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Superposition Operator on Some 2-Normed Sequence Spaces

Bazı 2-Normlu Dizi Uzayları Üzerinde Superposition Operatörü

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Abstract

In this paper, we study on some characterization of superposition operators defined on 2-normed sequence spaces.

Keywords: Superposition operator, 2-normed sequence space

Öz

Bu çalışmada, 2-normlu dizi uzayları üzerinde tanımlı superposition operatörlerin bazı karakterizasyonu üzerinde çalıştık.

Anahtar Kelimeler: Superposition operatör, 2-normlu dizi uzayları

1.Introduction

Let $\mathbb{N}$ and w_X denote the set of all natural numbers and the set of all sequences defined on real vector space *X*, which has dimension greater than one, respectively. The definition 2-normed space was introduced by Gahler (1964) as following;

Let *X* be a real vector space of dimension greater than one. If the real valued function $\|.,.\|$ on $X \times X$ satisfying the following four conditions, then $\|.,.\|$ is called a 2-normed on *X*;

N1) $\|x,y\| = 0$ if and only if the vectors *x* and *y* are linearly dependent;

N2) $\|x,y\| = \|y,x\|$;

N3) $\|\alpha x,y\| = |\alpha| \|x,y\|$ for every $\alpha \in \mathbb{R}$;

N4) $\|x+z,y\| \leq \|x,y\| + \|z,y\|$ for every $x,y,z \in X$.

Let X be a 2-normed space. Define 2-normed sequence spaces as following;

$l^{2,p} := x = (x_k) \in w_X : \sum_{k=1}^{\infty} \|x_k, z\|^p < \infty$ for all $z \in X$,

$l^{2,\infty} := \left\{ x = (x_k) \in w_X : \sup_k \|x_k, z\| < \infty \text{ for all } z \in X \right\}$

and

$c_{2,0} := \left\{ x = (x_k) \in w_X : \lim_{k \to \infty} \|x_k, z\| = 0 \text{ for all } z \in X \right\}$.

It is easy to check that these spaces are Banach space with the norms

$\|x\|_{2,p} = \sum_{k=1}^{\infty} \|x_k, z\|^{p \, \frac{1}{p}}, \|x\|_{2,\infty} = \sup_k \|x_k, z\|$, for every $z \in X$, respectively.

The 2-normed spaces and more general n-normed spaces were studied by (Kim, Cho and White 1992), (Lewandowska 1999), (Lewandowska 2001) and many others. Recently, (Duyar et. al. 2016, Duyar et. al. 2017) studied on these spaces.

Let λ and μ be two sequence spaces. Recall that a superposition operator $P_g: \lambda \to \mu$ is defined by $P_g(x) = (g(k, x_k))$, where $g: \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ is a function such that $g(k,0) = 0$. The superposition operators were studied F. Dedagich (Dedagich 1987), S. Petrantuarat and Y. Kemprasit (Petrantuarat and Kemprasit 1997), E. Kolk (Kolk 2004), B. Sağır and N. Güngör (Sağır and Güngör 2015), O. Oğur (2017) and others.

In this paper, we study on superposition operator P_g acts from the spaces $c_{2,0}, \ell^{2,\infty}$ and $\ell^{2,p}, 1 \leq p < \infty$, to the space $\ell^{2,1}$ such that $P_g(x) = (g(k, x_k))$, where *g* acts from $\mathbb{N} \times X$ to *X*.

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2. Results

Throughout the results, we assume that the function $g(k,.)$ is bounded on bounded subsets of X for every $k \in \mathbb{N}$.

Theorem 1. $P_g: \ell^{2,\infty} \leftarrow \ell^{2,1}$ if and only if there exists $c=(c_k) \in \ell^{2,1}$ such that for every $\alpha > 0$ and for each positive integer k

$\|g(k,t),y\| \le \|c_k,y\|$ whenever $\|t,y\| \le \alpha$

for all $y \in X$.

Proof. Let $P_g: \ell^{2,\infty} \to \ell^{2,1}$. For arbitrary $\alpha > 0$ and $k \in \mathbb{N}$ let define following sets;

$A(\alpha) = \{t \in X: \|t,y\| \le \alpha \text{ for every } y \in X\}$ and

$B(k,\alpha) = \sup\{\|g(k,t),y\|: t \in A(\alpha)\}$.

