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On Certain Properties of Quadrapell Quaternions

Quadrapell Kuaterniyonlarının Belirli Özellikleri Üzerine

Emrah Polatlı*

Bülent Ecevit University, Department of Mathematics, Zonguldak, Turkey

Abstract

In this paper, we introduce quadrapell quaternions. We give some combinatorial properties of these quaternions. In addition, we apply the binomial transform to these quaternions and derive some algebraic properties.

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Öz

Bu makalede, quadrapell kuaterniyonları tanıtılmıştır. Bu kuaterniyonların bazı kombinatorik özellikleri verilmiştir. İlave olarak, bu kuaterniyonlara binomiyal dönüşüm uygulanmış ve bazı cebirsel özellikler türetilmiştir.

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Anahtar Kelimeler: Binomiyal Dönüşüm,Üreteç Fonksiyon, Quadrapell Sayıları, Quadrapell Kuaterniyonları

1. Introduction

There are several integer sequences studied by many mathematicians in the literature. An interesting one of them (called the quadrapell sequence) was given by Taşçı [7] as follows:

$$D_n = D_{n-2} + 2D_{n-3} + D_{n-4} \quad (n \geq 4)$$

where $D_0 = D_1 = D_2 = 1$ and $D_3 = 2$. The first few terms of this sequence are 1,1,1,2,4,5,9,15,23,38,62.

The characteristic equation of quadrapell recurrence relation is

$$x^4 - x^2 - 2x - 1 = 0.$$

Note that the roots of this equation are $\alpha = (1+\sqrt{5})/2, \beta = (1-\sqrt{5})/2, \gamma = (-1+\sqrt{3}i)/2$, and $\delta = (-1-\sqrt{3}i)/2$.

In [7], Taşçı also gave generating function, a Binet-like formula and some summation formulas for the quadrapell numbers.

*Corresponding author: emrahpolatli@gmail.com

Emrah Polatlı orcid.org/0000-0002-2349-8978

In [6], given a sequence $A = \{a_n\}_{n=0}^{\infty}$, the binomial transform B of a sequence A is denoted by $B(A) = \{b_n\}$ and defined by

$$b_n = \sum_{i=0}^{n} \binom{n}{i} a_i .$$

Recently, Kızılateş et al. [4] have been investigated some properties of binomial sums of quadrapell sequences and quadrapell matrix sequences. At this point, we refer the reader to [1, 2, 5, 6] for more details and interesting properties of the binomial transformation and its generalizations.

A quaternion p with real components a_0, a_1, a_2, a_3 and basis $1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ is a hypercomplex number of the form

$$p = a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}, \ (a_0 1 = a_0)$$

where

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1,$$
$$\boldsymbol{ij} = \boldsymbol{k} = -\boldsymbol{ji}, \boldsymbol{jk} = \boldsymbol{i} = -\boldsymbol{kj}, \boldsymbol{ki} = \boldsymbol{j} = -\boldsymbol{ik}.$$

Motivated by the above definition, A. F. Horadam [3] defined the nth Fibonacci and nth Lucas quaternions as follows:

$$Q_n = F_n + F_{n+1}\boldsymbol{i} + F_{n+2}\boldsymbol{j} + F_{n+3}\boldsymbol{k}$$

and

$$K_n = L_n + L_{n+1}\boldsymbol{i} + L_{n+2}\boldsymbol{j} + L_{n+3}\boldsymbol{k}$$

where F_n and L_n are the nth Fibonacci and nth Lucas numbers respectively.

Inspired by the above studies, in this paper, we introduce quadrapell quaternions. We give some fundamental properties of these quaternions. In addition, we apply the binomial transform to these quaternions and derive some algebraic properties.

2. Fundamental Properties of Quadrapell Quaternions

In this section, we give a definition, the generating function, a Binet-like formula, and some summation formulas for the quadrapell quaternion sequence.

Definition 2.1. For $n \geq 0$, nth quadrapell quaternion is defined by

$$w_n = D_n + D_{n+1}\boldsymbol{i} + D_{n+2}\boldsymbol{j} + D_{n+3}\boldsymbol{k}$$

where D_n is the nth quadrapell number.

