Karaelmas Science and Engineering Journal
Journal home page: http://fbd.beun.edu.tr
DOI: 10.7212/zkufbd.v9i1.1213

Research Article
Received / Geliş tarihi : 20.11.2017
Accepted / Kabul tarihi : 19.12.2017

Coupled Fixed Point Theorems for Nonlinear Contractions in Ordered Uniform Spaces

Sıralı Düzgün Uzaylarda Lineer Olmayan Büzülmeler İçin İkili Sabit Nokta Teoremleri

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Abstract

In this paper, we have proved some coupled coincidence and coupled common fixed point theorems for two mappings defined on the ordered uniform spaces by using the order relation on uniform spaces presented by Altun and Imdad (2009).

Keywords: Coupled fixed point, Non-linear contractions, Ordered uniform spaces

Öz

Bu çalışmada, Altun and Imdad (2009) tarafından düzgün uzaylar üzerinde verilmiş olan sıralama bağıntısını kullanarak sıralı düzgün uzaylar üzerindeki iki dönüşüm için bazı ikili çakışık ve ikili ortak sabit nokta teoremleri ispat edilmiştir.

Anahtar Kelimeler: İkili sabit nokta, Lineer olmayan büzülmeler, Sıralı düzgün uzaylar

1. Introduction

Recently, fixed point and common fixed point theorems in uniform space have been investigated intensively (e.g. Aamri and El Moutawakil 2004a, b, 2005, Agarwal et al. 2004, Altun and Imdad 2009, Olatinwo 2008, Turkoglu 2010a, b, 2008, Turkoglu and Binbasioglu 2011). The greater part of these theorems are given for contractive or contractive type mappings. Lately, a new E-distance function concept on uniform spaces has been introduced by Aamri and El Moutawakil (2004). Then Altun and Imdad (2009) defined a partial ordering on uniform spaces. They benefited from the concept of E-distance while making this definition. The Banach contraction theorem has a very important place in fixed point theory. Boyd and Wong (1969) extended the Banach contraction theorem to nonlinear contraction mappings. So many authors prove significant fixed point theorems (Ciric 2008, Guo and Lakshmikantham 1987, Lakshmikantham and Ciric 2009, Ran and Reurings 2004, Samet 2010, Turkoglu and Binbasioglu 2015). Recently, Bhaskar and Lakshmikantham (2006), Nieto and Lopez (2005)a, b, (2007), Ran and Reurings (2004), Agarwal et al. (2008) and Abbas et al. (2010) presented some important results for contractions in partially ordered metric spaces.

In this paper, we use the partial ordering on uniform spaces and in this way we extend some results for a mixed monotone linear contractive mapping. Also we generalize the idea of a mixed monotone mapping. We give some coupled coincidence and coupled common fixed point theorems for two mappings.

2. Preliminaries

Firstly, we will give the following definitions. Because these concepts will be used in the sequel of our work.

Definition 2.1 Let (X,ϑ) be a uniform space. A function $p\colon X\times X\to\mathbb{R}^+$ is said to be an E-distance if

(p_1) For any $V\in\vartheta$ there exists $\delta>0$ such that $p(c,a)\le\delta$ and $p(c,b)\le\delta$ for some $c\in X$, imply $(a,b)\in V$, (p_2) $p(a,b)\le p(a,c)+p(c,b), \forall a,b,c\in X$ (Aamri and El Moutawakil 2004).

Lemma 2.2 Let (X,ϑ) be a Hausdorff uniform space and p be an E-distance on X. Let $\{a_n\}$ and $\{b_n\}$ be arbitrary sequences in X and $\{\alpha_n\},\{\beta_n\}$ be sequences in $\mathbb{R}^+$

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converging to 0. Then, for $a,b,c \in X$, the following holds:

a) If $p(a_n,b) \le \alpha_n$ and $p(a_n,c) \le \beta_n$ for all $n \in \mathbb{N}$, then $b = c$. Also, $p(a,b) = 0$ and $p(a,c) = 0$, then $b = c$.

b) If $p(a_n,b_n) \le \alpha_n$ and $p(a_n,c) \le \beta_n$ for all $n \in \mathbb{N}$, then $\{b_n\}$ converges to c.

c) If $p(a_n,a_m) \le \alpha_n$ for all $m > n$, then $\{a_n\}$ is a p-Cauchy sequence in (X,ϑ).

