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Directed Suborbital Graphs on the Poincare Disk

Poincare Dairesi Üzerinde Yönlendirilmiş Altyörüngesel Çizgeler

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Abstract

In this paper we investigate suborbital graphs of a special congruence subgroup of modular group. And this directed graphs is drawn in Poincare disk.

Keywords: Circuit, Imprimitive action, Orbit, Stabilizer, Suborbial graphs

AMS- Mathematical Subject Classification Number: 05C05, 05C20, 11F06, 20H05, 20H10.

Öz

Bu çalışmada, modüler grubun özel bir kongrüans alt grubunun alt yörüngesel çizgeleri araştırıldı ve bu yönlendirilmiş çizgeler Poincare dairesinde çizildi.

Anahtar Kelimeler: Devre, İmprimitif hareket, Yörünge, Sabitleyen, Altyörüngesel çizgeler

1. Introduction

Let $\mathbb{H}:=\{z\in\mathbb{C}:Imz>1\}$ be the complex upper half plane. $SL(2,\mathbb{Z}):=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}: a,b,c,d\in\mathbb{Z}\, and\, ad-bc=1\right\}$ is sometimes denoted by $\Gamma(1)$. And also we recall that in many papers authors use the projective special linear group $PSL(2,\mathbb{Z})\cong SL(2,\mathbb{Z})/\{\pm I\}$ instead of $SL(2,\mathbb{Z})$. The group

$$PSL(2,\mathbb{Z}):=\Gamma=\left\{\begin{array}{l} T(z)=\dfrac{az+b}{cz+d}: a,b,c,d\in\mathbb{Z}\, and\\ ad-bc=1\end{array}\right\}$$

is known the modular group. We say $SL(2,\mathbb{Z})$ and its subgroups of finite index modular groups.

Lemma 1. Γ has an action on $\mathbb{H}$ defined by $\gamma z=\dfrac{az+b}{cz+d}$ for $\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma$ and $z\in\mathbb{H}$.

Proof. If $z\in\mathbb{H}$ and $\gamma\in\Gamma$, then

$$\mathrm{Im}(\gamma z)=\mathrm{Im}\left(\frac{az+b}{cz+d}\right)=\frac{\det(\gamma)Imz}{|cz+d|^2}=\frac{Imz}{|cz+d|^2}>0.$$

Therefore $\gamma z\in\mathbb{H}$. Let $\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix},\beta=\begin{pmatrix} e & f\\ g & h\end{pmatrix}\in\Gamma$ and let $z\in\mathbb{H}$. Then $\beta(\gamma z)=\beta\left(\dfrac{az+b}{cz+d}\right)=\begin{pmatrix} ea+fc & eb+fd\\ ga+hc & gb+hd\end{pmatrix}z=(\beta\gamma)z$. As well, for all $z\in\mathbb{H}$, $\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}z=\dfrac{1.z+0}{0.z+1}=z$. Hence the identity element of Γ fixes all $z\in\mathbb{H}$.

Theorem 1. $SL(2,\mathbb{Z})$ is generated by two elements $\tau=\begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix}$ and $\omega=\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}$.

Proof. Let $\wedge$ be the subgroup of $SL(2,\mathbb{Z})$ generated by τ and ω. Suppose $\wedge\neq SL(2,\mathbb{Z})$. Since $\omega\tau^{-1}\omega^{-1}=\begin{pmatrix} 1 & 0\\ 1 & 1\end{pmatrix}$ and $\omega^2=-I$ all elements of the form $\begin{pmatrix} a & 0\\ c & d\end{pmatrix}$ of $SL(2,\mathbb{Z})$ are contained in $\wedge$. Therefore if we put $b_0=\min\left\{|b|:\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{Z})\setminus\wedge\right\}$, then $b_0\neq 0$. Take an element $\gamma_0=\begin{pmatrix} a_0 & b_0\\ c_0 & d_0\end{pmatrix}$ of $SL(2,\mathbb{Z})\setminus\wedge$, and an integer n so that $|a_0-nb_0|<b_0$. Since $\gamma_0\omega^{-1}\tau^n=\begin{pmatrix} -b_0 & a_0-nb_0\\ -d_0 & c_0-nd_0\end{pmatrix}$, w get $\gamma_0\omega^{-1}\tau^n\in\wedge$ by the assumption on b_0. Hence $\gamma_0\in\wedge$, this is a contradiction.

