Karaelmas Fen ve Müh. Derg. 8(2):626-629, 2018

Karaelmas Science and Engineering Journal
Journal home page: http://fbd.beun.edu.tr
DOI: 10.7212%2Fzkufbd.v8i2.1382

Research Article
Received / Geliş tarihi : 14.08.2018
Accepted / Kabul tarihi : 05.09.2018

A Boundary Value Problem for an Integro-Differential Equation

Bir İntegro-Diferensiyel Denklem İçin Bir Sınır Değer Problemi

İsmet Gölgeleyen, Mustafa Yildiz[1,*]

Zonguldak Bulent Ecevit University, Faculty of Arts and Sciences, Department of Mathematics, Zonguldak, Turkey

Abstract

In this work, we consider a boundary value problem for an integro-differential equation. We prove the existence and uniqueness of the solution of the problem by using a priori estimates and Galerkin method.

Keywords: Boundary value problem, Existence, Integro-differential equation, Uniqueness

Öz

Bu çalışmada, bir integro-diferensiyel denklem için bir sınır değer problemi ele alınmıştır. Problemin çözümünün varlığı ve tekliği ön değerlendirmeler ve Galerkin metodu kullanılarak ispatlanmıştır.

Anahtar Kelimeler: Sınır değer problemi, Varlık, İntegro-diferensiyel denklem, Teklik

1. Introduction

In this work, we consider the integro-differential equation

$$Lu \equiv x\Delta u + ku_x + \int_D K(x,y,\xi,\eta)\,u(\xi,\eta)\,d\xi d\eta = xf(x,y) \qquad (1)$$

in the domain $D = \{(x,y) \mid x > 0, y \in \mathbb{R}^n, G(x,y) < 0\}$ with the boundary condition

$$u(x,y)\,|_{\Gamma} = u_0(x,y), \qquad (2)$$

where $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y_1^2} + \cdots + \frac{\partial^2 u}{\partial y_n^2}, k > \frac{1}{2}$ is a constant, the kernel function $K(x,y,\xi,\eta) \neq 0$ and continuous on the domain $D \times D$, $|K(x,y,\xi,\eta)| < M$ for $M \in \mathbb{R}$, the boundary ∂D is defined by $G(x,y) = 0$ and $\Gamma = \partial D / \{(0,y)\}$.

We deal with the problem of determination of the real valued function $u(x,y), x \in \mathbb{R}, y \in \mathbb{R}^n$, from equation (1) that satisfies condition (2) in the domain *D*. More clearly, we have been given a domain *D* and a solution of equation (1) on a part of boundary of *D*, and we need to investigate the solvability of the problem in *D*.

Equation (1) is an elliptic integro-differential equation and associated with equilibrium or steady-state processes, (Dennis et al. 2015, Mikhailov 1978). Solvability of various direct and inverse problems for different type of equations were studied in Amirov (2001), Gölgeleyen (2010), Lavrentiev et al. (1986), Mikhailov (1978), Reddy (2013) and Yildiz (1995).

We introduce some notations that will be used in the sequel. For a bounded domain $D, C^m(D)$ is the Banach space of the functions that are *m* times continuously differentiable in *D* for all $m \geq 0$; $L_2(D)$ is the space of measurable functions that are square integrable in *D*; $H^k(D)$ is the Sobolev space, (Adams and Fournier 2003, Reddy (2013)).

We first investigate the uniqueness of the solution of Problem (1)-(2).

2. Uniqueness of the Solution of the Problem

Theorem 1. Let us assume that $k > \frac{1}{2}(1+\beta)$, where β is a positive constant such that $\beta > \mu diam(D)$, $\mu = \iint_{D\times D} K^2(x,y,\xi,\eta)\,dxdyd\xi d\eta$ and $diam(D)$ is the diameter of the domain *D*. Then problem (1)-(2) has at most one solution in the space $H^2(D)$.

Proof.

In order to prove uniqueness of the solution of the problem,

*Corresponding author: mustafa.yildiz@beun.edu.tr

İsmet Gölgeleyen orcid.org/0000-0002-3602-7219
Mustafa Yildiz orcid.org/0000-0003-3367-7176

it is sufficient to show that homogeneous problem

$$x\Delta u + ku_x + \int_D K(x,y,\xi,\eta)u(\xi,\eta)d\xi d\eta = 0, \quad (3)$$

$$u(x,y)\mid_\Gamma = 0, \quad (4)$$

has only trivial solution in the space $H^2(D)$. Since $C^2(\overline{D})$ is dense in $H^2(D)$, we will prove the theorem in $C^2(\overline{D})$. Therefore, assuming that $u(x,y) \in C^2(\overline{D})$, we multiply equation (3) by u_x, then by using the identities

