Karaelmas Science and Engineering Journal
Journal home page: http://fbd.beun.edu.tr
DOI: 10.7212%2Fzkufbd.v8i1.1038

Research Article
Received / Geliş tarihi : 19.06.2017
Accepted / Kabul tarihi : 02.04.2018

Covarient Derivatives of Structures with Respect to Lifts on Tangent Bundle $T(M)$

Tangent Demet T(M) Üzerinde Liftlere Göre Yapıların Kovaryant Türevleri

Haşim Çayır*

Giresun University, Department of Mathematics, Faculty of Arts and Sciences, Giresun, Turkey

Abstract

The differential geometry of tangent bundles was studied by several authors, for example: D. E. Blair (Blair 1976), V. Oproiu (Oproiu 1973), A. Salimov (Salimov 2013), Yano and Ishihara (1973) and among others. It is well known that differant structures defined on a manifold M can be lifted to the same type of structures on its tangent bundle. Our goal is to study covariant derivatives of almost contact structure and almost paracontact structure with respect to X^V, X^C and X^H on tangent bundle $T(M)$.

Keywords: Almost contact structure, Almost paracontact structure, Covariant derivative, Horizontal lift

Öz

Tanjant demetin diferensiyel geometrisi, D. E. Blair (Blair 1976), V. Oproiu (Oproiu 1973), A. Salimov (Salimov 2013), Yano ve Ishihara (1973) ve diğer birçok bilim adamları tarafından çalışıldı. Bir manifold üzerinde tanımlanan farklı yapılar, onun tanjant demeti üzerindeki aynı tip yapılara taşınabilindiği bilinmektedir. Bizim bu çalışmadaki amacımız tanjant demet üzerinde almost kontakt yapı ve almost parakontakt yapıların X^C, X^V ve X^H ye göre kovaryant türevlerini hesaplamaktır.

Anahtar Kelimeler: Almost kontakt yapı, Almost parakontakt yapı, Covariant türev, Horizontal lift

1. Introduction

The tangent bundles of differentiable manifolds are very important in many areas of mathematics and physics. The geometry of tangent bundles goes back to the fundamental paper (Sasaki 1958) of Sasaki published in 1958. Cotangent bundle is dual of the tangent bundle. Because of this duality, some of the geometric results are similar to each other. The most significant difference between them is construction of lifts (see Yano and Ishihara 1973 for more details).

Let M be an n-dimensional differentiable manifold of class C^∞ and $T_p(M)$ be the tangent space of M at a point p of M. Then the set (Yano and Ishihara 1973)

$$T(M) = \underset{p\in M}{U} T_p(M) \tag{1.1}$$

is called as the tangent bundle over the manifold M. For any point $\overline{P}$ of $T(M)$, the correspondence $\overline{P}$ determines the bundle projection $\pi : T(M) \to M$, thus $\pi(\overline{P}) = P$ where $\pi : T(M) \to M$ defines the bundle projection of $T(M)$ over M. The set $\pi^{-1}(p)$ is called the fibre over $p \in M$ and M the base space. Suppose that the base space M is covered by a system of coordinate neighbour-hoods $\{U; x^h\}$, where (x^h) is a system of local coordinates defined in the neighbour-hood U of M. The open set $\pi^{-1}(U) \subset T(M)$ is naturally differentiably homeomorphic to the direct product $U \times R^n, R^n$ being the n-dimensional vector space over the real field R, in such a way that a point $\overline{P} \in T_p(M)\,(p \in U)$ is represented by an ordered pair (P, X) of the point $p \in U$, and a vector $X \in R^n$, whose components are given by the cartesian coordinates (y^h) of $\overline{P}$ in the tangent space $T_p(M)$ with respect to the natural base $\{\partial_h\}$, where $\partial_h = \frac{\partial}{\partial x^h}$. Denoting by (x^h) the coordinates of $p = \pi(\overline{P})$ in U and establishing the correspondence $(x^h, y^h) \to \overline{p} \in \pi^{-1}(U)$, we can introduce a system of local coordinates (x^h, y^h) in the open set $\pi^{-1}(U) \subset T(M)$. Here we call (x^h, y^h) the coordinates in $\pi^{-1}(U)$ induced from (x^h) or simply, the induced coordinates in $\pi^{-1}(U)$.