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It is known that a discontinuous group is discrete. Γ acts properly discontinuously on $\mathbb{H}$, that is, for any two distinct points $x,y \in \mathbb{H}$, there exist open neighbourhoods U,V containing x,y respectively such that the number of group elements $g \in \Gamma$ with $gU \cap V \neq \phi$ is finite. For such an action there is a notion of fundamental domain: a subset $\mathbb{F}$ of $\mathbb{H}$ such that **(i)** $\mathbb{H} = \cup \gamma \mathbb{F}$, for all $\gamma \in \Gamma$. **(ii)** There is an open set U so that $\mathbb{F} = \overline{U}$. **(iii)** U and γU are either identical or disjoint. We recall that a fundamental domain for the action of Γ on $\mathbb{H}$ is given by the definition;

Definition 1. The set $\mathbb{F} = \left\{ z \in \mathbb{H} : |z| \geq 1 \, and \, |Rez| \leq \frac{1}{2} \right\}$ shown in Figure 1. is a fundamental domain of Γ.

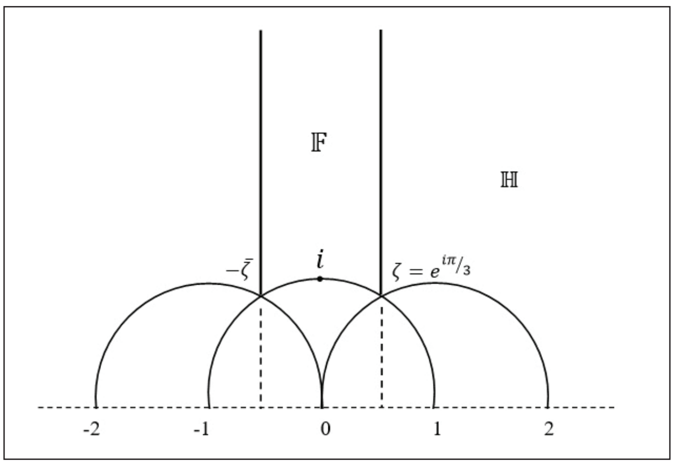

Figure 1. Fundamental domain for Γ.

Theorem 2. Any elliptic point of Γ is equivaent to i or ζ. The point i is an elliptic point of order 2 and $\Gamma_i = \left\{ \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$. The point ζ is an elliptic point of order 3 and $\Gamma_\zeta = \left\{ +\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \pm\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \right\}$.

Proof. It is obvious that interior points of a fundamental domain are ordinary points. Thus any elliptic point must be equivalent to a boundary point of the fundamental domain $\mathbb{F}$. Since Γ contains $\tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, the boundary points of $\mathbb{F}$, other that the three points i, ζ and $-\overline{\zeta}$ are also ordinary points. Observing that the interior angle of F at i is π, we see the order of i is at most 2. Since $\omega i = i$, and $\omega^2 = -1$, the point i is indeed an elliptic point of order 2. Since $\tau(-\overline{\zeta}) = \zeta$ and the interior angles of F at ζ and $-\overline{\zeta}$ are both $\frac{\pi}{3}$, the order of ζ is at most 3. Now we note $\tau\omega = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, $(\tau\omega)^2 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ and $(\tau\omega)^3 = -I$. As $\tau\omega$ fixes ζ, ζ is an elliptic point of order 3, and $-\overline{\zeta}$ is equivalent to ζ.

Remark 1. The set of the cusps of Γ is $P^1 := \mathbb{Q} \cup \{\infty\}$ and all cusps of Γ are equivalent.

Proof. It is clear that the point ∞ is a cusp of Γ. Let x be a cusp of Γ, and $x \neq \infty$. Because x is a double root of a quadratic equation with rational coefficients, x is a rational number. Coversely, let x be a rational number, and $x = \frac{a}{c}$ its reduced fractional expression. Then we can take integers *b, d* so that $ad - bc = 1$. Put $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\gamma \in \Gamma$ and $\gamma\infty = x$. Therefore x is Γ equivalent to ∞.

Now we explain congruence modular groups. Because they are very important number theory, algebraic graph theory and combinatorial group theory.