Let (X,ϑ) be a uniform space equipped with E-distance p. A sequence in X is p-Cauchy if it satisfies the usual metric condition. There are several concepts of completeness in this setting (Aamri and El Moutawakil 2004).

Definition 2.3 Let (X,ϑ) be a uniform space and p be an E-distance on X. Then

a) X said to be S-complete if for every p-Cauchy sequence $\{a_n\}$ there exists $a \in X$ with $\lim_{n\to\infty} p(a_n,a) = 0$,

b) X said to be p-Cauchy complete if for every p-Cauchy sequence $\{a_n\}$ there exists $a \in X$ with $\lim_{n\to\infty} a_n = a$ with respect to $\tau(\vartheta)$,

c) $f{:}X \to X$ is p continuous if $\lim_{n\to\infty} p(a_n,a) = 0$ implies $\lim_{n\to\infty} p(fa_n,fa) = 0$,

d) $f{:}X \to X$ is $\tau(\vartheta)$ continuous if $\lim_{n\to\infty} a_n = a$ with respect to $\tau(\vartheta)$ implies $\lim_{n\to\infty} fa_n = fa$ with respect to $\tau(\vartheta)$ (Aamri and El Moutawakil 2004a, b, 2005).

Remark 2.4 Let (X,ϑ) be a Hausdorff uniform space and $\{a_n\}$ be a p-Cauchy sequence. Suppose that X is S-complete, then there exists $a \in X$ such that $\lim_{n\to\infty} p(a_n,a) = 0$. Then Lemma 2.2 (b) gives that $\lim_{n\to\infty} a_n = a$ with respect to the topology $\tau(\vartheta)$ which shows that S-completeness implies p-Cauchy completeness (Aamri and El Moutawakil 2004).

Lemma 2.5 Let (X,ϑ) be a Hausdorff uniform space, p be E-distance on X and $\varphi{:}X \to \mathbb{R}$. Define the relation "$\preceq$" on X as follows;

$$a \preceq b \Leftrightarrow a = b \text{ or } p(a,b) \le \varphi(a) - \varphi(b).$$

Then "$\preceq$" is a (partial) order on X induced by φ (Altun and Imdad 2009).

Definition 2.6 We call an element $(a,b) \in X \times X$ a coupled fixed point of a mapping T if $T(a,b) = a, T(b,a) = b$ (Bhaskar and Lakshmikantham 2006).

Definition 2.7 An element $(a,b) \in X \times X$ is called a coupled coincidence point of a mapping $T{:}X \times X \to X$ and $g{:}X \times X \to X$ if $T(a,b) = g(a), t(b,a) = g(b)$ (Lakshmikantham and Ciric 2009).

Definition 2.8 Let X be a non-empty set and $T{:}X \times X \to X$ and $g{:}X \to X$. We say the mappings T and g commute if $g(T(a,b)) = T(g(a),g(b))$ for all $a,b \in X$ (Lakshmikantham and Ciric 2009).

Definition 2.9 Let (X,ϑ) be a uniform space, "$\preceq$" is an order on X and $T{:}X \times X \to X$. We say that T has the mixed monotonicity property if T is monotone nondecreasing in a and is monotone nonincreasing in b, that is for any $a,b \in X$,

$$a_1,a_2 \in X, a_1 \preceq a_2 \Rightarrow T(a_1,b) \preceq T(a_2,b) \tag{1}$$

and

$b_1,b_2 \in X, b_1 \preceq b_2 \Rightarrow T(a,b_2) \succeq T(a,b_1)$ (2) (Bhaskar and Lakshmikantham 2006).

Definition 2.10 Let (X,ϑ) be a uniform space, "$\preceq$" is an order on X and $T{:}X \times X \to X$ and $g{:}X \to X$. We say T has the mixed g-monotone property if T is monotone g-nondecreasing in a and is monotone g-nonincreasing in b, that is for any $a,b \in X$,

$$a_1,a_2 \in X, g(x_1) \preceq g(a_2) \Rightarrow T(a_1,b) \preceq T(a_2,b). \tag{3}$$

and

$b_1,b_2 \in X, g(b_1) \preceq g(b_2) \Rightarrow T(a,b_2) \preceq T(a,b_1)$ (4) (Boyd and Wong 1969).