$$xu_{xx}u_x = \frac{1}{2}(xu_x^2)_x - \frac{1}{2}u_x^2,$$

$$xu_{y_iy_i}u_x = (xu_{y_i}u_x)_{y_i} - \frac{1}{2}(u_{y_i}^2 x)_x + \frac{1}{2}u_{y_i}^2, i = 1,2,...,n$$

we obtain

$$\frac{1}{2}\sum_{i=1}^n u_{y_i}^2 + \left(k - \frac{1}{2}\right)u_x^2 + \frac{1}{2}(xu_x^2)_x - \frac{1}{2}\sum_{i=1}^n (xu_{y_i}^2)_x + \sum_{i=1}^n (xu_{y_i}u_x)_{y_i} + u_x\int_D K(x,y,\xi,\eta)u(\xi,\eta)d\xi d\eta = 0. \quad (5)$$

For the last term in equation (5), by the well-known inequality $2ab \le a^2 + b^2$, we have

$$\left|u_x\int_D K(x,y,\xi,\eta)u(\xi,\eta)d\xi d\eta\right| \le \frac{1}{2}\left[\beta u_x^2 + \frac{1}{\beta}\left(\int_D K(x,y,\xi,\eta)u(\xi,\eta)d\xi d\eta\right)^2\right]$$

for arbitrary $\beta > 0$.

By the Schwarz inequality, we obtain

$$\left(\int_D K(x,y,\xi,\eta)u(\xi,\eta)d\xi d\eta\right)^2 \le \int_D K^2(x,y,\xi,\eta)d\xi d\eta \int_D u^2(\xi,\eta)d\xi d\eta$$

and so

$$\left|u_x\int_D K(x,y,\xi,\eta)u(\xi,\eta)d\xi d\eta\right| \le \frac{1}{2}\left(\beta u_x^2 + \frac{1}{\beta}\int_D K^2(x,y,\xi,\eta)d\xi d\eta \int_D u^2(\xi,\eta)d\xi d\eta\right). \quad (6)$$

By using condition (4) and relations (5), (6), we have

$$0 = \int_D Lu\, u_x dD \ge \int_D \left(\frac{1}{2}|\nabla_y u|^2 + \left(k - \frac{1}{2}\right)u_x^2\right)dD - \frac{1}{2}\left(\beta\int_D u_x^2 dD + \frac{1}{\beta}\iint_{D\times D} K^2(x,y,\xi,\eta)dxdyd\xi d\eta \int_D u^2(\xi,\eta)d\xi d\eta\right). \quad (7)$$

On the other hand, we can write

$$\begin{aligned} 0 &= \int_D Lu\, u_x dD \ge \int_D \left(\frac{1}{2}|\nabla_y u|^2 + \left(k - \frac{1}{2}\right)u_x^2\right)dD - \frac{1}{2}\left(\beta\int_D u_x^2 dD + \frac{1}{\beta}\mu\int_D u^2(\xi,\eta)d\xi d\eta\right) \\ &= \int_D \left(\frac{1}{2}|\nabla_y u|^2 + \left(k - \frac{1}{2}\right)u_x^2 - \frac{1}{2}\beta u_x^2 - \frac{1}{2\beta}\mu u^2\right)dD \\ &= \int_D \left(\frac{1}{2}|\nabla_y u|^2 + \left(k - \frac{1}{2} - \frac{1}{2}\beta\right)u_x^2 - \frac{1}{2\beta}\mu u^2\right)dD, \end{aligned} \quad (8)$$

where

$$\mu = \iint_{D\times D} K^2(x,y,\xi,\eta)dxdyd\xi d\eta.$$

Here we choose β such that

$$\frac{\mu\, diam(D)}{2\beta} < \frac{1}{2}.$$

Then by using the hypothesis of the theorem and the Poincare-Rellich inequality, from inequality (8),

we see that

$$0 = \int_D Lu\, u_x dD \ge \int_D \left(\frac{1}{2}\sum_{i=2}^n u_{y_i}^2\right)dD. \quad (9)$$

By (9) we have $u_{y_i} = 0, (i = 2,3,...,n)$, in the domain D. Since $u|_\Gamma = 0$, we conclude that $u \equiv 0$ which implies that homogeneous problem (3)-(4) has zero solution in $C^2(\overline{D})$. Therefore the solution of problem (1)-(2) is unique in $C^2(\overline{D})$.