For a positive integer N, we define subgroups $\Gamma_0(N), \Gamma_1(N)$ and $\Gamma(N)$ of $SL(2,\mathbb{Z})$ by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : c \equiv 0 \, (\mathrm{mod}\, N) \right\},$$

$$\Gamma_1(N) = \left\{ \begin{array}{l} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : c \equiv 0 \, (\mathrm{mod} N), \\ a \equiv d \equiv 1 \, (\mathrm{mod} N) \end{array} \right\},$$

$$\Gamma(N) = \left\{ \begin{array}{l} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : b \equiv c \equiv 0 \, (\mathrm{mod} N), \\ a \equiv d \equiv 1 \, (\mathrm{mod} N) \end{array} \right\}.$$

We note that $SL(2,\mathbb{Z}) = \Gamma_0(1) = \Gamma_1(1) = \Gamma(1)$, and $\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset SL(2,\mathbb{Z})$. Further if $M \mid N$, then $\Gamma_0(N) \subset \Gamma_0(M), \Gamma_1(M)$, and $\Gamma(N) \subset \Gamma(M)$. These subgroups are modular groups since $|\Gamma(1) : \Gamma(N)| < \infty$. We call $\Gamma(N)$ a principal congruence modular group, and also $\Gamma_0(N), \Gamma_1(N)$ modular groups of Hecke type. We call N the level of $\Gamma_0(N)$, $\Gamma_1(N)$ and $\Gamma(N)$. A modular group containing a principal congruence modular group is called a congruence modular group. For an element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z})$, we define an element $\lambda_N(\gamma) = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}$, where $\bar{a} \equiv a \, (\mathrm{mod}\, N), \bar{b} \equiv b \, (\mathrm{mod}\, N), \bar{c} \equiv c \, (\mathrm{mod}\, N), \bar{d} \equiv d \, (\mathrm{mod}\, N)$. Then λ_N induces a homomorphism of $SL(2,\mathbb{Z})$ into $SL(2,\mathbb{Z}/N)$. It is easily seen that λ_N is surjective and $Ker(\lambda_N) = \Gamma(N)$, in particular $\Gamma(N)$ is a normal subgroup of $\Gamma(1)$.

Corollary 1. The mapping $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to d \, (\mathrm{mod} N)$ induces an isomorphism $\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^*$.

Now let $N = \prod_p p^e$ be the expression as a product of prime numbers. Then $\mathbb{Z}/N\mathbb{Z}$ is isomorphic to $\prod_p (\mathbb{Z}/p^e\mathbb{Z})$ by the correspondence $a \to \prod_p (a \bmod p^e)$, so that $M_2(\mathbb{Z}/N\mathbb{Z}) \cong \prod_p M_2(\mathbb{Z}/p^e\mathbb{Z})$ through the correspondence: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \prod_p \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \bmod p^e\right)$.

It is clear that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \bmod p^e \in SL(2,\mathbb{Z}/p^e\mathbb{Z})$ is obtained. Conversely, suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \bmod p^e \in SL(2,\mathbb{Z}/p^e\mathbb{Z})$ for all prime factors p of N. Then $ad - bc \equiv 1 (\bmod p^e)$, so that $ad - bc \equiv 1 (\bmod N)$. Therefore $SL(2,\mathbb{Z}/N\mathbb{Z}) \cong \prod_p SL(2,\mathbb{Z}/p^e\mathbb{Z})$.

Since the following lemma is well known, we only give the statement;

Lemma 2. For a positive integer N, we have

i. $|GL(2,/N\mathbb{Z})| = \phi(N)\,|SL(2,\mathbb{Z}/N\mathbb{Z})|$

ii. $|SL(2,\mathbb{Z}/N\mathbb{Z})| = N^3 \prod_{p|N}\left(1 - \frac{1}{p^2}\right)$. Here $\phi(N)$ is the Euler function.

2. Main Calculation and Results

The group $\Gamma_0^{\pm}(N) := \Gamma_0(N) \cup \Gamma_0(N)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ will likewise be called the congruence group. That is

$$\Gamma_0^{\pm}(N) = \left\{\begin{pmatrix} a & b \\ cN & d \end{pmatrix} \cup \begin{pmatrix} -a & b \\ -cN & d \end{pmatrix} : ad - bcN = \pm 1\right\}.$$

For all element of the set of p^1 can be represented as a reduced fraction $\frac{x}{y}$ with $x, y \in \mathbb{Z}$ and $(x,y) = 1$. Since $\frac{x}{y} = \frac{-x}{-y}$, this representation is not unique. We represent ∞ as $\frac{1}{0} = \frac{-1}{0}$. The action of the matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0^{\pm}(N)$ on $\frac{x}{y}$ is $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \frac{x}{y} \to \frac{\alpha x + \beta y}{\gamma x + \delta y}$. The action of a matrix on $\frac{x}{y}$ and on $\frac{-x}{-y}$ is identical.