If g is the identity mapping, then Definition 2.10 reduces to Definition 2.9.

3. Main Results

Theorem 3.1 Let (X,ϑ) be a Hausdorff uniform space, "$\preceq$" is an order on X and suppose there is an E-distance p on X such that (X,p) is a p-Cauchy complete uniform space. Let $T{:}X \times X \to X$ be a $\tau(\vartheta)$ continuous mapping having the mixed monotone property on X. Assume that there exists a $k \in [0,1)$ with

$$p(T(a,b),T(u,v)) \le \frac{k}{2}[p(a,u) + p(b,v)]$$

for all comparable *a, u* and all comparable *b, v*. If there exist $a_0, b_0 \in X$ such that

$$a_0 \preceq T(a_0,b_0) \text{ ve } b_0 \succeq T(b_0,a_0),$$

then there exist $a,b \in X$ such that $a = T(a,b)$ ve $b = T(b,a)$ (Turkoglu and Binbasioglu 2015).

Theorem 3.2 Let (X,ϑ) be a Hausdorff uniform space, "$\preceq$" is an order on X and suppose there is an E-distance p on X such that (X,p) is a p-Cauchy complete uniform space. Assume that X has the following property:

a) If a non-decreasing sequence $\{a_n\} \to a$, then $a_n \preceq a$ for every n;

b) If a non-increasing sequence $\{b_n\} \to b$, then $b \preceq b_n$ for every n.

Let $T{:}X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exists a $k \in [0,1)$ with $p(T(a,b),T(u,v)) \leq \frac{k}{2}[p(a,u)+p(b,v)]$ for all comparable a, u and b, v. If there exist $a_0, b_0 \in X$ such that $a_0 \preceq T(a_0,b_0)$ and $b_0 \succeq T(b_0,a_0)$ then there exist $a,b \in X$ such that $a = T(a,b)$ and $b = T(b,a)$.

Proof. Since $a_0 \preceq T(a_0,b_0) = a_1$ and $b_0 \succeq T(b_0,a_0) = b_1$ letting $a_2 = T(a_1,b_1)$ and $b_2 = T(b_1,a_1)$, we denote

$$T^2(a_0,b_0) = T(T(a_0,b_0),T(b_0,a_0)) = T(a_1,b_1) = a_2$$

and

$$T^2(b_0,a_0) = T(T(b_0,a_0),T(a_0,b_0)) = T(b_1,a_1) = b_2.$$

With this notation, we now have, due to the mixed monotone property of T,

$$a_2 = T^2(a_0,b_0) = T(a_1,b_1) \succeq T(a_0,b_0) = a_1$$

and

$$b_2 = T^2(b_0,a_0) = T(b_1,a_1) \preceq T(b_0,a_0) = b_1.$$

Further, for $n = 1,2,...$ we let

$$a_{n+1} = T^{n+1}(a_0,b_0) = T(T^n(a_0,b_0),T^n(b_0,a_0))$$

and

$$b_{n+1} = T^{n+1}(b_0,a_0) = T(T^n(b_0,a_0),T^n(a_0,b_0)).$$

We can easily verify that

$$a_0 \preceq T(a_0,b_0) = a_1 \preceq T^2(a_0,b_0) = a_2 \preceq ... \preceq T^{n+1}(a_0,b_0) = a_{n+1} \preceq ...$$

and

$$b_0 \succeq T(b_0,a_0) = b_1 \succeq T^2(b_0,a_0) = b_2 \succeq ... \succeq T^{n+1}(b_0,a_0) = b_{n+1} \succeq ... .$$

Now, we claim that, for $n \in \mathbb{N}$,

$$p(T^{n+1}(a_0,b_0),T^n(a_0,b_0)) \leq \frac{k^n}{2}[p(T(a_0,b_0),a_0)+p(T(b_0,a_0),b_0)] \tag{5}$$

$$p(T^{n+1}(b_0,a_0),T^n(b_0,a_0)) \leq \frac{k^n}{2}[p(T(b_0,a_0),b_0)+p(T(a_0,b_0),a_0)]. \tag{6}$$

Indeed, for $n = 1$ using $T(a_0,b_0) \succeq a_0$ and $T(b_0,a_0) \preceq b_0$, we get

$$p(T^2(a_0,b_0),T(a_0,b_0)) = p(T(T(a_0,b_0),T(b_0,a_0)),T(a_0,b_0)) \leq \frac{k}{2}[p(T(a_0,b_0),a_0)+p(T(b_0,a_0),b_0)].$$