We denote by $\Im_s^r(M)$ the set of all tensor fields of class C^∞ and of type (r, s) in M. We now put $\Im(M) = \sum_{r,s=0}^{\infty} \Im_s^r(M)$, which is the set of all tensor fields in M. Similarly, we denote by $\Im_s^r(T(M))$ and $\Im(T(M))$ respectively the corresponding sets of tensor fields in the tangent bundle $T(M)$.

*Corresponding author: hasim.cayir@giresun.edu.tr

Haşim Çayır orcid.org/0000-0003-0348-8665

1.1. Vertical Lifts

If f is a function in M, we write f^v for the function in $T(M)$ obtained by forming the composition of $\pi: T(M) \to M$ and $f: M \to R$, so that

$$f^v = fo\pi . \tag{1.2}$$

Thus, if a point $\overline{P} \in \pi^{-1}(U)$ has induced coordinates then (x^h, y^h), then

$$^v(\overline{P}) = f^v(x,y) = fo\pi(\overline{P}) = f(p) = f(x) . \tag{1.3}$$

Thus, the value of f^v $\overline{P}$ is constant along each fibre $T_p(M)$ and equal to the value $f(p)$. We call f^v the vertical lift of the function f (Yano and Ishihara 1973).

Let $\overline{X} \in \Im_0^1(T(M))$ be such that $f^v = 0$ for all $f \in \Im_0^0(M)$. Then we say that $\overline{X}$ is a vertical vector field. Let $\begin{pmatrix} \overline{X}^h \\ \overline{X}^{\bar{h}} \end{pmatrix}$ be components of $\overline{X}$ with respect to the induced coordinates. Then $\overline{X}$ is vertical if and only if its components in $\pi^{-1}(U)$ satisfy

$$\begin{pmatrix} \overline{X}^h \\ \overline{X}^{\bar{h}} \end{pmatrix} = \begin{pmatrix} 0 \\ X^{\bar{h}} \end{pmatrix} \tag{1.4}$$

Suppose that $X \in \Im_0^1(M)$ so that is a vector field in M. We define a vector field X^v in $T(M)$ by

$$X^v(\imath\omega) = (\omega X)^v \tag{1.5}$$

ω being an arbitrary 1-form in M. We cal X^v the vertical lift of X (Yano and Ishihara 1973).

Let $\overline{\omega} \in \Im_1^0(T(M))$ be such that $\overline{\omega}(X)^v = 0$ for all $X \in \Im_0^1(M)$. Then we say that $\overline{v}$ is a vertical 1-form in T(M). We define the vertical lift w^v of the 1-form ω by

$$\omega^v = (\omega_i)^v (dx^i)^v \tag{1.6}$$

in each open set $\pi^{-1}(U)$, where $(U; x^h)$ is coordinate neighbourhood in M and ω is given by $\omega = \omega_i dx^i$ in U. The vertical lift ω^v of ω with local expression $\omega = \omega_i dx^i$ has components of the form

$$\omega^v : (\omega^i, 0) \tag{1.7}$$

with respect to the induced coordinates in $T(M)$.

Vertical lifts to a unique algebraic isomorphism of the tensor algebra $\Im(M)$ into the tensor algebra $\Im(T(M))$ with respect to constant coefficients by the conditions

$$(P \otimes Q)^V = P^V \otimes Q^V, (P+R)^V = P^V + R^V \tag{1.8}$$

P, Q and R being arbitrary elements of $\Im(M)$. The vertical lifts F^V of an element $F \in \Im_1^1(M)$ with lokal components F_i^h has components of the form (Yano and Ishihara 1973)

$$F^V : \begin{matrix} 0 & 0 \\ F_i^h & 0 \end{matrix} .$$

Vertical lift has the following formulas (Omran et al. 1984, Yano and Ishihara 1973):

$$\begin{aligned} &(fX)^v = f^v X^v, I^v X^v = 0, \eta^v(X^v) = 0 \\ &(f\eta)^v = f^v \eta^v, X^v, Y^v = 0, \varphi^v X^v = 0 \\ &X^v f^v = 0, X^v f^v = 0 \end{aligned} \tag{1.9}$$

hold good, where

$f \in \Im_0^0(M_n), X, Y \in \Im_0^1(M_n), \eta \in \Im_1^0(M_n), \varphi \in \Im_1^1(M_n), I = id_{M_n}$.

1.2. Complete Lifts

If f is a function in M, we write f^c for the function in $T(M)$ defined by

$$f = \imath(df) \tag{1.10}$$

and call f^c the comple lift of the function f. The complete lift f^c of a function f has the local expression

$$f^c = y^i \partial_i f = \partial f \tag{1.11}$$

with respect to the induced coordinates in $T(M)$, where ∂f denotes $y^i \partial_i f$.

