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Hyers-Ulam Stability for the Wave Equation

Dalga Denklemi İçin Hyers-Ulam Kararlılık

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Abstract

In this paper we consider one dimensional wave equation of the form

$$\frac{\partial^2 U}{\partial t^2} = c^2 \frac{\partial^2 U}{\partial x^2}.$$

We study the stability of the considered equation in Hyers- Ulam sense. Our technique depends on Laplace transform method.

Keywords: Wave equation, Hyers-Ulam stability, Laplace transform

Öz

Bu makalede

$$\frac{\partial^2 U}{\partial t^2} = c^2 \frac{\partial^2 U}{\partial x^2}.$$

biçiminde bir boyutlu dalga denklemini düşündük. Düşünülen denklemin Hyers-Ulam kararlılığını çalıştık. Kullandığımız teknik Laplace dönüşüm metodudur.

Anahtar Kelimeler: Dalga denklemi, Hyers-Ulam kararlılık, Laplace dönüşümü.

1. Introduction

In 1940, the problem with stability of functional differential equation posed by Ulam, was partially resolved by Hyers Later, the result of Hyers (1941) has been generalized by Rassias (1978). After then, mathematicians have investigated Hyers-Ulam stability for several differential equations. Ger (1998) were the first authors who investigated the Hyers-Ulam stability of the first order linear differential equation $y'(t) = y(t)$. They proved that if a differentiable function $y: I \to \Re$ satisfies $|y'(t) - y(t)| \le \varepsilon$ for all $t \in I$, then there exist a differentiable function $g: I \to \Re$ satisfying $g'(t) = g(t)$ for any $t \in I$ such that $|y(t) - g(t)| \le 3\varepsilon$ for every $t \in I$.

We should mention the earliest results on the topic or some results obtained for the linear partial differential equation of first or second order by and the references therein.

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In 2009, Jung investigated the Hyers-Ulam stability of linear partial differential of first order equations

$$au_x(x,y) + bu_y(x,y) + g(y)\,u(x,y) + h(y) = 0$$

and

$$au_x(x,y) + bu_y(x,y) + g(x)\,u(x,y) + h(x) = 0$$

in the cases of $a \le 0, b > 0$ and $a > 0.b \leqslant 0, (a, b \in \Re)$, respectively.

In 2011, Gordji proved the Hyers-Ulam- Rassias stability of the following nonlinear partial differential equations

$$\gamma_x(x,t) = f(x,t,\gamma(x,t)),$$
$$a\gamma_x(x,t) + b\gamma_t(x,t) = f(x,t,\gamma(x,t)),$$
$$p(x,t)\,\gamma_{xx}(x,t) + q(x,t)\,\gamma_x(x,t) = f(x,t,\gamma(x,t)),$$

and

$$p(x,t)\,\gamma_{xt}(x,t) + q(x,t)\,\gamma_t(x,t) + p_t(x,t)\,\gamma_x(x,t) = f(x,t,\gamma(x,t)),$$

respectively, by using Banach's Contraction Principle.

After that, in 2012, Lungu and Popa discussed the Hyers-Ulam stability of first order partial differential equation of the form

$$p(x,y)\frac{\partial u}{\partial x}+q(x,y)\frac{\partial u}{\partial y}=p(x,y)r(x)u+f(x,y).$$

In 2013, Quarawani proved Hyers-Ulam Rassias Stability for one dimensional heat equation applying Fourier transform.

Motivated by the papers Quarawani and the references therein, in this paper we consider the Hyers-Ulam stability of the wave equation

$$\frac{\partial^2 U}{\partial t^2}=c^2\frac{\partial^2 U}{\partial x^2},-\infty<x_0<x<\infty,t>0 \tag{1.1}$$

where C is positive real constant and $(x,t)\in D, D=(x_0,x]\times(0,\infty)$.

2. Preliminaries

In this study, we use Laplace and inverse Laplace transformation for show that the equation (1.1) has Hyers-Ulam stability.