Similarly,

$$p(T^2(b_0,a_0),T(b_0,a_0)) = p(T(T(b_0,a_0),T(a_0,b_0)),T(b_0,a_0)) \leq \frac{k}{2}[p(T(a_0,b_0),a_0)+p(T(b_0,a_0),b_0)].$$

Now, assume that (5) and (6) hold. Using $T^{n+1}(a_0,b_0) \succeq T^n(a_0,b_0)$ and $T^{n+1}(b_0,a_0) \preceq T^n(b_0,a_0)$ we get

$$\begin{aligned} p(T^{n+2}(a_0,b_0),T^{n+1}(a_0,b_0)) &= p(T(T^{n+1}(a_0,b_0),T^{n+1}(b_0,a_0)) \\ &\quad ,T(T^n(a_0,b_0),T^n(b_0,a_0))) \\ &\leq \frac{k}{2}[p(T^{n+1}(a_0,b_0),T^n(a_0,b_0)) \\ &\quad +p(T^{n+1}(b_0,a_0),T^n(b_0,a_0))] \\ &\leq \frac{k^{n+1}}{2}[p(T(a_0,b_0),a_0)+p(T(b_0,a_0),b_0)]. \end{aligned}$$

Similarly, one can show that

$$p(T^{n+2}(b_0,a_0),T^{n+1}(b_0,a_0)) \leq \frac{k^{n+1}}{2}[p(T(b_0,a_0),b_0)+p(T(a_0,b_0),a_0)].$$

This implies that $\{T^n(a_0,b_0)\}$ and $\{T^n(b_0,a_0)\}$ are p-Cauchy sequences in X.

Indeed, let $m > n$, then

$$\begin{aligned} p(T^m(a_0,b_0),T^n(a_0,b_0)) &\leq p(T^m(a_0,b_0),T^{m-1}(a_0,b_0)) \\ &\quad ... + p(T^{n+1}(a_0,b_0),T^n(a_0,b_0)) \\ &\leq \frac{k^{m-1}+...+k^n}{2}[p(T(a_0,b_0),a_0)+p(T(b_0,a_0),b_0)] \\ &\leq \frac{k^n}{2(1-k)}[p(T(a_0,b_0),a_0)+p(T(b_0,a_0),b_0)]. \end{aligned}$$

Similarly, we can verify that $\{T^n(b_0,a_0)\}$ is also a p-Cauchy sequence.

Since X is a p-Cauchy complete uniform space, there exist $a,b \in X$ such that

$$\lim_{n\to\infty} T^n(a_0,b_0) = a \text{ and } \lim_{m\to\infty} T^m(b_0,a_0) = b.$$

Let $\delta_1 > 0, \delta'_1 > 0$ and $\delta'_2 > 0$. Since $\{T^n(a_0,b_0)\} \to a$ and $\{T^m(b_0,a_0)\} \to b$ there exist $n_1, n_2 \in \mathbb{N}$ such that, for all $n \geq n_1$ and $m \geq n_2$, we have

$p(T^n(a_0,b_0),a) < \delta_1$ and
$p(a,T^n(a_0,b_0)) < \delta'_1$, $p(b,T^n(b_0,a_0)) < \delta'_2$. Taking $n \in \mathbb{N}$, $n \geq \max\{n_1,n_2\}$ and using $T^n(a_0,b_0) \preceq a, T^n(b_0,a_0)\} \succeq b,$ we get

$$p(T(a,b),a) \leq p(T(a,b),T^{n+1}(a_0,b_0)) + p(T^{n+1}(a_0,b_0),a$$
$$= p(T(a,b),T(T^n(a_0,b_0),T^n(b_0,a_0))) + p(T^{n+1}(a_0,b_0),a)$$
$$\leq \frac{k}{2}[p(a,T^n(a_0,b_0)) + p(b,T^n(b_0,a_0))] + p(T^{n+1}(a_0,b_0),a)$$
$$\leq p(a,T^n(a_0,b_0)) + p(b,T^n(b_0,a_0)) + p(T^{n+1}(a_0,b_0),a)$$
$$< \delta_1' + \delta_2' + \delta_1$$

So if we take $\varepsilon = \delta_1' + \delta_2' + \delta_1$, this implies that $T(a,b) = a$. Similarly, we can show that $p(T(b,a),b) < \varepsilon$, implying that $T(b,a) = b$.