Suppose that $X \in \Im_0^1(M)$. We define a vector field X^c in $T(M)$ by

$$X^c f^c = (Xf)^c , \tag{1.12}$$

f being an arbitrary function in M and call X^c the complete lift of X in $T(M)$ (Das Lovejoy 1993, Yano and Ishihara 1973). The complete lift X^c of X with components x^h in M has components

$$X^c = \begin{matrix} X^h \\ \partial X^h \end{matrix} \tag{1.13}$$

with respect to the induced coordinates in $T(M)$.

Suppose that $\omega \in \Im_1^0(M)$, then a 1-form ω^c in $T(M)$ deffined by

$$\omega^c(X^c) = (\omega X)^c \tag{1.14}$$

X being an arbitrary vector field in M. We call ω^c the complete lift of ω. The complete lift ω^c of ω with components ω_i in M has components of the form

$$\omega^c : (\partial\omega_i, \omega_i) \tag{1.15}$$

with respect to the induced coordinates in $T(M)$ (Das Lovejoy 1993).

The complete lifts to a unique algebra isomorphism of the tensor algebra $\Im(M)$ into the tensor algebra $\Im(T(M))$ with respect to constant coefficients, is given by the conditions

$$(P\otimes Q)^C=P^C\otimes Q^V+P^V\otimes Q^C,(P+R)^C=P^C+R^C, \quad (1.16)$$

where P, Q and R being arbitrary elements of $\Im(M)$. The complete lifts F^C of an element $F\in\Im_1^1(M)$ with local components F_i^h has components of the form

$$F^C: \begin{matrix} F_i^h & 0 \\ \partial F_i^h & F_i^h \end{matrix}.$$

In addition, we know that the complete lifts are deffined by (Omran et al. 1984, Yano and Ishihara 1973):

$$\begin{aligned}
&(fX)^c=f^cX^v+f^vX^c=(Xf)^c,\\
&X^cf^c=(Xf)^v,\eta^v(x^c)=(\eta(x))^v,\\
&X^vf^c=(Xf)^v,\varphi^vX^c=(\varphi X)^v,\\
&\varphi^cX^v=(\varphi X)^v,(\varphi X)^c=\varphi^cX^c,\\
&\eta^v(X^c)=(\eta(X))^c,\eta^c(X^v)=(\eta(X))^v,\\
&X^v,Y^c=[X,Y]^v,I^c=I,I^vX^c=X^v,X^c,Y^c=[X,Y]^c.
\end{aligned} \quad (1.17)$$

1.3. Horizontal Lifts

The horizontal lift f^H of $f\in\Im_0^0(M)$ to the tangent bundle $T(M)$ is given by

$$f^H=f^C-\nabla_\gamma f, \quad (1.18)$$

where

$$\nabla_\gamma f=\gamma\nabla f. \quad (1.19)$$

Let $X\in\Im_0^1(M)$. Then the horizontal lift X^H of X deffined by

$$X^H=X^C-\nabla_\gamma X \quad (1.20)$$

in $T(M)$, where

$$\nabla_\gamma X=\gamma\nabla X. \quad (1.21)$$

The horizontal lift X^H of X has the components

$$X^H: \begin{matrix} X^h \\ -\Gamma_i^hX^i \end{matrix} \quad (1.22)$$

with respect to the induced coordinates in $T(M)$, where

$$\Gamma_i^h=y^i\Gamma_{ji}^h. \quad (1.23)$$

Let $\omega\in\Im_1^0(M)$ with affine connection ∇. Then the horizontal lift ω^H of ω is deffined by

$$\omega^H=\omega^C-\nabla_\gamma\omega \quad (1.24)$$

in $T(M)$, where $\nabla_\gamma\omega=\gamma\nabla\omega$. The horizontal lift ω^H of ω has component of the form

$$\omega^H:(\Gamma_i^h\omega_h,\omega_i) \quad (1.25)$$

with respect to the induced coordinates in $T(M)$.