We now explain imprimitivity of the action on $\Gamma_0^{\pm}(N)$ on p^1. $(\Gamma_0^{\pm}(N), P^1)$ is transitive permutation group, comprising of a group $\Gamma_0^{\pm}(N)$ acting on a set P^1 transitively. $\vartheta_1, \vartheta_2 \in P^1$ satisfy $\vartheta_1 \approx \vartheta_2$ then $\gamma(\vartheta_1) \approx \gamma(\vartheta_2)$ for all $\gamma \in \Gamma_0^{\pm}(N)$. In this case equivalence relation $\approx$ on P^1 is invariant and equivalence classes form blocks. We say $(\Gamma_0^{\pm}(N), P^1)$ imprimitive, if P^1 admits some invariant equivalence relation different from the identity relation and the universal relation. Otherwise $(\Gamma_0^{\pm}(N), P^1)$ is primitive. These two relations are supposed to be trivial relations. Also $\approx$ relation of equivalence classes are called orbits of action.

Lemma 3. Let (G, Ω) be a transitive permutation group. (G, Ω) is primitive if and only if G_σ is a maximal subgroup of G for each $\sigma \in \Omega$.

Proof. It is clear that from book of Biggs and White 1979.

Consequently we understand that if $G_\sigma < H < G$ then Ω is imprimitive. So we use the transitivity, for all element of Ω has the form $g(\sigma)$ for some $g \in G$. Therefore one of the non trivial G invariant equivalence relation on Ω is given as follows:

$g_1(\sigma) \approx g_2(\sigma)$ if and only if $g_{-1}^{1} g_2 \in H$.

The number of the blocks is the index $\Psi = |G{:}H|$. We can apply these ideas to the case where G is the $\Gamma_0^{\pm}(N)$ and Ω is P^1. We have the following lemmas:

Lemma 4. $\Gamma_0^{\pm}(N)$ acts transitively on P^1.

Proof. We can show that the orbit containing ∞ is P^1. If $\frac{a}{b} \in P^1$ then as $(a,b) = 1$ there exist $x, y \in \mathbb{Z}$ with $ay - bx = 1$. We can state the element $\begin{pmatrix} a & x \\ b & y \end{pmatrix}$ of $\Gamma_0^{\pm}(N)$ sends ∞ to $\frac{a}{b}$.

Lemma 5. The stabilizer of ∞ in P^1 is the set of $\left\{\begin{pmatrix} \pm 1 & \lambda_1 \\ 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} \pm 1 & \lambda_2 \\ 0 & \mp 1 \end{pmatrix} : \lambda_1, \lambda_2 \in \mathbb{Z}\right\}$ denoted by $\Gamma_\infty^{\pm}(N)$.

Proof. Because of the action is transitive, stabilizer of any two points conjugate. Therefore we can only look at the stabilizer of ∞ in $\Gamma_0^{\pm}(N)$. Let $T_1 := \begin{pmatrix} a & b \\ cN & d \end{pmatrix}, ad - bcN = 1$. Thus

$T_1(\infty) = \begin{pmatrix} a & b \\ cN & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

then $a = 1, c = 0, d = 1$ and $b = \lambda_1 \in \mathbb{Z}$.

Therefore $\begin{pmatrix} a & b \\ cN & d \end{pmatrix} = \begin{pmatrix} 1 & \lambda_1 \\ 0 & 1 \end{pmatrix}$ is obtained. Again let

$T_2 := \begin{pmatrix} -a & b \\ -cN & d \end{pmatrix}, ad - bcN = -1$. So

$T_2(\infty) = \begin{pmatrix} -a & b \\ -cN & d \end{pmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

then $a = 1, c = 0, d = -1$ and $b = \lambda_2 \in \mathbb{Z}$.

$\begin{pmatrix} -a & b \\ -cN & d \end{pmatrix} = \begin{pmatrix} 1 & \lambda_2 \\ 0 & -1 \end{pmatrix}$ is achieved.

Similarly we can prove other cases. That is,

$\Gamma_\infty^{\pm}(N) = \left\{\begin{pmatrix} \pm 1 & \lambda_1 \\ 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} \pm 1 & \lambda_2 \\ 0 & \mp 1 \end{pmatrix} : \lambda_1, \lambda_2 \in \mathbb{Z}\right\}$. Moreover it is easily seen that $\Gamma_\infty^{\pm}(N) < \Gamma_0(N) < \Gamma_0^{\pm}(N)$ is satisfied.