Theorem 3.3 Let (X,ϑ) be a Hausdorff uniform space, "$\preceq$" is an order on X and suppose there is an E-distance p on X such that (X,p) is a p-Cauchy complete uniform space. Assume that there is a function $\phi:[0,\infty) \to [0,\infty)$ with $\phi(t) < t$ and $\lim_{r \to t^+} \phi(r) < t$ for each $t > 0$ and also suppose that $T:X \times X \to X$ and $g:X \to X$ are defined such that T has the mixed g-monotone property and

$$p(T(a,b),T(u,v)) \leq \phi \frac{p(g(a),g(u)) + p(g(b),g(v))}{2} \quad (7)$$

for all $a,b,u,v \in X$ for which $g(a),g(u)$ are comparable and $g(b), g(v)$ are comparable. Suppose that $T(X \times X) \subseteq g(X)$, g is $\tau(\vartheta)$-continuous and commutes with T and also suppose either

a) T is $\tau(\vartheta)$-continuous

or

b) X has the following property:

If a non-decreasing sequence $\{a_n\} \to a$, then $a_n \preceq a$ for every n; (8)

If a non-increasing sequence $\{b_n\} \to b$, then $b \preceq b_n$ for every n. (9)

If there exist $a_0,b_0 \in X$ such that $g(a_0) \preceq T(a_0,b_0)$ and $g(b_0) \succeq T(b_0,a_0)$, then there exist $a,b \in X$ such that $g(a) = T(a,b)$ and $g(b) = T(b,a)$, that is, T and g have a coupled coincidence. Furthermore, suppose that for every $(a,b),(b^*,a^*) \in X \times X$ there exists a $(u,v) \in X \times X$ such that $(T(u,v),T(v,u))$ is comparable to $(T(a,b),T(b,a))$ and $(T(a^*,b^*),T(b^*,a^*))$. Then T and g have a unique coupled fixed point, that is, there exists a unique $(a,b) \in X \times X$ such that $a = g(a) = T(a,b)$ and $b = g(b) = T(b,a)$.

Proof. Since $g(a) = T(a,b)$ and $g(b) = T(b,a)$ in (Turkoglu and Binbasioglu (2015), Theorem 2), by commutativity of T and g we have

$$g(g(a)) = g(T(a,b)) = T(g(a),g(b))$$

and

$$g(g(b)) = g(T(b,a)) = T(g(b),g(a)).$$

Denote $g(a) = c, g(b) = d$. Then from the above equality,

$g(c) = T(c,d)$ and $g(d) = T(d,c)$.

Thus (c,d) is a coupled coincidence point.

Now, we shall show that if (a,b) and (a^*,b^*) are coupled coincidence points, that is if $g(a) = T(a,b), g(b) = T(b,a)$ and $g(a^*) = T(a^*,b^*), g(b^*) = T(b^*,a^*)$, then $g(a) = g(a^*)$ and $g(b) = g(b^*)$.

By assumption there is a $(u,v) \in X \times X$ such that $(T(u,v),T(v,u))$ is comparable with $(T(a,b),T(b,a))$ and $(T(a^*,b^*),T(b^*,a^*))$. Put $u_0 = u, v_0 = v$ and choose $u_1, v_1 \in X$ so that $g(u_1) = T(u_0,v_0)$ and $g(v_1) = T(v_0,u_0)$. Then we can inductively define sequences $\{g(u_n)\}$ and $\{g(v_n)\}$ such that $g(u_{n+1}) = T(u_n,v_n)$ and $g(v_{n+1}) = T(v_n,u_n)$. Moreover, set $a_0 = a$, $b_0 = b, a_0^* = a^*, b_0^* = b^*$ and on the same way, define the sequences $\{g(a_n)\},\{g(b_n)\}$ and $\{g(a_n^*)\},\{g(b_n^*)\}$. Then it is easy to show that $g(a_n) = T(a,b), g(b_n) = T(b,a)$ and

$g(a_n^*) = T(a^*,b^*), g(b_n^*) = T(b^*,a^*)$ for all $n \geq 1$.

