On Some Difference Equations of Exponential Form

Üstel Biçimdeki Bazı Fark Denklemleri Üzerine

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Abstract

In this paper, the local asymptotic behavior of positive solutions of some exponential difference equations

$$x_{n+1} = \frac{x_n + x_{n-k}}{1 + x_{n-k}e^{x_{n-k}}}, k \in \mathbb{N}, \ n = 0,1,2,...$$

are investigated where the initial conditions are arbitrary positive real numbers. Furthermore, some numerical examples are presented to verify our results.

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Öz

Bu makalede, başlangıç koşulları keyfi pozitif reel sayılar olmak üzere bazı

$$x_{n+1} = \frac{x_n + x_{n-k}}{1 + x_{n-k}e^{x_{n-k}}}, k \in \mathbb{N}, \ n = 0,1,2,...$$

üstel rasyonel fark denklemlerinin pozitif çözümlerinin lokal asimptotik davranışı araştırılmıştır. Ayrıca, sonuçlarımızı doğrulamak için bazı nümerik örnekler verilmiştir.

Anahtar Kelimeler: Fark denklem, Lokal kararlılık, Pozitif çözümler, Denge noktası

1. Introduction

In recent years, there has been a great interest in studying nonlinear difference equations and systems. Also, it is very interesting to investigate the behavior of solutions of nonlinear difference equations and to discuss the local asymptotic stability of their equilibrium points, see (5,6,7,10,12). We can see many papers and books concerning theory and applications of difference equations, see (2,3,8,9). Difference equations especially exponential type of difference equations have many applications in biology, biomathematics, bioengineering, population dynamics, genetics, etc. Therefore, the behavior of positive solutions of the difference equations of exponential form has been a great role in the theory of difference equations.

Firstly, El-Metwally et al. (4) studied the boundedness character, asymptotic behavior, periodic character of the positive solutions and stability of the positive equilibrium of the following population model:

$$x_{n+1} = a + bx_{n-1}e^{-x_n}, \ n = 0,1,2,... \tag{1.1}$$

where *a*, *b* are positive constants and the initial values x_{-1}, x_0 are arbitrary non-negative numbers. Furthermore, authors considered *a* is the immigration rate and *b* the population growth rate. This equation may be viewed as a model in Mathematical Biology which was suggested by the people from the Harward School of Public Health; studying the population dynamics of one species x_n.

In (11), Ozturk et al. investigated the global attractivity, the boundedness and the periodic nature of the exponential rational difference equation

$$y_{n+1} = \frac{\alpha + \beta e^{-y_n}}{\gamma + y_{n-1}}, \ n = 0,1,2,... \tag{1.2}$$

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where the parameters α, β and γ are positive real numbers and the initial conditions y_{-1}, y_0 are arbitrary non-negative numbers.

Bozkurt (1) obtained results concerning the local and global behavior of positive solutions of the following difference equation:

$$y_{n+1} = \frac{\alpha e^{-y_n} + \beta e^{-y_{n-1}}}{\gamma + \alpha y_n + \beta y_{n-1}}, n = 0, 1, 2, \ldots \tag{1.3}$$

where the parameters α, β, γ and the initial conditions are arbitrary positive numbers.

Motivated by the above studies, our aim in this paper is to investigate the local asymptotic behavior of positive solutions of the following exponential rational difference equation:

$$x_{n+1} = \frac{x_n + x_{n-k}}{1 + x_{n-k} e^{x_{n-k}}}, k \in \mathbb{N}, n = 0, 1, 2, \ldots \tag{1.4}$$

2. Preliminaries and Theorems

Definition 2.1 *Let I be an interval of real numbers* $\mathbb{R}$ *and let* $f : I \times I \to I$ *be a continuous function. Consider the difference equation*

$$x_{n+1} = f(x_n, x_{n-1}), \ n = 0, 1, 2, \ldots \tag{2.1}$$

where the initial conditions $x_{-1}, x_0 \in I$. *We say that* $\bar{x}$ *is an equilibrium point of Eq.(2.1) if* $\bar{x} = f(\bar{x}, \bar{x})$.

Definition 2.2

(i) The equilibrium $\bar{x}$ *of Eq.(2.1) is called locally stable if for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $x_{-1}, x_0 \in I$ *with* $|x_0 - \bar{x}| + |x_{-1} - \bar{x}| < \delta$, *then* $|x_n - \bar{x}| < \varepsilon$ *for all* $n \geq -1$.

(ii) The equilibrium $\bar{x}$ *of Eq.(2.1) is called locally asymptotically stable if it is locally stable, and if there exists* $\gamma > 0$ *such that* $x_{-1}, x_0 \in I$ *with* $|x_0 - \bar{x}| + |x_{-1} - \bar{x}| < \gamma$, *then* $\lim_{n \to \infty} x_n = \bar{x}$.