Suppose there is given a tensor field

$$S=S_{k\ldots j}^{i\ldots h}\frac{\partial}{\partial x^i}\otimes\ldots\otimes\frac{\partial}{\partial x^h}\otimes dx^k\otimes\ldots\otimes dx^j \quad (1.26)$$

in M with affine connection ∇ and a tensor field $\nabla_\gamma S$ defined by

$$\nabla_\gamma S=y^l\nabla_lS_{k\ldots j}^{i\ldots h}\frac{\partial}{\partial y^i}\otimes\ldots\otimes\frac{\partial}{\partial y^h}\otimes dx^k\otimes\ldots\otimes dx^j \quad (1.27)$$

with respect to the induced coordinates (x^h,y^h) in $\pi^{-1}(U)$ in $T(M)$. In addition, we define a tensor field γ_XS in $\pi^{-1}(U)$ by

$$\gamma_XS=(X^lS_{lk\ldots j}^{i\ldots h})\frac{\partial}{\partial y^i}\otimes\ldots\frac{\partial}{\partial y^h}\otimes dx^k\otimes\ldots\otimes dx^j$$

and a tensor field γS in $\pi^{-1}(U)$ by

$$\gamma S=(y^lS_{lk\ldots j}^{i\ldots h})\frac{\partial}{\partial y^i}\otimes\ldots\frac{\partial}{\partial y^h}\otimes dx^k\otimes\ldots\otimes dx^j$$

with respect to the induced coordinates (x^h,y^h), U being an arbitrary coordinate neighborhood in M^n. Then we have

$$\gamma_XS=(S_X)^V$$

for any $X\in\Im_0^1(M_n)$ and $S\in\Im_s^0(M_n)$ or $\Im_s^l(M_n)$, where $S_X\in\Im_{s-1}^0(M_n)$ or $\Im_{s-1}^1(M_n)$ (Yano and Ishihara 1973).

The horizontal lift of a tensor field of arbitrary type in to is defined by

$$S^H=S^C-\nabla_\gamma S. \quad (1.28)$$

For any $P,Q\in T(M)$, we have

$$\begin{aligned}
&\nabla_\gamma(P\otimes Q)=(\nabla_\gamma P)\otimes Q^V+P^V\otimes(\nabla_\gamma Q),\\
&(P\otimes Q)^H=P^H\otimes Q^V+P^V\otimes Q^H.
\end{aligned} \quad (1.29)$$

Differantial transformation of algebra $T(M)$, deffined by

$$D=\nabla_X:T(M_n)\to T(M_n),X\in\Im_0^l(M_n),$$

is called as covarient derivation with respect to vector field X if

$$\begin{aligned}
&\nabla_{fX+gY}t=f\nabla_Xt+g\nabla_Yt,\\
&\nabla_Xf=Xf,
\end{aligned} \quad (1.30)$$

where $\forall f,g\in\Im_0^0(M_n),\forall X,Y\in\Im_0^1(M_n),\forall t\in\Im(M_n)$.

On the other hand, a transformation deffined by

$$\nabla:\Im_0^1(M_n)\times\Im_0^1(M_n)\to\Im_0^1(M_n),$$

is called as affin connection (Salimov 2013, Yano and Ishihara 1973). We also know that the horizontal lifts are deffined by (Omran et al. 1984, Yano and Ishihara 1973)

$$\begin{aligned}
&I^H=I,I^HX^v=X^V,I^vX^H=X^v,I^HX^H=X^H,\\
&X^Hf^v=(Xf)^v,(fX)^H=f^vX^H,\omega^H(X^H)=0,\\
&\omega^v(X^H)=(\omega(X))^v,\omega^H(X^v)=(\omega(X))^v,\\
&F^HX^v=(FX)^v,F^HX^H=(FX)^H.
\end{aligned} \quad (1.31)$$

In addition, the horizontal lift of an affine connection ∇ in M_n to $T(M_n)$, denoted by ∇^H, defined by

$$\begin{aligned}&\nabla^H_{X^V} Y^V = 0, \nabla^H_{X^V} Y^H = 0,\\ &\nabla^H_{X^H} Y^V = (\nabla_X Y)^V, \nabla^H_{X^H} Y^H = (\nabla_X Y)^H\end{aligned} \tag{1.32}$$

for any $X, Y \in \Im^1_0(M_n)$.