Since $(T(a,b),T(b,a)) = (g(a_1),g(b_1)) = (g(a),g(b))$ and $(T(u,v),T(v,u)) = (g(u_1),g(v_1))$ are comparable, then $g(a) \leq g(u_1)$ and $g(b) \geq g(v_1)$. It is easy to show that $(g(a),g(b))$ and $(g(u_n),g(v_n))$ are comparable, that is, $g(a) \leq g(u_n)$ and $g(b) \geq g(v_n)$ for all $n \geq 1$.

So,

$$p(g(a),g(u_{n+1})) = p(T(a,b),T(u_n,v_n))$$
$$\leq \phi \frac{p(g(a),g(u_n)) + p(g(b),g(v_n))}{2}$$
$$p(g(b),g(v_{n+1})) = p(T(b,a),T(v_n,v_n))$$
$$\leq \phi \frac{p(g(b),g(v_n)) + p(g(a),g(u_n))}{2}.$$

Adding we get

$$\frac{p(g(a),g(u_{n+1})) + p(g(b),g(v_{n+1}))}{2}$$
$$\leq \phi \frac{p(g(a),g(u_n)) + p(g(b),g(v_n))}{2}.$$

Hence it follows

$$\frac{p(g(a),g(u_{n+1})) + p(g(b),g(v_{n+1}))}{2}$$
$$\leq \phi^n \frac{p(g(a),g(u_1)) + p(g(b),g(v_1))}{2}.$$

for each $n \geq 1$. It is known that $\phi(t) < t$ and $\lim_{r \to t^+} \phi(r) < t$ imply $\lim_{n \to \infty} \phi^n(t) = 0$ for each $t > 0$.

Thus from above inequality $\lim_{n\to\infty} p(g(a),g(u_{n+1}))=0$ and $\lim_{n\to\infty} p(g(b),g(v_{n+1}))=0$.

Similarly one can prove that $\lim_{n\to\infty} p(g(a^*),g(u_{n+1}))=0$ and $\lim_{n\to\infty} p(g(b^*),g(v_{n+1}))=0$.

By the triangle inequality,

$p(g(a),g(a^*))\le p(g(a),g(u_{n+1}))+p(g(u_{n+1}),g(a^*))\to 0$ as $n\to\infty$,

$p(g(b),g(b^*))\le p(g(b),g(v_{n+1}))+p(g(v_{n+1}),g(b^*))\to 0$ as $n\to\infty$.

Hence $g(a)=g(a^*)$ and $g(b)=g(b^*)$.

Since $g(a)=T(a,b)$ and $g(b)=T(b,a)$, by the commutativity of T and g we have

$g(g(a))=g(T(a,b))=T(g(a),g(b))$ and
$g(g(b))=g(T(b,a))=T(g(b),g(a))$.

Denote $g(a)=c, g(b)=d$. Then we obtain that
$g(c)=T(c,d)$ and $g(d)=T(d,c)$.

Thus (c,d) is a coupled coincidence point. Then from $g(a)=g(a^*)$ and $g(b)=g(b^*)$ with $a^*=c$ and $b^*=d$ it follows $g(c)=g(a)$ and $g(d)=g(b)$, that is, $g(c)=c$ and $g(d)=d$. Therefore $c=g(c)=T(c,d)$ and $d=g(d)=T(d,c)$. So (c,d) is a coupled common fixed point of T and g.

To prove the uniqueness, assume that (p,q) is another coupled common fixed point. Then we have $p=g(p)=g(c)=c$ and $q=g(q)=g(d)=d$.

Corollary 3.4. Let (X,ϑ) be a Hausdorff uniform space, "$\preceq$" is an order on X and suppose there is an E-distance p on X such that (X,p) is a p-Cauchy complete uniform space. Suppose that $T{:}X\times X\to X$ and $g{:}X\to X$ are defined, such that T has the mixed g-monotone property and assume that there exists a $k\in[0,1)$ with

$$p(T(a,b),T(u,v))\le \frac{k}{2}[p(g(a),g(u))+p(g(b),g(v))]$$

for all $a,b,u,v\in X$ for which $g(a),g(u)$ are comparable and $g(b),g(v)$ are comparable. Suppose that $T(X\times X)\subseteq g(X)$, g is $\tau(\vartheta)$-continuous and commutes with T and also either

a) T is $\tau(\vartheta)$-continuous

or

b) X has the following properties:

If a non-decreasing sequence $\{a_n\}\to a$ then $a_n\preceq a$ for all n, (10)

If a non-increasing sequence $\{b_n\}\to b$ then $b\preceq b_n$ for all n. (11)

If there exist $a_0,b_0\in X$ such that $g(a_0)\preceq T(a_0,b_0)$ and $g(b_0)\succeq T(b_0,a_0)$, then there exists $a,b\in X$ such that $g(a)=T(a,b)$ and $g(b)=T(b,a)$, that is, T and g have a coupled coincidence.