Definition 2.1. Equation (1.1) is said to be stable in Hyers-Ulam sense if there exists $K > 0$ such that for every function $U{:}D\to\Re$ satisfying

$$\left|\frac{\partial^2 U}{\partial t^2}-c^2\frac{\partial^2 U}{\partial x^2}\right|\leq\varepsilon$$

for all $(x,t)\in D)$ there exists a solution $U_0{:}D\to\Re$ of (1.1) with the property

$$|U(x,t)-U_0(x,t)|\leq K\varepsilon.$$

3. Main Results

Theorem 3.1. Let ε be positive constant. If the function U satisfies the differential inequality

$$|U_{tt}-c^2U_{xx}|\leq\varepsilon \tag{3.1}$$

for all $(x,t)\in D$, then there exists a solution $U_0{:}D\to\Re$ of equation (1.1) such that

$$|U(x,t)-U_0(x,t)|\leq c\varepsilon, c>0, c\in\Re.$$

Proof. Let $Z(x,t)=U_t-cU_x$ for any $(x,t)\in D$. Then it follows that

$$|Z_t+cZ_x|=|U_{tt}-c^2U_{xx}|.$$

Then from inequality (3.1), we get

$$|Z_t+cZ_x|\leq\varepsilon. \tag{3.2}$$

Making use of (3.2), we can write

$$-\varepsilon\leq Z_t+cZ_x\leq\varepsilon.$$

If we apply the Laplace operator of above inequality, we get

$$L(-\varepsilon)\leq L(Z_t)+L(cZ_x)\leq L(\varepsilon)$$

$$-\frac{\varepsilon}{s}\leq\int_0^\infty e^{-st}\frac{\partial Z}{\partial t}dt+c\int_0^\infty e^{-st}\frac{\partial Z}{\partial x}dt\leq\frac{\varepsilon}{s}$$

$$-\frac{\varepsilon}{s}\leq\lim_{p\to\infty}\int_0^p e^{-st}\frac{\partial Z}{\partial t}dt+c\frac{d}{dx}\int_0^\infty e^{-st}Zdt\leq\frac{\varepsilon}{s}$$

$$-\frac{\varepsilon}{s}\leq S\int_0^\infty e^{-st}Z(x,t)dt-Z(x,0)+c\frac{dz}{dx}\leq\frac{\varepsilon}{s}.$$

Then, we have

$$-\frac{\varepsilon}{s}\leq c\frac{dz}{dx}+sz(x,s)-Z(x,0)\leq\frac{\varepsilon}{s},$$

where $z=z(x,s)=L\{Z(x,t)\}$. Since c > 0, by dividing the above inequality by c,

$$-\frac{\varepsilon}{cs}\leq\frac{dz}{dx}+\frac{s}{c}z(x,s)-\frac{1}{c}Z(x,0)\leq\frac{\varepsilon}{cs}.$$

Multiplying the above estimate by the function $e^{\frac{s}{c}(x-x_0)}$, we obtain

$$-\frac{\varepsilon}{cs}e^{\frac{s}{c}(x-x_0)}\leq e^{\frac{s}{c}(x-x_0)}\frac{dz}{dx}+\frac{s}{c}e^{\frac{s}{c}(x-x_0)}z(x,s)-\frac{1}{c}e^{\frac{s}{c}(x-x_0)}Z(x,0)$$

$$\leq\frac{\varepsilon}{cs}e^{\frac{s}{c}(x-x_0)}.$$

or, equivalently

$$-\frac{\varepsilon}{cs}e^{\frac{s}{c}(x-x_0)}\leq\frac{d}{dx}[e^{\frac{s}{c}(x-x_0)}z]-\frac{1}{c}e^{\frac{s}{c}(x-x_0)}Z(x,0)\leq\frac{\varepsilon}{cs}e^{\frac{s}{c}(x-x_0)}.$$

For any x_0, integrating above inequality from x_0 to x, we get

$$-\frac{\varepsilon}{cs}\int_{x_0}^x e^{\frac{s}{c}(u-x_0)}du\leq e^{\frac{s}{c}(x-x_0)}z(x,s)-z(x_0,s)$$

$$-\frac{1}{c}\int_{x_0}^x Z(u,0)e^{\frac{s}{c}(u-x_0)}du\leq\frac{\varepsilon}{cs}\int_{x_0}^x e^{\frac{s}{c}(u-x_0)}du$$