(iii) The equilibrium $\bar{x}$ *of Eq.(2.1) is called a global attractor if for every* $x_{-1}, x_0 \in I$ *we have* $\lim_{n \to \infty} x_n = \bar{x}$.

(iv) The equilibrium $\bar{x}$ *of Eq.(2.1) is called globally asymptotically stable if it is locally stable and a global attractor.*

(v) The equilibrium $\bar{x}$ *of Eq.(2.1) is called unstable if it is not stable.*

Definition 2.3 *The linearized equation of (2.1) about the equilibrium point* $\bar{x}$ *is*

$$y_{n+1} = py_n + qy_{n-1} \tag{2.2}$$

where

$p = \frac{\partial f}{\partial x_n}(\bar{x}, \bar{x})$ *and* $q = \frac{\partial f}{\partial x_{n-1}}(\bar{x}, \bar{x})$. *Then the characteristic equation of (2.2) is*

$$\lambda^2 - p\lambda - q = 0. \tag{2.3}$$

The following result, known as the Linearized Stability Theorem, is very useful in determining the local stability character of the equilibrium point $\bar{x}$ of Eq.(2.1).

Theorem 2.1 (Linearized Stability).

Suppose that the function F is a continuously differentiable function defined on some open neighborhood of an equilibrium point $\bar{x}$. *Then, the following statements are true:*

(i) If all roots of (2.3) have absolute value less than one, then the equilibrium point of (2.1) is locally asymptotically stable.

(ii) If at least one of the roots of (2.3) has absolute value greater than one, then the equilibrium point $\bar{x}$ *of (2.1) is unstable.*

3. Local Stability of Eq.(1.4)

Firstly, we show that Eq.(1.4) has a unique positive equilibrium point $\bar{x}$. The equilibrium points of Eq.(1.4) are the solutions of the equation

$$\bar{x} = \frac{\bar{x} + \bar{x}}{1 + \bar{x} e^{\bar{x}}}.$$

Set

$$f(x) = \frac{x + x}{1 + x e^x} - x$$

then we have $f\left(\frac{1}{2}\right) > 0$ and $\lim_{x \to \infty} f(x) = -\infty$.

Moreover, it follows from the

$$f'(x) = \frac{2 - 2x^2 e^x}{(1 + x e^x)^2} - 1 < 0$$

that Eq.(1.4) has a unique positive equilibrium point $\bar{x}$.

Theorem 3.1 *The positive equilibrium point* $\bar{x}$ *of Eq.(1.4) is locally asymptotically stable.*

Proof. *The positive equilibrium point* $\bar{x}$ *of Eq.(1.4) is the solution of the equation*

$$\bar{x} = \frac{\bar{x} + \bar{x}}{1 + \bar{x} e^{\bar{x}}} \Leftrightarrow 1 + \bar{x} e^{\bar{x}} = 2 \Leftrightarrow \bar{x} e^{\bar{x}} = 1$$

and it is evaluated numerically as 0.56714. The linearized equation associated with Eq.(1.4) about the

equilibrium point $\bar{x}$ *is*

$$x_{n+1} - \frac{1}{1 + \bar{x} e^{\bar{x}}} x_n - \frac{(1 - \bar{x} e^{\bar{x}} - 2\bar{x}^2 e^{\bar{x}})}{(1 + \bar{x} e^{\bar{x}})^2} x_{n-k} = 0. \tag{2.4}$$

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Case $k = 1$

The characteristic equation of Eq.(2.4) with $k = 1$ about the equilibrium point $\bar{x}$ is

$$\lambda^2 - (0.5)\lambda + 0.28357 = 0. \tag{2.5}$$

The roots of Eq.(2.5) are $\lambda_{1,2} = 0.25 \pm 0.47018\,i$ and the absolute value of each root is less than one. It follows from the Theorem 2.1 that the positive equilibrium point $\bar{x}$ of Eq.(1.4) with $k = 1$, is locally asymptotically stable.

Case $k = 2$

The characteristic equation of Eq.(2.4) with $k = 2$ about the equilibrium point $\bar{x}$ is

$$\lambda^3 - (0.5)\lambda^2 + 0.28357 = 0. \tag{2.6}$$

The roots of Eq.(2.6) are
$\lambda_{1,2} = 0.51289 + 0.52562i, \lambda_3 = 0.52578$ and the absolute value of each root is less than one. It follows from the Theorem 2.1 that the positive equilibrium point $\bar{x}$ of Eq.(1.4) with $k = 2$, is locally asymptotically stable.

Case $k = 3$

The characteristic equation of Eq.(2.4) with $k = 3$ about the equilibrium point $\bar{x}$ is

$$\lambda^4 - (0.5)\lambda^3 + 0.28357 = 0. \tag{2.7}$$

The roots of Eq.(2.7) are $\lambda_{1,2} = 0.66294 \pm 0.48501i$ and $\lambda_{3,4} = -0.41294 \pm 0.49976i.$.