Proof. Taking $\phi(t)=kt$ with $k\in[0,1)$ in the above theorem, we obtain this corollary.

Now, we shall prove the existence and uniqueness theorem of a coupled common fixed point. Note that if (X,p) is a partially ordered uniform space, then we endow the product $X\times X$ with the following partial order:

for $(a,b),(u,v)\in X\times X, (a,b)\preceq(u,v)\Leftrightarrow a\preceq u, v\preceq b$.

Corollary 3.5. Let (X,ϑ) be a Hausdorff uniform space, "$\preceq$" is an order on X and suppose there is an E-distance p on X such that (X,p) is a p-Cauchy complete uniform space. Assume that there is a function $\phi{:}[0,\infty)\to[0,\infty)$ with $\phi(t)<t$ and $\lim_{r\to t^+}\phi(r)<t$ for each $t>0$ and let $T{:}X\times X\to X$ be a mapping having the mixed monotone property on X and

$$p(T(a,b),T(u,v))\le\phi\frac{p(a,u)+p(b,v)}{2}$$

for each comparable a,u and each comparable b,v.

Also suppose that

c) T is $\tau(\vartheta)$-continuous

or

d) X has the following properties:

If a non-decreasing sequence $\{a_n\}\to a$ then $a_n\preceq a$ for all n, (12)

If a non-increasing sequence $\{b_n\}\to b$ then $b\preceq b_n$ for all n. (13)

If there exist $a_0,b_0\in X$ such that $a_0\preceq T(a_0,b_0)$ and $b_0\succeq T(b_0,a_0)$ then there exist $a,b\in X$ such that $a=T(a,b)$ and $b=T(b,a)$.

Furthermore, if ϕ is a non-increasing function and a_0,b_0 are comparable, then $a=b$, that is, $a=T(a,a)$.

Proof. Following the proof of above theorem with $g=I$ (I is denote the identity mapping), we only have to show that $a=T(a,a)$.

Since ϕ is a non-increasing function by the triangle inequality

$$
\begin{aligned}
p(a,b) &\leq p(a,a_{n+1}) + p(a_{n+1},b_{n+1}) + p(b_{n+1},b) \\
&= p(T(a_n,b_n),T(b_n,a_n)) + p(a,a_{n+1}) + p(b_{n+1},b) \\
&\leq \phi\frac{p(a_n,b_n)+p(b_n,a_n)}{2} + p(a,a_{n+1}) + p(b_{n+1},b) \\
&\leq \phi\frac{p(a_n,a_n)}{2} + p(a,a_{n+1}) + p(b_{n+1},b)
\end{aligned}
$$

Taking the limit as $n \to \infty$ we get, as $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$,

$p(a,b) \leq \phi(p(a_n,a_n)) < \phi(\varepsilon) < \varepsilon.$

Hence $p(a,b) = 0$. Therefore $a = b$, that is, $a = T(a,a)$.

Corollary 3.6. Let (X,ϑ) be a Hausdorff uniform space, "$\preceq$" is an order on X and suppose that there is an E-distance p on X such that (X,p) is a p-Cauchy complete uniform space. Suppose that $T{:}X\times X \to X$ is a mapping having the mixed monotone property on X. Assume that there exists a $k \in [0,1)$ with

$p(T(a,b),T(u,v)) \leq \frac{k}{2}[p(a,u)+p(b,v)]$ for each comparable a,u and each comparable b,v. Also suppose that

e) T is $\tau(\vartheta)$-continuous

or

f) X has the following properties:

If a non-decreasing sequence $\{a_n\} \to a$ then $a_n \preceq a$ for all n, (14)

If a non-increasing sequence $\{b_n\} \to b$ then $b \preceq b_n$ for all n. (15)

If there exist $a_0, b_0 \in X$ such that $a_0 \preceq T(a_0,b_0)$ and $b_0 \succeq T(b_0,a_0)$, then there exist $a,b \in X$ such that $a = T(a,b)$ and $b = T(b,a)$.