Let $\approx$ denote the $\Gamma_0^{\pm}(N)$ invariant equivalence relation on P^1 by $\Gamma_0(N)$, let $v = \frac{r}{s}$ and $w = \frac{x}{y}$ be elements of P^1. Then there are the elements $g_1 := \begin{pmatrix} r & \varsigma_1 \\ s & \varsigma_2 \end{pmatrix}$ and $g_2 := \begin{pmatrix} x & \varrho_1 \\ y & \varrho_2 \end{pmatrix}$ in $\Gamma_0^{\pm}(N)$ such that $v = g_1(\infty)$ and $w = g_2(\infty)$. So we have

$g_1(\infty) \approx g_2(\infty) \Longleftrightarrow g_1^{-1} g_2 \in \Gamma_0(N)$

and so from the above we can calculate that
$g_1^{-1}g_2 = \begin{pmatrix} * & * \\ ry-sx & * \end{pmatrix} \in \Gamma_0(N)$. Hence $ry - sx \equiv 0 \pmod N$ is obtained. And also the number of block is $|\Gamma_0^{\pm}(N):\Gamma_0(N)| = 2$. These blocks are

$$[\infty] := \left[\frac{1}{0}\right] = \left\{\frac{x}{y} \in P^1 : (x,y) = 1 \, and \, y \equiv 0 \pmod N\right\}$$

$$[0] := \left[\frac{0}{1}\right] = \left\{\frac{x}{y} \in P^1 : (x,y) = 1 \, and \, x \equiv 0 \pmod N\right\}$$

Definition 2. Let V be a nonempty set, the elements of which are called vertices. A directed graph $\sum$ is a pair (V,E) where E is a subset of $V \times V$. The elements of E are called edges. The directed graph $\sum$ is said to be finite if the vertex set V is finite. If $(\alpha,\beta) \in E$, this is indicated as $\alpha \to \beta$.

Definition 3. Let a sequence $v_1, v_2, ..., v_k$ of different vertices. Then the form

$$v_1 \to v_2 \to \cdots \to v_k \to v_1,$$

where $k \in \mathbb{N}$ and $k \geq 3$, is called a directed circuit in $\sum$. If $k = 2$, then we will say the configuration $v_1 \to v_2 \to v_1$ a self paired edge. If $k = 3$ or $k = 4$, then the circuit, directed or not, is called a triangle or quadrilateral. In a graph is a finite or infinite sequence of edges which connect a sequence of vertices which are all distinct from one another are called a path.

Let (G,V) be transitive permutation group. Then G acts on $V \times V$ by

$$\theta : G \times (V \times V) \to V \times V, \theta(g,(\alpha,\beta)) = (g(\alpha), g(\beta))$$

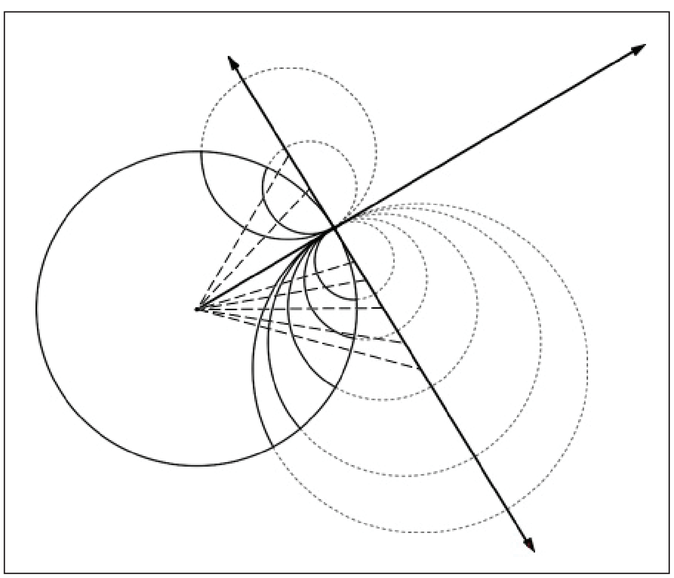

Figure 2. Poincare lines and points.

