proof. Since the isomorphism L, defined in the proof of Theorem 2, between the spaces $\ell(\Phi,p)$ and $\ell(p)$ is onto, the inverse image of the Schauder basis of the space $\ell(p)$ is a Schauder basis of the space $\ell(\Phi,p)$. Thus, the proof is trivial.

3. Some Duals of the Space $\ell(\Phi,p)$

In the present section, we give the theorems determining the $\alpha-,\beta-,\gamma-$ duals of the sequence space $\ell(\Phi,p)$.

The following three lemmas are essential for proving our theorems given in this section.

Lemma 1. [(Grosse-Erdmann 1993) Theorem 5.1.0 with $q_m=1$]

(i) Let $1<p_n\leq P<\infty$ for all $n\in\mathbb{N}$. Then, $T=(t_{mn})\in(\ell(p):\ell_1)$ if and only if there exists an integer $K>1$ such that

$$\sup_{N\in\aleph}\sum_n\left|\sum_{m\in N}t_{mn}K^{-1}\right|^{p_n'}<\infty. \tag{7}$$

(ii) Let $0<p_n\leqslant 1$ for all $n\in\mathbb{N}$. Then, $T=(t_{mn})\in(\ell(p):\ell_1)$ if and only if

$$\sup_{N\in\aleph}\sup_n\left|\sum_{m\in N}t_{mn}\right|^{p_n}<\infty. \tag{8}$$

Lemma 2. [(Grosse-Erdmann 1993), Theorem 1 (i)-(ii)]

(i) Let $1<p_n<\infty$ for all $n\in\mathbb{N}$. Then, $T=(t_{mn})\in(\ell(p):\ell_\infty)$ if and only if there exists an integer $K>1$ such that

$$\sup_m\sum_n|t_{mn}K^{-1}|^{p_n'}<\infty. \tag{9}$$

(ii) Let $0<p_n\leq 1$ for all $n\in\mathbb{N}$. Then, $T=(t_{mn})\in(\ell(p):\ell_\infty)$ if and only if

$$\sup_{m,n\in\mathbb{N}}|t_{mn}|^{p_n} \tag{10}$$

Lemma 3. [16, Corollary for Theorem 1] Let $0<p_n\leq\infty$ for all $n\in\mathbb{N}$. Then, $T=(t_{mn})\in(\ell(p):c)$ if and only if (9), (10) hold, and

$$\lim_m t_{mn}=c_n\ (n\in\mathbb{N}) \tag{11}$$

also holds.

Theorem 4. For every $N\in\aleph$, we write $N_n=N\cap\{m\in\mathbb{N}:n\mid m\}(n\in\mathbb{N})$. Let $K>1$ and define the sets D_1^α and D_2^α as follows:

$$D_1^\alpha = \left\{ s=(s_n)\in\omega: \sup_{N\in\aleph}\sup_n \left| \sum_{\substack{n|m \\ m\in N}} \frac{\mu\left(\frac{m}{n}\right)}{\varphi(m)} n s_m \right|^{p_n} < \infty \right\}$$

and

$$D_2^\alpha = \bigcup_{K>1} \left\{ s=(s_n)\in\omega: \sup_{N\in\aleph}\sum_n \left| \sum_{m\in N_n} \frac{\mu\left(\frac{m}{n}\right)}{\varphi(m)} n s_m K^{-1} \right|^{p_n'} < \infty \right\}.$$

(i) If $0 < p_n \le 1$ for all $n\in\mathbb{N}$, then $\{\ell(\Phi,p)\}^\alpha = D_1^\alpha$..

(ii) If $1 < p_n \le P < \infty$ for all $n\in\mathbb{N}$, then $\{\ell(\Phi,p)\}^\alpha = D_2^\alpha$.