As a consequence, we arrive at

$$-\frac{\varepsilon}{s^2}e^{\frac{s}{c}(x-x_0)}\leq e^{\frac{s}{c}(x-x_0)}z(x,s)-z(x_0,s)$$

$$-\frac{1}{c}\int_{x_0}^x Z(u,0)e^{\frac{s}{c}(u-x_0)}du\leq\frac{\varepsilon}{s^2}e^{\frac{s}{c}(x-x_0)}.$$

Since s > 0 we can write above equation

$$-\frac{\varepsilon}{s}e^{\frac{s}{c}(x-x_0)}\leq e^{\frac{s}{c}(x-x_0)}z(x,s)-z(x_0,s)$$

$$-\frac{1}{c}\int_{x_0}^x Z(u,0)e^{\frac{s}{c}(u-x_0)}du\leq\frac{\varepsilon}{s}e^{\frac{s}{c}(x-x_0)}.$$

By dividing the above inequality by $e^{\frac{s}{c}(x-x_0)}$, we obtain

$$-\frac{\varepsilon}{s}\leq z(x,s)-e^{-\frac{s}{c}(x-x_0)}z(x_0,s)-\frac{1}{c}\int_{x_0}^x Z(u,0)e^{\frac{s}{c}(u-x)}du\leq\frac{\varepsilon}{s}$$

Now, if we apply inverse Laplace transform, we get

$$L^{-1}(-\frac{\varepsilon}{s}) \leq L^{-1}(z(x,s)) - L^{-1}(e^{-\frac{s}{c}(x-X_0)}z(x_0,s))$$
$$-L^{-1}(\frac{1}{c}\int_{x_0}^{x} Z(u,0)e^{\frac{s}{c}(u-x)}du) \leq L^{-1}(\frac{\varepsilon}{s}). \tag{3.3}$$

For $L^{-1}(\frac{1}{c}\int_{x_0}^{x} Z(u,0)e^{\frac{s}{c}(u-x)}du)$, let

$$g(x,s) = \frac{1}{c}\int_{x_0}^{x} Z(u,0)e^{\frac{s}{c}(u-x)}du = L(G(x,t)) \tag{3.4}$$

and from this

$$\frac{dg}{dx}(x,s) = \frac{1}{c}Z(x,0)$$

Multiplying both sides above equality by $\frac{1}{s}$, we get

$$\frac{1}{s}\frac{dg}{dx}(x,s) = \frac{1}{cs}Z(x,0)$$

Applying inverse Laplace transform both sides above equation, we obtain

$$L^{-1}(\frac{1}{s}\frac{dg}{dx}(x,s)) = \frac{1}{c}Z(x,0). \tag{3.5}$$

Now, from properties of derivatives of inverse Laplace transformations, we can write

$$L^{-1}(\frac{dg}{dx}(x,s)) = \frac{\partial}{\partial x}G(x,t)$$

and from property of dividing by s of inverse Laplace transformation, we get

$$L^{-1}(\frac{1}{s}\frac{dg}{dx}(x,s)) = \int_{0}^{t} \frac{\partial}{\partial x}G(x,m)\,dm. \tag{3.6}$$

Then from (3.5) and (3.6), we obtain

$G(x,t) = M$,

where $M > 0$ is real constant. Similarly, for $L^{-1}(e^{-\frac{s}{c}(x-x_0)}z(x_0,s))$, we get

$$L^{-1}(e^{-\frac{s}{c}(x-x_0)}z(x_0,s)) = \begin{matrix} Z(x_0,t-\frac{1}{c}(x-x_0)), t > \frac{1}{c}(x-x_0) \\ Z(x_0,0), t < \frac{1}{c}(x-x_0) \end{matrix}.$$

Then we get from inequality (3.3),

$$|Z(x,t) - Z_0(x,t)| \leq \varepsilon$$

where

$$Z_0(x,t) = \begin{matrix} Z(x_0,t-\frac{1}{c}(x-x_0)), t > \frac{1}{c}(x-x_0) \\ Z(x_0,0), t < \frac{1}{c}(x-x_0) \end{matrix}.$$