where $g \in G$ and $\alpha, \beta \in V$. The orbits of this action are called suborbitals of G. The orbit containing (α,β) is denoted by $0(\alpha,\beta)$. From $0(\alpha,\beta)$ we can form a suborbital graph $\sum$. Its vertices are the elements of V, and if $(\gamma,\delta) \in 0(\alpha,\beta)$ there is a directed edge from γ to δ. As $\Gamma_0^{\pm}(N)$ acts transitively on P^1, it permutes the blocks transitively. Also there is a disjoint union of isomorphic copies of suborbital graphs. We say that edges of these graphs can be drawn as hyperbolic geodesic in the upper half-plane $\mathbb{H}$ and Poincare disk model $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Here we will draw these graphs on the Poincare disk model that points and lines are in Figure 2. Note that points on the circle are not in the hyperbolic plane. However they play an important role to determine our model. Euclidean points on the circle are called ideal points, omega points, vanishing points, or points at infinity. We recall that the area inside the unit circle must represent the infinite hyperbolic plane. This means that our standard distance formula will not work. We introduce a distance metric by

$$d\rho = \frac{2dr}{1-r^2}$$

where ρ represents the hyperbolic distance and r is the Euclidean distance from the center of the circle. Note that $d\rho \to \infty$ as $r \to 1$. This means that lines are going to have infinite extend. The relationship between the Euclidean distance of a point from the center of the circle and the hyperbolic distance is $\rho = \int_0^r \frac{2du}{1-u^2} = 2\arctan h\, r$. The hyperbolic distance from any point in the interior of $\mathbb{D}$ to the circle itself is infinite.

Let $F_{u,N} := F\left(\frac{1}{0}, \frac{u}{N}\right)$ and $Z_{u,N} := Z\left(\frac{0}{1}, \frac{u}{N}\right)$ denote the subgraphs in $\sum$ whose vertices are in the blocks $[\infty]$ and $[0]$ respectively. Similarly, we may write subgraphs for other blocks.

Theorem 3. Let $\frac{\alpha_1}{\gamma_1}$ and $\frac{\alpha_2}{\gamma_2}$ be in the block $[\infty]$. Then there is an edge $\frac{\alpha_1}{\gamma_1} \to \frac{\alpha_2}{\gamma_2}$ in $F_{u,N}$ if and only if $\alpha_2 \equiv \pm u\alpha_1 \pmod N$, $\gamma_2 \equiv \pm u\gamma_1 \pmod N$ and $\alpha_1\gamma_2 - \gamma_1\alpha_2 = \pm N$.

Proof. Let $\frac{\alpha_1}{\gamma_1} \to \frac{\alpha_2}{\gamma_2} \in F_{u,N}$, then there exists some $T := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0^{\pm}(N)$ such that $T\left(\frac{1}{0}\right) = \frac{\alpha}{\gamma} = \frac{\alpha_1}{\gamma_1}$ and $T\left(\frac{u}{N}\right) = \frac{\alpha u + \beta N}{\gamma u + \delta N} = \frac{\alpha_2}{\gamma_2}$. Hence $\alpha = \alpha_1, \gamma = \gamma_1$. Then these equations $\alpha_2 \equiv u\alpha_1 \pmod N$ and $\gamma_2 \equiv u\gamma_1 \pmod N$ are satisfied. So we have the matrix equation

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & N \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \gamma_1 & \gamma_2 \end{pmatrix}.$$

If we take determinant, it is easily seen that $\alpha_1\gamma_2-\gamma_1\alpha_2=N$. Again let $S:=\begin{pmatrix}-\alpha & \beta\\ -\gamma & \delta\end{pmatrix}\in\Gamma_0^{\pm}(N)$. Then $S\left(\frac{1}{0}\right)=\frac{-\alpha}{-\gamma}=\frac{\alpha_1}{\gamma_1}$ and $S\left(\frac{u}{N}\right)=\frac{-\alpha u+\beta N}{-\gamma u+\delta N}=\frac{\alpha_2}{\gamma_2}$. Hence $\alpha=-\alpha_1,\gamma=-\gamma_1$. So $\alpha_2\equiv-u\alpha_1(\bmod N)$ and are obtained. Also

$$\begin{pmatrix}-\alpha & \beta\\ -\gamma & \delta\end{pmatrix}\begin{pmatrix}1 & u\\ 0 & N\end{pmatrix}=\begin{pmatrix}\alpha_1 & \alpha_2\\ \gamma_1 & \gamma_2\end{pmatrix}$$

and then $\alpha_1\gamma_2-\gamma_1\alpha_2=-N$.