2. Results

Let an n-dimensional diferentiable manifold M_n be endowed with a tensor field φ of type (1,1), a vector field ξ and a 1-form η, I the identity and let them satisfy

$$\varphi^2 = -I + \eta \otimes \xi, \quad \varphi(\xi) = 0, \quad \eta o \varphi = 0, \quad \eta(\xi) = 1. \tag{2.1}$$

Then (φ, ξ, η) define almost contact structure on M (Blair 1976, Çayır and Köseoğolu 2016, Şahin and Akyol 2014, Yano and Ishihara 1973). From (2.1), we get on taking horizontal and vertical lifts

$$\begin{aligned}&(\varphi^H)^2 = -I + \eta^v \otimes \xi^H + \eta^H \otimes \xi^v,\\ &\varphi^H \xi^v = 0, \varphi^H \xi^H = 0, \eta^v o \xi^H = 0,\\ &\eta^H o \varphi^H = 0, \eta^v(\xi^v) = 0, \eta^v(\xi^H) = 1,\\ &\eta^H(\xi^v) = 1, \eta^H(\xi^H) = 0.\end{aligned} \tag{2.2}$$

We now define a (1,1) tensor field J on $T(M_n)$ by

$$J = \varphi^H - \xi^v \otimes \eta^v + \xi^H \otimes \eta^H. \tag{2.3}$$

Then it is easy to show that $J^2 X^v = -X^v$ and $J^2 X^H = -X^H$, which give that J is an almost contact structure on $T(M_n)$. We get from (2.3)

$$\begin{aligned}&JX^v = (\varphi X)^v + (\eta(X)\xi)^H,\\ &JX^H = (\varphi X)^H - (\eta(X)\xi)^v\end{aligned}$$

for any $X \in \Im^1_0(M_n)$ (Omran et al. 1984).

Theorem 2.1. For the horizontal lift ∇^H of an affine connection ∇ in M_n to $T(M_n)$, $\eta(Y) = 0$ and $J \in \Im^1_1(T(M_n))$ defined by (2.3), we have

i) $(\nabla^H_{X^H} J) Y^v = ((\nabla_X \varphi) Y)^v + ((\nabla_X \eta) Y)^v \xi^H$,

ii) $(\nabla^H_{X^H} J) Y^H = ((\nabla_X \varphi) Y)^H - ((\nabla_X \eta) Y)^v \xi^v$,

iii) $(\nabla^H_{X^v} J) Y^v = 0$,

iv) $(\nabla^H_{X^v} J) Y^H = 0$,

where $X, Y \in \Im^1_0(M_n)$, a tensor field $\varphi \in \Im^1_1(M_n)$, a vector field $\Im \in \Im^1_0(M_n)$ and a 1-form $\eta \in \Im^0_1(M_n)$.

Proof. For $J = \varphi^H - \xi^v \otimes \eta^v + \xi^H \otimes \eta^H$ and $\eta(Y) = 0$, we get

i)

$$\begin{aligned}(\nabla^H_{X^H} J) Y^v &= \nabla^H_{X^H} J Y^v - J \nabla^H_{X^H} Y^v\\ &= \nabla^H_{X^H} (\varphi Y)^v + (\eta(Y)\xi)^H - (\varphi^H - \xi^v \otimes \eta^v + \xi^H \otimes \eta^H) \nabla^H_{X^H} Y^v\\ &= \nabla^H_{X^H} (\varphi Y)^v + \nabla^H_{X^H} (\eta(Y)\xi)^H - (\varphi^H - \xi^v \otimes \eta^v + \xi^H \otimes \eta^H) \nabla^H_{X^H} Y^v\\ &= (\nabla_X \varphi Y)^v + \nabla^H_{X^H} (\eta(Y)\xi)^H - \varphi^H (\nabla_X Y)^v + (\eta^v (\nabla_X Y)^v) \xi^v\\ &\quad -(\eta^H (\nabla_X Y)^v) \xi^H\\ &= ((\nabla_X \varphi) Y)^v + (\varphi \nabla_X Y)^v - (\varphi(\nabla_X Y))^v - (\eta(\nabla_X Y))^v \xi^H\\ &= ((\nabla_X \varphi) Y)^v + ((\nabla_X \eta) Y)^v \xi^H.\end{aligned}$$

ii)

$$\begin{aligned}(\nabla^H_{X^H} J) Y^H &= \nabla^H_{X^H} J Y^H - J \nabla^H_{X^H} Y^H\\ &= \nabla^H_{X^H} (\varphi Y)^H - (\eta(Y)\xi)^v - (\varphi^H - \xi^v \otimes \eta^v + \xi^H \otimes \eta^H) \nabla^H_{X^H} Y^H\\ &= (\nabla_X \varphi Y)^H - \nabla^H_{X^H} (\eta(Y)\xi)^v - (\varphi^H - \xi^v \otimes \eta^v + \xi^H \otimes \eta^H) \nabla^H_{X^H} Y^H\\ &= (\nabla_X \varphi Y)^H - \nabla^H_{X^H} (\eta(Y)\xi)^v - (\varphi \nabla_X Y)^H + (\eta^v (\nabla_X Y)^H) \xi^v\\ &\quad -(\eta^H (\nabla_X Y)^H) \xi^H\\ &= ((\nabla_X \varphi) Y)^H + (\varphi \nabla_X Y)^H - (\varphi(\nabla_X Y))^H + (\eta(\nabla_X Y))^v \xi^v\\ &= ((\nabla_X \varphi) Y)^H - ((\nabla_X \eta) Y)^v \xi^v.\end{aligned}$$