We need to show that $(B(k,\alpha)) \in \ell^{2,1}$ for every $\alpha > 0$. Suppose the contrary, i. e. there exists $\alpha_1 > 0$ such that $(B(k,\alpha)) \notin \ell^{2,1}$. It means that $\sum_{k=1}^{\infty} B(k,\alpha_1) = \infty$. Thus, there exist natural numbers n_i, where $n_0 = 0 < n_1 < n_2 < ... < n_i < ...$, such that $\sum_{k=n_{i-1}+1}^{n_i} B(k,\alpha_1) > 1$ and ε_i such that

$$\sum_{k=n_{i-1}+1}^{n_i} B(k,\alpha_1) - (n_i - n_{i-1})\varepsilon_i > 1. \tag{2.1}$$

Take $i \in \mathbb{N}$. Then, we have $0 \le B(k,\alpha_1) < \infty$ for every natural numbers $k \in [n_{i-1}+1, n_i]$. By definition of $B(k,\alpha_1)$, there exists $x_k \in A(\alpha_1)$ such that

$$\|g(k,x_k),y\| > B(k,\alpha_1) - \varepsilon_i. \tag{2.2}$$

Thus, using (2.1) and (2.2), we have

$$\sum_{k=n_{i-1}+1}^{n_i} \|g(k,x_k),y\| > \sum_{k=n_{i-1}+1}^{n_i} B(k,\alpha_1) - \sum_{k=n_{i-1}+1}^{n_i} \varepsilon_i > 1$$

for every $y \in X$, which shows that $g(k,x_k) \notin \ell^{2,1}$. Also, since $x_k \in A(\alpha_1)$ for all $k \in \mathbb{N}$, we write $\|x_k,y\| \le \alpha_1$ for every $y \in X$, which implies that $(x_k) \in \ell^{2,\infty}$. This contradicts with the assumption $P_g: \ell^{2,\infty} \to \ell^{2,1}$.

Conversely, assume that there exists $c=(c_k) \in \ell^{2,1}$ such that for every $\alpha > 0$ and for each positive integer k

$\|g(k,t),y\| \le \|c_k,y\|$ whenever $\|t,y\| \le \alpha$

for all $y \in X$. Take $(x_k) \in \ell^{2,\infty}$. Then, there exists a real number $M > 0$ such that $\|x_k,y\| \le M$ for every positive integer k and all $y \in X$. By assumption, we have $c=(c_k) \in \ell^{2,1}$ such that $\|g(k,x_k),y\| \leqslant \|c_k,y\|$ for every positive integer k. Thus, we have

$$\sum_{k=1}^{\infty} \|g(k,x_k),y\| \le \sum_{k=1}^{\infty} \|c_k,y\| < \infty$$

which shows that $P_g: \ell^{2,\infty} \to \ell^{2,1}$.

Example 2. Let $X=\mathbb{R}^2$ and let define a 2-norm on $X=\mathbb{R}^2$ such that $\|x,y\| = |x_1y_2 - x_2y_1|$, where $x=(x_1,x_2), y=(y_1,y_2) \in \mathbb{R}^2$. Also, let define a function $g: \mathbb{N} \times \mathbb{R}^2 \to \mathbb{R}^2$ such that $g(k,t) = \frac{|t_1|}{4^k}, 0$, where $t=(t_1,t_2) \in \mathbb{R}^2$. Given $\alpha > 0$ and $t \in \mathbb{R}^2$ such that $\|t,z\| \leqslant \alpha$ for all $z \in \mathbb{R}^2$. Then, since it is true for $z^* = (0,1) \in \mathbb{R}^2$, we write $\|t,z^*\| = |t_1.1 - t_2.0| = |t_1| \le \alpha$. Thus, we have

$$\|g(k,t),z\| = \left|\frac{|t_1|}{4^k}.z_2 - 0.z_1\right| \le \left|\frac{\alpha}{4^k}.z_2 - 0.z_1\right|$$

for every $z \in \mathbb{R}^2$. Choose $c_k = \left(\frac{\alpha}{4^k}, 0\right) \in \mathbb{R}^2$ for every k and so $(c_k) \in \ell^{2,1}$. By theorem 1, we get $P_g: \ell^{2,\infty} \to \ell^{2,1}$.

Theorem 3. $P_g: c_{2,0} \to \ell^{2,1}$ if and only if there exist a real number $\alpha > 0$ and $(c_k) \in \ell^{2,1}$ such that

$\|g(k,t),y\| \le \|c_k,y\|$ whenever $\|t,y\| \le \alpha$

for every positive number k and every $y \in X$.