Conversely, we suppose that $\alpha_2\equiv u\alpha_1(\bmod N),$, $\gamma_2\equiv u\gamma_1(\bmod N)$ and $\alpha_1\gamma_2-\gamma_1\alpha_2=N$. Then there exist integers θ_1 and θ_2 such that $\alpha_2=u\alpha_1+\theta_1 N$ and $\gamma_2=u\gamma_1+\theta_2 N$. In this case

$$\begin{pmatrix}\alpha_1 & \theta_1\\ \gamma_1 & \theta_2\end{pmatrix}\begin{pmatrix}1 & u\\ 0 & N\end{pmatrix}=\begin{pmatrix}\alpha_1 & u\alpha_1+\theta_1 N\\ \gamma_1 & u\gamma_1+\theta_2 N\end{pmatrix}=\begin{pmatrix}\alpha_1 & \alpha_2\\ \gamma_1 & \gamma_2\end{pmatrix}$$

is held. Since $\alpha_1\gamma_2-\gamma_1\alpha_2=N$ from determinants we get $\alpha_1\theta_2-\gamma_1\theta_1=1$. Consequently, $\begin{pmatrix}\alpha_1 & \theta_1\\ \gamma_1 & \theta_2\end{pmatrix}\in\Gamma_0^{\pm}(N)$ and $\frac{\alpha_1}{\gamma_1}\rightarrow\frac{\alpha_2}{\gamma_2}\in F_{u,N}$. Similarly we may show other cases.

Theorem 4. The graph $F_{u,N}$ contains directed triangles if and only if $u^2\pm u+1\equiv 0(\bmod N)$.

Proof. Firstly suppose that $F_{u,N}$ has a triangle $\frac{k_0}{l_0}\rightarrow\frac{m_0}{n_0}\rightarrow\frac{x_0}{y_0}\rightarrow\frac{k_0}{l_0}$. It can be easily shown that $\Gamma_0(N)$ permutes the vertices and edges of $F_{u,N}$ transitively. So we assume that the above triangle is transformed under $\Gamma_0(N)$ to the $\frac{1}{0}\rightarrow\frac{u}{N}\rightarrow\frac{x_0}{y_0}\rightarrow\frac{1}{0}$. Without loss of generality, from the edge of $\frac{u}{N}\overset{<}{\rightarrow}\frac{x_0}{y_0 N}$ the equation of $x_0\equiv-u^2(\bmod N)$ and from the $uy_0N-Nx_0=-N$ equation, $x_0=uy_0+1$ is achieved. For $y_0=1$ case, $\frac{u}{N}\rightarrow\frac{x_0}{N}$ and $x_0=u+1$ and eventually $\frac{u}{N}\rightarrow\frac{u+1}{N}$ is found. And also $u+1\equiv-u^2(\bmod N)$ then $u^2+u+1\equiv 0(\bmod N)$. Again $y_0=2$ can not be true because for $\frac{x_0}{2N}\rightarrow\frac{1}{0}$ there is not an edge condition. Similarly if we take $\frac{u}{N}\overset{>}{\rightarrow}\frac{x_0}{y_0 N}$ holds then we conclude that $u^2-u+1\equiv 0(\bmod N)$ is satisfied. Consequently we have $u^2\pm u+1\equiv 0(\bmod N)$. On the other hand suppose that $u^2\pm u+1\equiv 0(\bmod N)$. Then, using Theorem 3, we see that $\frac{1}{0}\rightarrow\frac{u}{N}\rightarrow\frac{u\pm 1}{N}\rightarrow\frac{1}{0}$ is a triangle in $F_{u,N}$.

Now we will give examples for to understand the theorem. Hence, there are the hyperbolic triangles the following shape.

Example 1. For $(u,N)=(2,7)$ and $(u,N)=(-2,7)$ hyperbolic directed triangles in $F_{2,7}$ and $F_{-2,7}$ on the Poincare disk model are given in Figure 3.

Triangle circuits: $\frac{1}{0}\leftrightarrow\frac{2}{7}\leftrightarrow\frac{1}{7}\leftrightarrow\frac{1}{0},\frac{1}{0}\leftrightarrow\frac{2}{7}\leftrightarrow\frac{3}{7}\leftrightarrow\frac{1}{0}$ and $\frac{1}{0}\leftrightarrow\frac{-2}{7}\leftrightarrow\frac{-1}{7}\leftrightarrow\frac{1}{0},\frac{1}{0}\leftrightarrow\frac{-2}{7}\leftrightarrow\frac{-3}{7}\leftrightarrow\frac{1}{0}$

Corollary 2. Actually $F_{u,N}$ contains hyperbolic triangle if and only if the group $\Gamma_0(N)$ contains elliptic element $\varphi_1=\begin{pmatrix}-u & \frac{u^2+u+1}{N}\\ -N & u+1\end{pmatrix}$ of order 3 in $\Gamma_0(N)$. It is obvious that $\varphi_1(\infty)=\frac{u}{N},\varphi_1\left(\frac{u}{N}\right)=\frac{u+1}{N}$ and $\varphi_1\left(\frac{u+1}{N}\right)=\infty$.