iii)

$$\begin{aligned}(\nabla^H_{X^v} J) Y^v &= \nabla^H_{X^v} J Y^v - J \nabla^H_{X^v} Y^v\\ &= \nabla^H_{X^v} (\varphi Y)^v + (\eta(Y)\xi)^H - (\varphi^H - \xi^v \otimes \eta^v + \xi^H \otimes \eta^H) \nabla^H_{X^v} Y^v\\ &= \nabla^H_{X^v} (\varphi Y)^v + \nabla^H_{X^v} (\eta(Y)\xi)^H\\ &= \nabla^H_{X^v} (\varphi Y)^v\\ &= 0.\end{aligned}$$

iv)

$$\begin{aligned}(\nabla^H_{X^v} J) Y^H &= \nabla^H_{X^v} J Y^H - J \nabla^H_{X^v} Y^H\\ &= \nabla^H_{X^v} (\varphi Y)^H - (\eta(Y)\xi)^v - (\varphi^H - \xi^v \otimes \eta^v + \xi^H \otimes \eta^H) \nabla^H_{X^v} Y^H\\ &= \nabla^H_{X^v} (\varphi Y)^H - \nabla^H_{X^v} (\eta(Y)\xi)^v\\ &= \nabla^H_{X^v} (\varphi Y)^H\\ &= 0,\end{aligned}$$

where $\eta \nabla_X Y = \nabla_X \eta(Y) - (\nabla_X \eta) Y$ and $\varphi Y \in \Im^1_0(M_n)$.

Corollary 2.1. If we put $Y = \xi$, i.e. $\eta(\xi) = 1$ and ξ has the conditions of (2.1), then we get different results

i) $(\nabla^H_{X^H} J) \xi^v = ((\nabla_X \xi)^H + ((\nabla_X \varphi) \xi)^v + ((\nabla_X \eta) \xi)^v \xi^H$,

ii) $(\nabla^H_{X^H} J) \xi^H = -((\nabla_X \xi)^v + ((\nabla_X \varphi) \xi)^H - ((\nabla_X \eta) \xi)^v \xi^v$,

iii) $(\nabla^H_{X^v} J) \xi^v = 0$,

iv) $(\nabla^H_{X^v} J) \xi^H = 0$.

Let an n-dimensional diferentiable manifold M_n be endowed with a tensor field φ of type (1,1), a vector field ξ and a 1-form η, I the identity and let them satisfy

$$\varphi^2 = I - \eta \otimes \xi, \varphi(\xi) = 0, \eta o \varphi = 0, \ \eta(\xi) = 1. \tag{2.4}$$

Then (φ,ξ,η) define almost paracontact structure on M_n (Çayır 2016a, 2016b, 2016c, Salimov and Çayır 2013, Şahin and Akyol 2014, Yano and Ishihara 1973). From (2.4), we get on taking complete and vertical lifts (Çayır 2015, Omran et al. 1984)

$$\begin{aligned}&(\varphi^H)^2 = I-\eta^v\otimes\xi^H-\eta^H\otimes\xi^v,\\ &\varphi^H\xi^v=0,\varphi^H\xi^H=0,\eta^v o\xi^H=0,\\ &\eta^H o\varphi^H=0,\eta^v(\xi^v)=0,\eta^v(\xi^H)=1,\\ &\eta^H(\xi^v)=1,\eta^H(\xi^H)=0.\end{aligned} \quad (2.5)$$

We now define a (1,1) tensor field J on $T(M_n)$ by

$$J=\varphi^H-\xi^v\otimes\eta^v-\xi^H\otimes\eta^H. \quad (2.6)$$

Then it is easy to show that $J^2X^v=X^v$ and $J^2X^c=X^c$, which give that J is an almost product structure on $T(M_n)$. We get from (2.6)

$$JX^v=(\varphi X)^v-(\eta(X)\xi)^H,$$

$$JX^H=(\varphi X)^H-(\eta(X)\xi)^v$$

for any $X\in\Im_0^1(M_n)$.