Next, we shall prove the existence of the solution of Problem (1)-(2). If $u_0 \in C^2(\Gamma)$ and $\Gamma \in C^2$, then there exists a function $w \in C^2(\overline{D})$ such that $w|_\Gamma = u_0$, (Mikhailov 1978). Thus, Problem (1)-(2) can be reduced to the following problem for a new unknown function v:

$$Lv = F, \quad (10)$$

$$v|_\Gamma = 0, \quad (11)$$

where

$$F(x,y) = xf(x,y) - x\Delta w - kw_x. - \int_D K(x,y,\xi,\eta)w(\xi,\eta)d\xi d\eta.$$

3. Existence of the Solution of the Problem

Theorem 2. Let $k > \frac{1}{2}\left(1 - \frac{\mu}{2\beta}diam(D)\right)$ and $F \in L_2(D)$, where β, μ and *diam*(*D*) are defined in the statement of Theorem 1. Then there exists a solution v of problem (10)-(11) in $H^1(D)$.

Proof. Let $\{\varphi_j(y)\}, j=0,1,2,...$ be a complete and linearly independent system in $L_2([-1,1]^n)$, then $(x-b)^i\varphi_j(y), i=1,...,N$, is also a complete system in $L_2(D=[a,b]\times[-1,1]^n)$, (Kolmogorov and Fomin 2012). Suppose that $D^1=[-1,1]^n$ and $\varphi_j(y)$ is zero on the boundary of D^1.

We shall investigate the approximate solution of problem (10)-(11) in the form

$$v_N(x,y)=\sum_{i,j=1}^N c_{ij}(x-b)^i\varphi_j(y).$$

The unknown coefficients $c_{ij}, (i,j=\overline{1,N})$ will be determined from the following system of linear algebraic equations which consists of N^2 equations:

$$\langle Lv_N,(x-b)^i\varphi_j(y)\rangle=\langle F,(x-b)^i\varphi_j(y)\rangle, i,j=\overline{1,N}. \quad (12)$$

In order to prove existence and uniqueness of the solution of system (12), it is sufficient to show that homogeneous version of the system has only zero solution.

Let us consider homogeneous form of system (12), that is, $F=0$. We multiply the $(i\text{-}1,j)$ th equation of the system by ic_{ij} and sum from 1 to N with respect to i and j, then we find

$$\langle Lv_N, v_{N_x}\rangle=0.$$

Then similar to inequality (9), we can write

$$0=\int_D Lv_N v_{N_x}dD\geq\frac{1}{2}\int_D\sum_{i=2}^N v_{Ny_i}^2 dD. \quad (13)$$

By (13), we have

$$v_N(x,y)=\sum_{i,j=1}^N c_{ij}(x-b)^i\varphi_j(y)=0.$$

Since the system $(x-b)^i\varphi_j(y)$ is linearly independent, we conclude that $c_{ij}=0$. This shows that system (12) has a unique solution for arbitrary $F\in L_2(D)$.

We now estimate v_N in terms of F. For this purpose, we multiply the $(i\text{-}1,j)$ th equation of the system by ic_{ij} and sum from 1 to N with respect to i and j, then we obtain

$$\langle Lv_N, v_{N_x}\rangle=\langle F, v_{N_x}\rangle.$$

By inequality (8), we see that

$$\begin{aligned}\langle F, v_{N_x}\rangle&=\int_D Lv_N v_{N_x}dD\\ &\geq\int_D\Big[\frac{1}{2}\Big(\sum_{i=2}^N v_{Ny_i}^2\Big)+\Big(k-\frac{1}{2}-\frac{1}{2}\beta\Big)v_{N_x}^2\\ &+\Big(\frac{1}{2}-\frac{\mu\, diam(D)}{2\beta}\Big)v_{Ny_1}^2\Big]dD\end{aligned} \quad (14)$$

From the Cauchy-Bunyakovskii inequality we have

$$|\langle F, v_{N_x}\rangle|\leq\frac{1}{2\varepsilon}\int_D F^2dD+\frac{1}{2}\varepsilon\int_D v_{N_x}^2 dD. \quad (15)$$

By (14) and (15), we get

$$\begin{aligned}&\int_D\Big[\frac{1}{2}\Big(\sum_{i=2}^N v_{Ny_i}^2\Big)+\Big(\frac{1}{2}-\frac{\mu\, diam(D)}{2\beta}\Big)v_{Ny_1}^2\\ &+\Big(k-\frac{1}{2}-\frac{1}{2}\beta-\frac{1}{2}\varepsilon\Big)v_{N_x}^2\Big]dD\\ &\leq\frac{1}{2\varepsilon}\int_D F^2 dD.\end{aligned} \quad (16)$$

Here we choose ε in (15) such that $k-\frac{1}{2}-\frac{1}{2}\beta-\frac{1}{2}\varepsilon>0$, then by (16), we obtain

$$\int_D(|\nabla_y v_N|^2+v_{N_x}^2)dD\leq C\int_D F^2dD.$$

Here $C>0$ is a constant which depends on ε,β and $diam(D)$ but independent of N.