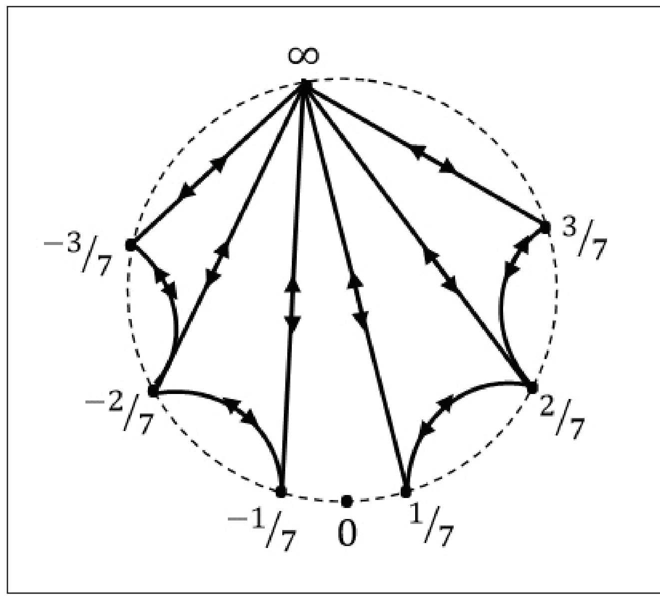

Figure 3. Hyperbolic directed triangles in $F_{2,7}$ and $F_{-2,7}$.

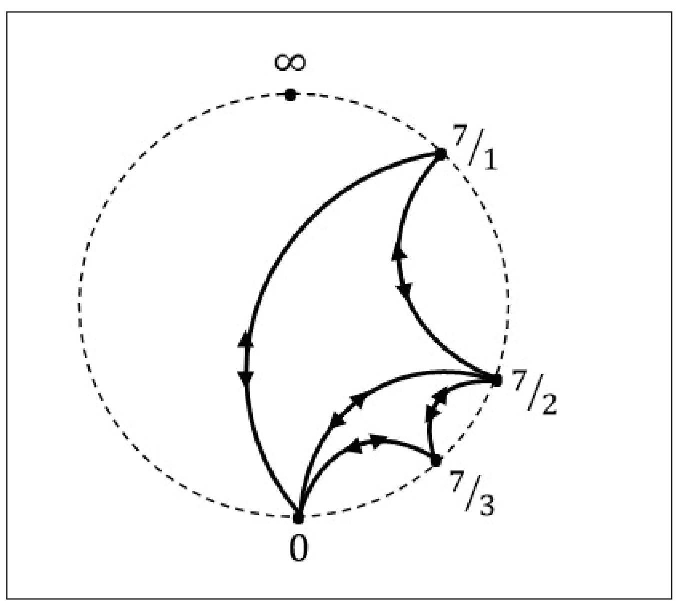

Figure 4. Hyperbolic directed triangles in $Z_{2,7}$.

Therefore by the mapping the φ_1 transform vertices to each other.

Example 2. Similarly hyperbolic triangles in subgraph $Z_{2,7}$ whose vertices form the block [0] is given in Figure 4.

Triangle circuits: $\frac{0}{1} \leftrightarrow \frac{7}{2} \leftrightarrow \frac{7}{1} \leftrightarrow \frac{0}{1}, \frac{0}{1} \leftrightarrow \frac{7}{2} \leftrightarrow \frac{7}{3} \leftrightarrow \frac{0}{1}$.

Corollary 3. Again we may easily seen that $Z_{u,N}$ contains hyperbolic triangle if and only if the group $\Gamma_0(N)$ contains elliptic element $\varphi_2 = \begin{pmatrix} u+1 & -N \\ \frac{u^2+u+1}{N} & -u \end{pmatrix}$ of order 3 in $\Gamma_0(N)$. That is $\varphi_2^3 = -I$. It is obvious that $\varphi_2(0) = \frac{N}{u}, \varphi_2\left(\frac{N}{u}\right) = \frac{N}{u+1}$ and $\varphi_2\left(\frac{N}{u+1}\right) = 0$. Hence by the mapping the φ_2 transform vertices to each other.

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