Theorem 2.2. For horizontal lift ∇^H of an affine connection ∇ in M_n to $T(M_n)$, $\eta(Y)=0$ and $J\in\Im_1^1(T(M_n))$, defined by (2.6), we have

i) $(\nabla^H_{X^H}J)Y^v=((\nabla_X\varphi)Y)^v-((\nabla_X\eta)Y)^v\xi^H$,

ii) $(\nabla^H_{X^H}J)Y^H=((\nabla_X\varphi)Y)^H-((\nabla_X\eta)Y)^v\xi^v$,

iii) $(\nabla^H_{X^v}J)Y^v=0$,

iv) $(\nabla^H_{X^v}J)Y^H=0$,

where $X,Y\in\Im_0^1(M_n)$, a tensor field $\varphi\in\Im_1^1(M_n)$, a vector field ξ and a 1-form

$\eta\in\Im_1^0(M_n)$.

Proof. For $J=\varphi^H-\xi^v\otimes\eta^H-\xi^H\otimes\eta^H$ and $\eta(Y)=0$, we get

i)

$$\begin{aligned}&(\nabla^H_{X^H}J)Y^v=\nabla^H_{X^H}JY^v-J\nabla^H_{X^H}Y^v\\ &=\nabla^H_{X^H}(\varphi Y)^v-(\eta(Y)\xi)^H-(\varphi^H-\xi^v\otimes\eta^v-\xi^H\otimes\eta^H)\nabla^H_{X^H}Y^v\\ &=\nabla^H_{X^H}(\varphi Y)^v-\nabla^H_{X^H}(\eta(Y)\xi)^H-(\varphi^H-\xi^v\otimes\eta^v-\xi^H\otimes\eta^H)\nabla^H_{X^H}Y^v\\ &=(\nabla_X\varphi Y)^v-\nabla^H_{X^H}(\eta(Y)\xi)^H-\varphi^H(\nabla_XY)^v\\ &+(\eta^v(\nabla_XY)^v)\xi^v+(\eta^H(\nabla_XY)^v)\xi^H\\ &=((\nabla_X\varphi)Y)^v+(\varphi\nabla_XY)^v-(\varphi(\nabla_XY))^v+(\eta(\nabla_XY))^v\xi^H\\ &=((\nabla_X\varphi)Y)^v-((\nabla_X\eta)Y)^v\xi^H.\end{aligned}$$

ii)

$$\begin{aligned}&(\nabla^H_{X^H}J)Y^H=\nabla^H_{X^H}JY^H-J\nabla^H_{X^H}Y^H\\ &=\nabla^H_{X^H}(\varphi Y)^H-(\eta(Y)\xi)^v-(\varphi^H-\xi^v\otimes\eta^v-\xi^H\otimes\eta^H)\nabla^H_{X^H}Y^H\\ &=(\nabla_X\varphi Y)^H-\nabla^H_{X^H}(\eta(Y)\xi)^v-(\varphi^H-\xi^v\otimes\eta^v-\xi^H\otimes\eta^H)\nabla^H_{X^H}Y^H\\ &=(\nabla_X\varphi Y)^H-\nabla^H_{X^H}(\eta(Y)\xi)^v-(\varphi\nabla_XY)^H\\ &+(\eta^v(\nabla_XY)^H)\xi^v+(\eta^H(\nabla_XY)^H)\xi^H\\ &=((\nabla_X\varphi)Y)^H+(\varphi\nabla_XY)^H-(\varphi(\nabla_XY))^H+(\eta(\nabla_XY))^v\xi^v\\ &=((\nabla_X\varphi)Y)^H-((\nabla_X\eta)Y)^v\xi^v.\end{aligned}$$

iii)

$$\begin{aligned}&(\nabla^H_{X^v}J)Y^v=\nabla^H_{X^v}JY^v-J\nabla^H_{X^v}Y^v\\ &=\nabla^H_{X^v}(\varphi Y)^v-(\eta(Y)\xi)^H-(\varphi^H-\xi^v\otimes\eta^v-\xi^H\otimes\eta^H)\nabla^H_{X^v}Y^v\\ &=\nabla^H_{X^v}(\varphi Y)^v-\nabla^H_{X^v}(\eta(Y)\xi)^H\\ &=\nabla^H_{X^v}(\varphi Y)^v\\ &=0.\end{aligned}$$