Proof. Let $P_g: c_{2,0} \to \ell^{2,1}$. Define the following sets;

$A(\alpha) = \{t \in X: \|t,y\| \le \alpha \text{ for every } y \in X\}$ and

$B(k,\alpha) = \sup\{\|g(k,t),y\|: t \in A(\alpha)\}$

for arbitrary $\alpha > 0$ and every $k \in \mathbb{N}$. By definition of these sets, we have $\|g(k,t),y\| \le B(k,\alpha)$ whenever $t \in A(\alpha)$. Let suppose that $(B(k,\alpha)) \notin \ell^{2,1}$ for every $\alpha > 0$. Then, we write $\sum_{k=1}^{\infty} B(k,2^i) = \infty$ for every $i \in \mathbb{N}$. Thus, there exists a subsequence $n_0 = 0 < n_1 < n_2 < ... < n_i < ...$ such that $\sum_{k=n_{i-1}+1}^{n_i} B(k,2^i) > 1$ and positive real number ε_i such that

$$\sum_{k=n_{i-1}+1}^{n_i} B(k,2^i) - (n_i - n_{i-1})\varepsilon_i > 1. \tag{2.3}$$

Take $i \in \mathbb{N}$. Then, for every $k \in \mathbb{N}$ with $n_{i-1}+1 \le k \le n_i$ we have $0 \le B(k,2^{-i}) < \infty$. By definition of $B(k,2^{-i})$, there exists $x_k \in A(2^{-i})$ such that

$$\|g(k,x_k),y\| > B(k,2^i) - \varepsilon_i. \tag{2.4}$$

By (2.3) and (2.4), we get

$$\sum_{k=n_{i-1}+1}^{n_i} \|g(k,x_k),y\| > \sum_{k=n_{i-1}+1}^{n_i} B(k,2^{-i}) - \sum_{k=n_{i-1}+1}^{n_i} \varepsilon_i > 1,$$

which shows that $(g(k,x_k)) \notin l^{2,1}$. Also, since $(x_k) \in A(2^{-i})$ for every $k \in [n_{i-1}, n_i]$ we have $\|x_k,y\| \le 2^{-i}$ and so $(x_k) \in c_{2,0}$. This contradicts the assumption that $P_g: c_{2,0} \to \ell^{2,1}$. Thus, there exists $\alpha_1 > 0$ such that $(B(k,\alpha_1)) \in \ell^{2,1}$. By choosing $c_k = B(k,\alpha_1)$, the necessity is completed.

Conversely, assume that there exist a real number $\alpha > 0$ and $(c_k) \in \ell^{2,1}$ such that $\| g(k,t), y \| \le \| c_k, y \|$ whenever $\| t, y \| \le \alpha$ for every positive number k and every $y \in X$. Let $x = (x_k) \in c_{2,0}$. Then, for this $\alpha > 0$, there exists $n_0 \in \mathbb{N}$ such that $\| x_k, y \| \le \alpha$ for every $k \ge n_0$. By assumption, there exists $c = (c_k) \in \ell^{2,1}$ such that $\| g(k,x_k), y \| \le \| c_k, y \|$ for every $k \ge n_0$. Thus, we get

$$\sum_{k=n_0}^{\infty} \| g(k,x_k), y \| \le \sum_{k=n_0}^{\infty} \| c_k, y \| \le \sum_{k=1}^{\infty} \| c_k, y \|,$$

which shows that $(g(k,x_k)) \in \ell^{2,1}$.

Example 4. Let $X = \mathbb{R}^2$ and let take a 2-norm on $X = \mathbb{R}^2$ such that $\| x, y \| = | x_1 y_2 - x_2 y_1 |$, where $x = (x_1, x_2), y = (y_1 y_2) \in \mathbb{R}^2$. Also, let define a function $g: \mathbb{N} \times \mathbb{R}^2 \to \mathbb{R}^2$ such that $g(k,t) = \frac{| t_1 (t_1 - 1) |}{2^k}, 0$, where $t = (t_1, t_2) \in \mathbb{R}^2$. Given arbitrary $t = (t_1, t_2) \in \mathbb{R}^2$ such that $\| (t_1, t_2), y \| \le 1$. Here we chose $\alpha = 1$. By definition of 2-norm, last inequality is true for $(0, 1 \in \mathbb{R}^2)$. Thus, we have