It is easy to see that absolute value of each root is less than one. This implies that the positive equilibrium point $\bar{x}$ of Eq.(1.4) with $k = 3$, is locally asymptotically stable.

Thus, for all $k \in \mathbb{N}$ values, we obtain the characteristic equation of Eq.(2.4) about the

equilibrium point $\bar{x}$ is

$$\lambda^n - (0.5)\lambda^{n-1} + 0.28357 = 0. \tag{2.8}$$

Set $f(\lambda) = \lambda^n$ and $g(\lambda) = -(0.5)\lambda^{n-1} + 0.28357$.

Then, by Rouche's theorem, $f(\lambda)$ and $f(\lambda) + g(\lambda)$ have same number of zeroes in an open unit disc

$|\lambda| < 1$. Hence, all the roots of (2.8) satisfies $|\lambda| < 1$, and it follows from Theorem 2.1 that the unique positive equilibrium point $\bar{x}$ of Eq.(1.4) is locally asymptotically stable.

Hence, one can obtained the desired results and the proof is complete.

4. Numerical Examples

In this section, some numerical examples are presented to show the positive equilibrium point $\bar{x}$ of Eq.(1.4) is locally asymptotically stable.

Example 4.1 Consider the difference equation (1.4) with $k = 1$ and the initial conditions
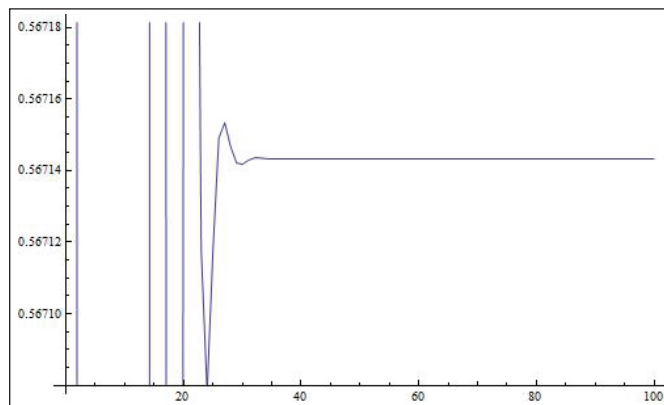
$$x_{-1} = 0.7, x_0 = 10.$$

Figure 1. The positive solutions of Eq.(1.4) with $k = 1$.

Example 4.2 Consider the difference equation (1.4) with $k = 2$ and the initial conditions
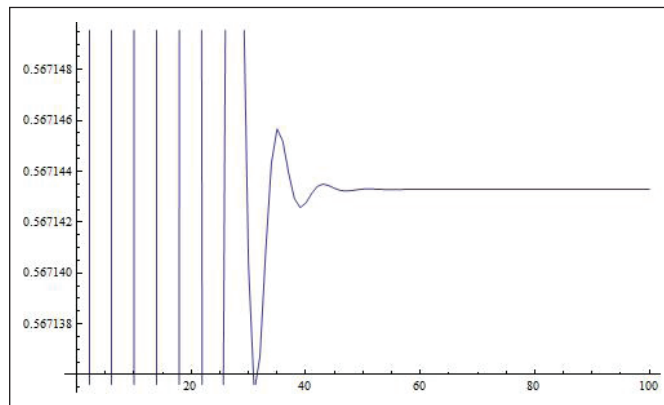
$$x_{-2} = 0.5, x_{-1} = 0.2, x_0 = 0.4.$$

Figure 2. The positive solutions of Eq. (1.4) with $k = 2$.

Example 4.3 Consider the difference equation (1.4) with $k = 3$ and the initial conditions

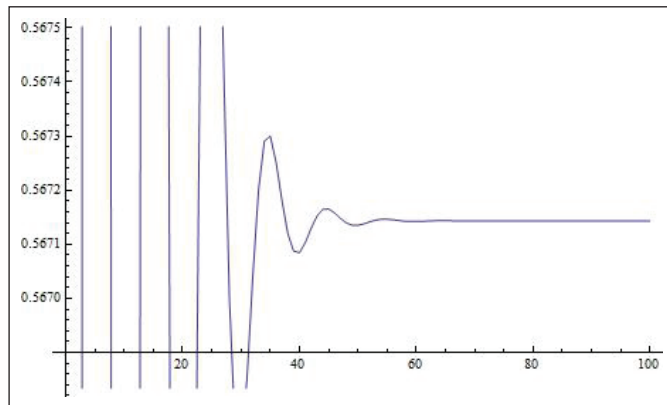$$x_{-3} = 0.1, x_{-2} = 0.5, x_{-1} = 0.2, x_0 = 0.4.$$

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Figure 3. The positive solutions of *Eq.(1.4)* with $k = 3$.

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