iv)

$$\begin{aligned}&(\nabla^H_{X^v}J)Y^H=\nabla^H_{X^v}JY^H-J\nabla^H_{X^v}Y^H\\ &=\nabla^H_{X^v}(\varphi Y)^H-(\eta(Y)\xi)^v-(\varphi^H-\xi^v\otimes\eta^v-\xi^H\otimes\eta^H)\nabla^H_{X^v}Y^H\\ &=\nabla^H_{X^v}(\varphi Y)^H-\nabla^H_{X^v}(\eta(Y)\xi)^v\\ &=\nabla^H_{X^v}(\varphi Y)^H\\ &=0,\end{aligned}$$

where $\eta\nabla_XY=\nabla_X\eta(Y)-(\nabla_X\eta)Y),\varphi Y\in\Im_0^1(M_n)$.

Corollary 2.2. If we put $Y=\xi$, i.e. $\eta(\xi)=1$ and ξ has the conditions of (2.4), then we have

i) $(\nabla^H_{X^H}J)\xi^v=-((\nabla_X\xi)^H+((\nabla_X\varphi)\xi)^v-((\nabla_X\eta)\xi)^v\xi^H$,

ii) $(\nabla^H_{X^H}J)\xi^H=-((\nabla_X\xi)^v+((\nabla_X\varphi)\xi)^H-((\nabla_X\eta)\xi)^v\xi^v$,

iii) $(\nabla^H_{X^v}J)\xi^v=0$,

iv) $(\nabla^H_{X^v}J)\xi^H=0$.

3. Discussion

In this paper, we get the covarient derivatives of almost contact structure and almost paracontact structure with respect to X^V,X^C and X^H on tangent bundle $T(M)$. In addition, this covarient derivatives which obtained shall be studied for some special values in almost contact structure and almost paracontact structure.

4. References

Blair, DE. 1976. Contact Manifolds in Riemannian Geometry, Lecture Notes in Math. 509, Springer Verlag, New York.

Çayır, H. 2015. Some Notes on Lifts of Almost Paracontact Structures. *Am. Rev. of Math. and Stat.,* 3(1): 52-60 .

Çayır, H. 2016a. Lie derivatives of Almost Contact Structure and Almost Paracontact Structure with respect to X^V and X^C on Tangent Bundle $T(M)$. *Proc. of the Inst. of Math. and Mech.,* 42(1): 38-49.

Çayır, H. 2016b. Tachibana and Vishnevskii Operators Applied to X^V and X^C in Almost Paracontact Structure on Tangent Bundle $T(M)$. *Ordu Üniv. Bil. ve Tek. Der.,* 6(1): 67-82.

Çayır, H. 2016c. Tachibana and Vishnevskii Operators Applied to X^V and X^H and in Almost Paracontact Structure on Tangent Bundle $T(M)$. *New Tr. in Math. Sci.,* 4(3): 105-115.

Çayır, H., Köseoğolu, G. 2016. Lie Derivatives of Almost Contact Structure and Almost Paracontact Structure With Respect to X^C and X^V on Tangent Bundle $T(M)$. *New Tr. in Math. Sci.,* 4(1): 153-159.

Das Lovejoy, S. 1993. Fiberings on almost r-contact manifolds. *Pub. Math., Deb., Hung.,* 43: 161-167.

Oproiu, V. 1973. Some Remarkable Structures and Connexions, Defined on the Tangent Bundle. *Ren. di Math.,* 3(6) VI.

Omran, T., Sharffuddin, A., Husain, SI. 1984. Lift of Structures on Manifolds. *Pub. de 1'Ins. Math., Nou. ser.,* 360(50): 93 – 97.

Salimov, AA. 2013. Tensor Operators and Their applications, Nova Sci. Publ., New York, 186 pp .

Salimov, AA., Çayır, H. 2013. Some Notes On Almost Paracontact Structures. *C. R. de 1'Aced. Bulg. Des Sci.,* 66(3): 331-338.

Sasaki, S. 1958. On The Differantial Geometry of Tangent Boundles of Riemannian. Manifolds, *Toh. Math. J.,* 10: 338-358.

Şahin, B., Akyol, MA. 2014. Golden maps between Golden Riemannian manifolds and constancy of certain maps. *Math. Com.,* 19: 333–342.

Yano, K., Ishihara, S. 1973. Tangent and Cotangent Bundles, Marcel Dekker Inc., New York, 419 pp.