$$\| (t_1, t_2), (0,1) \| = | t_1 | \le 1.$$

Therefore, we get

$$\begin{aligned} \| g(k,t), y \| &= \left| \frac{| t_1 (t_1 - 1 |}{2^k} y_2 - 0.y_1 \right| \le \left| \frac{| t_1 + 1 |}{2^k} y_2 - 0.y_1 \right| \\ &\le \left| \frac{2}{2^k} y_2 - 0.y_1 \right| \end{aligned}$$

where $(y_1, y_2) \in \mathbb{R}^2$. If we choose $c_k = \frac{1}{2^{k-1}} 0$, then $(c_k) \in \ell^{2,1}$. By theorem 3, we get $P_g: c_{2,0} \to \ell^{2,1}$.

Theorem5. $P_g: \ell^{2,p} \to \ell^{2,1}$ if and only if there exist $\alpha > 0, \beta > 0$ and $(c_k) \in \ell^{2,1}$ such that $\| g(k,t), y \| \le \| c_k, y \| + \alpha \| t, y \|^p$ whenever $\| t, y \| \le \beta$

for every $y \in X$.

Proof. Let assume that there exist $\alpha > 0, \beta > 0$ and $(c_k) \in \ell^{2,1}$ such that $\| g(k,t), y \| \le \| c_k, y \| + \alpha \| t, y \|^p$ whenever $\| t, y \| \le \beta$ for every $y \in X$. Take $x = (x_k) \in \ell^{2,p}$. Since $\sum_{k=1}^{\infty} \| x_k, y \|^p < \infty$ for every $y \in X$, we have $\lim_{k \to \infty} \| x_k, y \|^p = 0$. Specially, for this positive number $\beta > 0$, there is $i \in \mathbb{N}$ such that $\| x_k, y \|^p \le \beta^p$ for $k \ge i$. Then, by assumption, there exists $(c_k) \in \ell^{2,1}$ such that $\| g(k,x_k), y \| \le \| c_k, y \| + \alpha \| x_k, y \|^p$ for $k \ge i$. Thus, we get

$$\begin{aligned} \sum_{k=i}^{\infty} \| g(k,x_k), y \| &\le \sum_{k=i}^{\infty} \| c_k, y \| + \alpha \sum_{k=i}^{\infty} \| x_k, y \|^p \\ &\le \sum_{k=1}^{\infty} \| c_k, y \| + \alpha \sum_{k=1}^{\infty} \| x_k, y \|^p \\ &< \infty \end{aligned}$$

which shows that $P_g: \ell^{2,p} \to \ell^{2,1}$.

Conversely, let $P_g: \ell^{2,p} \to \ell^{2,1}$ and given arbitrary positive numbers $\alpha > 0, \beta > 0$. Let define the following; for every $k \in \mathbb{N}$ and $y \in X$

$$A(k,\alpha,\beta) = \{ t \in X : \| t, y \|^p \le \min\{ \beta, \alpha^{-1} \| g(k,k), y \| \} \}$$

and

$$B(k,\alpha,\beta) = \sup\{ \| g(k,t), y \| : t \in A(k,\alpha,\beta) \}.$$

Then, we have $\| t, y \| \le \beta$ and $\| g(k,t), y \| \le B(k,\alpha,\beta)$ whenever $t \in A(k,\alpha,\beta)$. Also, if $\| t, y \| \le \beta$ and $t \notin A(k,\alpha,\beta)$, we have $\| g(k,t), y \| \le \alpha \| t, y \|^p$. Thus, we can write that $\| g(k,t), y \| \leqslant B(k,\alpha,\beta) + \alpha \| t, y \|^p$ whenever $\| t, y \| \le \beta$. We want to show that there exist $\alpha_1 > 0, \beta_1 > 0$ such that $(B(k,\alpha,\beta)) \in \ell^{2,1}$. Assume that $(B(k,\alpha,\beta)) \notin \ell^{2,1}$ for every $\alpha, \beta > 0$. By the assumption, we have $\sum_{k=1}^{\infty} B(k, 2^i, 2^{-i}) = \infty$ for all $i \in \mathbb{N}$. Then, there exist a subsequence $n_0 = 0 < n_1 < n_2 < \ldots < n_i < \ldots$ such that $\sum_{k=n_{i-1}+1}^{n_i} B(k, 2^i, 2^{-i}) > 1$ for all $i \in \bullet$ and positive number ε_i such that

$$\sum_{k=n_{i-1}+1}^{n_i} B(k, 2^i, 2^{-i}) - (n_i - n_{i-1}) \varepsilon_i > 1 \tag{2.5}$$