Proof. We prove only the case (ii) since the case (i) can be proved by analogy. Let us take any $s=(s_n)\in\omega$. Bearing in mind the relation between the sequences $u=(u_n)$ and $v=(v_n)$, we easily write that

$$s_m u_m = s_m \sum_{n\backslash m} \frac{\mu\left(\frac{m}{n}\right)}{\varphi(m)} n v_n = \sum_{n\backslash m} \left(\frac{\mu\left(\frac{m}{n}\right)}{\varphi(m)} n s_m \right) v_n = C_m(v) \quad (m\in\mathbb{N}), \tag{12}$$

where the matrix $C=(c_{mn})$ is defined by

$$C_{mn} = \begin{cases} \frac{\mu\left(\frac{m}{n}\right)}{\varphi(m)} n s_m, & n \mid m \\ 0, & otherwise \end{cases}$$

for all $n,m\in\mathbb{N}$. Thus, we obtain from combining (3.6) with the equation (7) of Lemma 1 (i) that $su=(s_n u_n)\in\ell_1$ whenever $u,p=(u_n)\in\ell(\Phi,p)$ if and only if $Cv\in\ell_1$ whenever $v=(v_n)\in\ell(p)$. This says that $s=(s_n)\in\{\ell(\Phi,p)\}^\alpha$ if and only if $C\in(\ell(p):\ell_1)$. Hence, this gives the result $\{\ell(\Phi,p)\}^\alpha = D_2^\alpha$.

Theorem 5. Let us define the sets D_1^β, D_2^β and D_3^β as follows:

$$D_1^\beta = \bigcup_{K>1} \left\{ s=(s_n)\in\omega: \sup_m \sum_{n=1}^m \left| \sum_{k=n,n|k}^m \frac{\mu\left(\frac{k}{n}\right)}{\varphi(k)} n s_k K^{-1} \right|^{p_n'} \right\},$$

$$D_2^\beta = \left\{ s=(s_n)\in\omega: \sup_{m,n\in\mathbb{N}} \left| \sum_{k=n,n|k}^m \frac{\mu\left(\frac{k}{n}\right)}{\varphi(k)} n s_k \right|^{p_n} \right\}$$

and

$$D_3^\beta = \left\{ s=(s_n)\in\omega: \lim_m \sum_{k=n,n|k}^m \frac{\mu\left(\frac{k}{n}\right)}{\varphi(k)} n s_k \text{ exists for } n\in\mathbb{N} \right\}.$$

Then, $\{\ell(\Phi,p)\}^\beta = D_1^\beta \cup D_2^\beta \cup D_3^\beta$.

Proof. Take any $s=(s_n)\in\omega$ and let $v=(v_n)$ be $\Phi-$transformation of the sequence $u=(u_n)$. Then, we can write

$$\begin{aligned} \sum_{n=1}^m s_n u_n &= \sum_{n=1}^m s_n \left(\sum_{k\backslash n} \frac{\mu\left(\frac{n}{k}\right)}{\varphi(n)} k v_k \right) \\ &= \sum_{n=1}^m \left(\sum_{k=n,n|k}^m \frac{\mu\left(\frac{k}{n}\right)}{\varphi(k)} n s_k \right) v_n \\ &= D_m(v) \end{aligned} \tag{13}$$

where the matrix $D=(d_{mn})$ is defined by

$$d_{mn} = \begin{cases} \sum_{k=n,n|k}^m \frac{\mu\left(\frac{k}{n}\right)}{\varphi(k)} n s_k, & 1\le n\le m \\ 0, & otherwise \end{cases}$$

for all $n,m\in\mathbb{N}$. Thus, we have from Lemma 3 with (3.7) that $su=(s_n u_n)\in cs$ whenever $u=(u_n)\in\ell(\Phi,p)$ if and only if $Dv\in c$ whenever $v=(v_n)\in\ell(p)$ which means that $s=(s_n)\in\{\ell(\Phi,p)\}^\beta$ if and only if $D\in(\ell(p):c)$. It follows from (9), (10) and (11) that

$\{\ell(\Phi,p)\}^\beta = D_1^\beta \cup D_2^\beta \cup D_3^\beta$.

Theorem 6.

$$\{\ell(\Phi,p)\}^\gamma = \begin{cases} D_2^\beta, & 0 < p_n \le 1 \text{ for all } n\in\mathbb{N} \\ D_1^\beta, & 1 < p_n \le P < \text{ for all } n\in\mathbb{N} \end{cases}.$$

Proof. This can be obtained in the similar way, as mentioned in the proof of Theorem 5 with Lemma 2 instead of Lemma 3. So, we omit the details.

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