Since $Z(x,t) = U_t - cU_x$, we have

$$|U_t(x,t) - cU_x(x,t) - Z_0(x,t)| \leq \varepsilon.$$

Using similar technique to the one above, we get

$$-\frac{\varepsilon}{s}e^{\frac{s}{c}(x-x_0)} \leq u(x,s) - e^{\frac{s}{c}(x-x_0)}u(x_0,s)$$
$$+\frac{1}{c}\int_{x_0}^{x} U(v,0)e^{\frac{s}{c}(x-v)}dv + \frac{1}{c}\int_{x_0}^{x} z_0(v,s)e^{\frac{s}{c}(x-v)}dv$$
$$\leq \frac{\varepsilon}{s}e^{\frac{s}{c}(x-x_0)},$$

where $L(Z_0(x,t)) = z_0(x,s)$. Now, applying inverse Laplace transformation, we get

$$-\varepsilon \leq U(x,t) - L^{-1}(e^{\frac{s}{c}(x-x_0)}u(x_0,s))$$
$$+L^{-1}(\frac{1}{c}\int_{x_0}^{x} U(v,0)e^{\frac{s}{c}(x-v)}dv) + L^{-1}(\frac{1}{c}\int_{x_0}^{x} z_0(v,s)e^{\frac{s}{c}(x-v)}dv) \leq \varepsilon.$$

We know that from (3.6)

$$L^{-1}(\frac{1}{c}\int_{x_0}^{x} U(v,0)e^{\frac{s}{c}(x-v)}dv) = M$$
$$L^{-1}(\frac{1}{c}\int_{x_0}^{x} z_0(v,s)e^{\frac{s}{c}(x-v)}dv) = \int_{x_0}^{x}\frac{Z_0(v,t)}{c}dv$$

and since $t > 0$,

$$L^{-1}(e^{\frac{s}{c}(x-x_0)}u(x_0,s)) = U(x_0,t-\frac{1}{c}(x_0-x))\ for$$
$$t > \frac{1}{c}(x_0-x).$$

As a consequently, we obtain

$$|U(x,t) - U_0(x,t)| \leq \varepsilon$$

where

$$U_0(x,t) = U(x_0,t-\frac{1}{c}(x_0-x)) - \int_{x_0}^{x}\frac{Z_0(v,t)}{c}dv,$$

for $t > \frac{1}{c}(x_0-x)$. This completes the proof.

Example 3.2. Consider the equation

$$\frac{\partial^2 U}{\partial t^2} = 16\frac{\partial^2 U}{\partial x^2}.$$

and let $t < \frac{1}{4}(x-x_0)$. If we set $Z(x,t) = U_t - 4U_x$, then we obtain

$$-\varepsilon \leq Z_t + 4Z_x \leq \varepsilon.$$

Applying Laplace tranform in the above equation, we get

$$-\frac{\varepsilon}{s} \leq z(x,s) - e^{-\frac{s}{4}(x-x_0)}z(x_0,s)$$
$$-\frac{1}{4}\int_{x_0}^{x} Z(m,0)e^{\frac{s}{4}(m-x)}dm \leq \frac{\varepsilon}{s},$$

and applying inverse Laplace transform, we obtain

$$|Z(x,t)-Z_0(x,t)|\leq \varepsilon, \qquad (3.7)$$

where

$$Z_0(x,t)=Z(x_0,0)$$

since $t<\frac{1}{4}(x-x_0)$.

Writing in (3.7), $Z(x,t)=U_t-4U_x$, we get

$$|U_t(x,t)-4U_x(x,t)-Z(x_0,t)|\leq \varepsilon,$$

and using similar technique in the Theorem 3.1. we obtain

$$|U(x,t)-U_0(x,t)|\leq \varepsilon,$$

where

$$U_0(x,t)=U(x_0,t-\frac{1}{4}(x_0-x))-\frac{Z(x_0,0)}{4}(x-x_0).$$

4. Conclusion

We consider type of second order and hyperbolic partial differential equation. We study Hyers-Ulam stability of this equation. We give an example to verify the obtained result. Our results have contributions to the topic and the related literature.

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