Thus, we have that $0 \le B(k, 2^i, 2^{-i}) < \infty$ for all $k \in [n_{i-1} + 1, n_i]$. By definition of $B(k, 2^i, 2^{-i})$, there exists $x_k \in A(k, 2^i, 2^{-i})$ such that

$$\| g(k,x_k), y \| > B(k, 2^i, 2^{-i}) - \varepsilon_i \tag{2.6}$$

for all $y \in X$. Using inequalities (2.5) and (2.6), we have

$$\sum_{k=n_{i-1}+1}^{n_i} \| g(k,x_k), y \| > \sum_{k=n_{i-1}+1}^{n_i} B(k, 2^i, 2^{-i}) - \sum_{k=n_{i-1}+1}^{n_i} \varepsilon_i > 1$$

for all $i \in \mathbb{N}$, which shows that $(g(k,x_k)) \notin \ell^{2,1}$ Also, since $x_k \in A(k, 2^i, 2^{-i})$ for all $k \in [n_{i-1} + 1, n_i]$, we have

$$\| x_k, y \| \le 2^i \text{ and } \| x_k, y \|^p \le 2^{-i} \| g(k,x_k), y \|$$

for all $y \in X$. Using the fact that $\sum_{k=n_{i-1}+1}^{n_i} B(k, 2^i, 2^{-i}) \le 1$, we get

$$\begin{aligned} \sum_{k=n_{i-1}+1}^{n_i} \| x_k, y \|^p &= \sum_{k=n_{i-1}+1}^{n_i - 1} \| x_k, y \|^p + \| x_{n_i}, y \|^p \\ &\le \sum_{k=n_{i-1}+1}^{n_i - 1} 2^{-i} \| g(k,x_k), y \| + (2^{-i})^p \\ &\le 2^{-i} \sum_{k=n_{i-1}+1}^{n_i - 1} B(k, 2^i, 2^{-i}) + 2^{-i} \\ &\le \frac{1}{2^{i-1}} \end{aligned}$$

which shows that $(x_k) \in \ell^{2,p}$. This contradicts the assumption that $P_g: \ell^{2,p} \to \ell^{2,1}$. By choosing $c_k = B(k, \alpha, \beta_1)$, we complete the proof.

3. References

Dedagich, F., Zabreiko PP. 1987. Operator superposition in the space ℓ_p. *Sib. Math. J.*, 28: 86-98.

Duyar, C., Kanber, O, Sağır, B. 2016. On n-normed Cesaro sequence space $Ces_{n,\varphi}$. *Int. Electron. J. Pure Appl. Math.*, 10(2): 151-159.

Duyar, C., Sağır, B., Oğur, O. 2017. On a class of n-normed double sequences related to p-summable double sequence space $\ell_p^{(2)}$. *E. J. Mathematical Anal. App.*, 5(1): 106-111.

Gahler, S. 1964. Lineare 2-normierte Raume, *Math. Nachr.*, 28: 1-43.

Kim, SS., Cho, YJ., White, A. 1992. Linear operators on linear 2-normed spaces. *Glasnik Matematicki*, 27(47): 63-70.

Kolk, E. 2004. Superposition operators on sequence spaces defined by φ-functions, *Demonstratio Math.*, 37: 159-175.

Lemandowski, Z. 1999. Linear operators on generalized 2-normed spaces. *Bull. Math. Soc. Sci. Math. Roumanie*, 42: 353-368.

Lemandowski, Z. 2001. Generalized 2-normed spaces. *Slupskie Prace Matematyczno-Fizyczne*, 1: 33-40.

Oğur, O., On characterization of boundedness of superposition operators on the Maddox space $C_{r0}(p)$ of double sequences. *New Trends Math. Sci.* 5(4):365-377.

Sağır, B., Güngör, N. 2015. Continuity of superposition operators on the double sequence spaces L_p, *Filomat*, 29(9): 2107-2118.

Sağır, B., Güngör, N. 2015. Locally boundedness and continuity of superposition operators on double sequence spaces. *J. Comput. Anal. App.*, 19(2): 365-377.