Proof. Taking $\phi(t) = kt$ with $k \in [0,1)$ in Corollary 3.5 we obtain Corollary 3.6.

4. Results

In this study, firstly we gave some basic definitions and theorems related to the concept of E-distance in uniform spaces. Using the order relation defined by Altun and Imdad (2009), we have shown that a T-mapping with mixed monotonicity and a nonlinear contraction condition has a fixed point under some conditions. Then we showed when a mapping T with mixed g-monotonicity would have a coupled fixed point. Finally, we proved three important corollaries related to this fixed point and coupled fixed point theorems.

5. References

Aamri, M., El Moutawakil, D. 2004. Common fixed point theorems for E- contractive or E-expansive maps in uniform spaces. *Acta Math. Acad. Pea. Nyir.* ,20: 83-91.

Aamri, M., El Moutawakil, D. 2005. Weak compatibility and common fixed point theorems for A-contractive and E-expansive maps in uniform spaces. *Serdica Math. J.*,31, 75-86.

Abbas, M., Ilic, D., Khan, MA. 2010. Coupled coincidence point and coupled common fixed point theorems in partially ordered metric spaces with w-distance. *Fixed Point Theo. Appl.*,Article ID:134897.

Agarwal, RP., O'Regan, D., Papageorgiou, NS. 2004. Common fixed point theory for multivalued contractive maps of Reich type in uniform spaces. *Appl. Anal.*,83(1): 37-47.

Agarwal, RP., El-Gebeily, MA., O'Regan, D. 2008. Generalized contractions in partially ordered metric spaces. *Appl. Anal.*,87: 109-116.

Altun, I., Imdad, M. 2009. Some fixed point theorems on ordered uniform spaces. *Filomat*,23(3): 15-22.

Bhaskar, TG. Lakshmikantham,V. 2006. Fixed point theorems in partially ordered metric spaces and applications. *Nonlinear Anal.*,65: 1379-1393.

Boyd, DW., Wong, JS. 1969. On nonlinear contractions. *Proc. Amer. Math. Soc.*,20: 458-464.

Ciric, LB. 2008. Fixed point theorems for multi-valued contractions in complete metric spaces. *J. Math. Anal. Appl.*,348(1): 499-507.

Guo, D., Lakshmikantham,V. 1987. Coupled fixed points of nonlinear operators with applications. *Nonlinear Anal.*,11, 623-632.

Lakshmikantham, V., Ciric, LB. 2009. Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. *Nonlinear Anal.*,70: 4341-4349

Nieto, JJ., Lopez, RR. 2005. Contractive mapping theorems in partially ordered sets and applications to ordinary diferential equations. *Order,* 22: 223-239.

Nieto, JJ., Lopez, RR. 2007. Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary diferential equations. *Acta Math. Sinica, Engl. Ser.*,23(12): 2205-2212.

Olatinwo, MO. 2008. On some common fixed point theorems of Aamri and El Moutawakil in uniform spaces. *Applied Math. E-Notes,*8: 254-262.

Ran, ACM., Reurings, MCB. 2004. A fixed point theorem in partially ordered sets and some applications to matrix equations. *Proc. Amer. Math. Soc.*,132: 1435-1443.

Samet,B. 2010. Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces. *Nonlinear Anal.*, 72: 4508-4517.

Turkoglu, D. 2010. Some common fixed point theorems for weakly compatible mappings in uniform spaces. *Acta Math. Hungar.*128(1-2): 165-174.

Turkoglu, D. 2008. Some fixed point theorems for hybrid contractions in uniform space. *Taiwanese J. Math.*,12(3): 807-820.

Turkoglu, D., Binbasioglu, D. 2011. Some fixed point theorems for Multivalued Monotone Mappings in Ordered Uniform Space. *Fixed Point Theo. Appl.*, Article ID:186237.

Turkoglu, D., Binbasioglu, D. 2015. Coupled Coincidence point theorems for Compatible Mappings in Ordered Uniform Space. *Miskolc Math. Not.*, 16(1): 527-541.