Then $\{v_N\}$ is bounded in $H^1(D)$. Since $H^1(D)$ is a Hilbert space, $\{v_N\}$ has a subsequence that converges weakly in $H^1(D)$. For simplicity, we again denote by $\{v_N\}$, that is, $v_N\rightharpoonup v$ in $H^1(D)$.

We write system (12) as

$$\begin{aligned}&\Big\langle x\Delta v_N+kv_{N_x}+\int_D K(x,y,\xi,\eta)v_N(\xi,\eta)d\xi d\eta,(x-b)^i\varphi_j(y)\Big\rangle\\ &=\langle F,(x-b)^i\varphi_j(y)\rangle,\ i,j=\overline{1,N}.\end{aligned} \quad (17)$$

By using the identities,

$$\begin{aligned}&\langle x\Delta v_N,(x-b)^i\varphi_j(y)\rangle=\int_D x\Delta v_N(x-b)^i\varphi_j(y)dD\\ &=\int_D x\Big(v_{N_{xx}}+\sum_{i=1}^n v_{N_{y_iy_i}}\Big)(x-b)^i\varphi_j(y)dD,\\ &xv_{N_{xx}}(x-b)^i\varphi_j(y)=\big(xv_{N_x}(x-b)^i\varphi_j(y)\big)_x\\ &-v_{N_x}(x-b)^i\varphi_j(y)-ixv_{N_x}(x-b)^{i-1}\varphi_j(y),\\ &\sum_{i=1}^n xv_{N_{y_iy_i}}(x-b)^i\varphi_j(y)=\sum_{i=1}^n\big(xv_{N_{y_i}}(x-b)^i\varphi_j(y)\big)_{y_i}\\ &-\sum_{i=1}^n xv_{N_{y_i}}(x-b)^i\varphi_{jy_i}(y),\end{aligned}$$

and the boundary condition $v_N|_\Gamma=0$, we can easily see that

$$\begin{aligned}&\int_D(Lv_N-F)(x-b)^i\varphi_j(y)dD=\\ &-\int_D\big[v_{N_x}(x-b)^i\varphi_j(y)+ixv_{N_x}(x-b)^{i-1}\varphi_j(y)\\ &+xv_{N_{y_i}}(x-b)^i\varphi_{jy_i}-kv_{N_x}(x-b)^i\varphi_j(y)\\ &+\int_D K(x,y,\xi,\eta)v_N(\xi,\eta)(x-b)^i\varphi_j(y)d\xi d\eta\Big]dD.\end{aligned}$$

Since $v_N\rightharpoonup v$ in $H^1(D)$ for $N\to\infty$, we have

$$\langle v,L^*\psi_j\rangle=\langle F,\psi_j\rangle$$

for $N \to \infty$ in the generalized functions sense where $\psi_j = (x-b)^i \varphi_j(y)$, or

$$\langle Lv - F, \psi_j \rangle = 0 . \tag{18}$$

Since $\{\psi_j\}$ is a complete system in $L_2(D)$, we conclude that

$$Lv - F = 0 ,$$

which implies that v is a solution of (10).

By the fact that $v_N \rightharpoonup v$ in $H^1(D)$ and $v_N|_\Gamma = 0$, we see that $v|_\Gamma = 0$. Thus, existence of the solution of problem (10)-(11) is proven.

4. References

Adams, RA., Fournier, JJ. 2003. Sobolev spaces (Vol. 140). Elsevier.

Amirov, AK. 2001. Integral Geometry and Inverse Problems for Kinetic Equations, VSP, Utrecht The Netherlands, 201 pp.

Dennis, MR., Glendinning, P., Martin, PA., Santosa, F., Tanner, J. 2015. The Princeton Companion to Applied Mathematics, Princeton University Press.

Gölgeleyen, I. 2010. On the Solution of an Inverse Problem for an Integro-differential Transport Equation. *CMES-Comp. Model. Eng. Sci., 64*(1): 71-89.

Kolmogorov, AN., Fomin, SV. 2012. Introductory Real Analysis, Courier Corporation.

Lavrent'ev, MM., Romanov, VG., Shishatskii, SP. 1986. Ill-Posed Problems of Mathematical Physics and Analysis. American Mathematical Society, Providence, RI, 290 pp.

Mikhailov, VP. 1978. Partial Differential Equations, Mir Publishers, Moscow, 396 pp.

Reddy, BD. 2013. Introductory Functional Analysis: With Applications to Boundary Value Problems and Finite Elements(Vol. 27), Springer Science & Business Media.

Yildiz, M. 1995. Some Problems in the Generalized Function Classes for the Elliptic Equation $Lu \equiv x\Delta u + ku_x = xf(x,y)$, Ph. D. Thesis, Ankara University.