Proposition 2.1. For $n \geq 0$, quadrapell quaternion sequence satisfies the following recurrence relation:

$$w_{n+4} = w_{n+2} + 2w_{n+1} + w_n$$

where $w_0 = 1 + \boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$, $w_1 = 1 + \boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k}$, $w_2 = 1 + 2\boldsymbol{i} + 4\boldsymbol{j} + 5\boldsymbol{k}$, and $w_3 = 2 + 4\boldsymbol{i} + 5\boldsymbol{j} + 9\boldsymbol{k}$.

Theorem 2.1. The generating function of the quadrapell quaternion sequence is

$$W(x) = \frac{w_0 + w_1 x + (w_2 - w_0)x^2 + (w_3 - w_1 - 2w_0)x^3}{1 - x^2 - 2x^3 - x^4}.$$

Proof. Let $W(x) = \sum_{n=0}^{\infty} w_n x^n$. Then we have

$$W(x) - x^2 W(x) - 2x^3 W(x) - x^4 W(x) = w_0 + w_1 x + (w_2 - w_0)x^2 + (w_3 - w_1 - 2w_0)x^3 + \sum_{n=4}^{\infty}(w_n - w_{n-2} - 2w_{n-3} - w_{n-4})x^n.$$

Since, for each $n \geq 4$, the coefficient of x^n is zero in the right-hand side of this equation, we easily get

$$W(x) = \frac{w_0 + w_1 x + (w_2 - w_0)x^2 + (w_3 - w_1 - 2w_0)x^3}{1 - x^2 - 2x^3 - x^4}$$

Theorem 2.2. For $n \geq 0$, we have

$$w_n = A\frac{\alpha^n - \beta^n}{\alpha - \beta} + B\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} + C\frac{\gamma^n - \delta^n}{\gamma - \delta} + D\frac{\gamma^{n+1} - \delta^{n+1}}{\gamma - \delta}$$

where

$$A = \frac{-1 + 2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}}{2}, B = \frac{2 + \boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}}{2}, C = \frac{1 - \boldsymbol{j} + \boldsymbol{k}}{2}, D = \frac{\boldsymbol{i} - \boldsymbol{j}}{2}.$$

Proof. By using the partial fraction decomposition, we obtain

$$W(x) = \frac{\frac{2 + \boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}}{2} + \left(\frac{-1 + 2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}}{2}\right)x}{1 - x - x^2} + \frac{\frac{\boldsymbol{i} - \boldsymbol{j}}{2} + \left(\frac{1 - \boldsymbol{j} + \boldsymbol{k}}{2}\right)x}{1 + x + x^2}$$

$$= \sum_{n=0}^{\infty}\left(A\frac{\alpha^n - \beta^n}{\alpha - \beta} + B\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} + C\frac{\gamma^n - \delta^n}{\gamma - \delta} + D\frac{\gamma^{n+1} - \delta^{n+1}}{\gamma - \delta}\right)x^n.$$

By the equality of the generating function we have

$$w_n = A\frac{\alpha^n - \beta^n}{\alpha - \beta} + B\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} + C\frac{\gamma^n - \delta^n}{\gamma - \delta} + D\frac{\gamma^{n+1} - \delta^{n+1}}{\gamma - \delta}.$$

Now we give some summation formulas for the quadrapell quaternions. We only prove the first claim. The others could be proven similarly.

Theorem 2.3. For $k \geq 0$, we have

(i) $\sum_{j=0}^{3k} w_j = w_{3k+2} - (\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k})$,

(ii) $\sum_{j=0}^{3k+1} w_j = w_{3k+3} - (2\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k})$,

(iii) $\sum_{j=0}^{3k+2} w_j = w_{3k+4} - (1 + \boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k})$,

(iv) $\sum_{j=0}^{3k} w_{2j} = w_{6k+1} - (\boldsymbol{j} + 2\boldsymbol{k})$,

(v) $\sum_{j=0}^{3k+1} w_{2j} = w_{6k+3} - (\boldsymbol{i} + 2\boldsymbol{k})$,

(vi) $\sum_{j=0}^{3k+2} w_{2j} = w_{6k+5} - (-1 + \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$.

Proof. From the definition of quadrapell quaternions, we have

$$\sum_{j=0}^{3k} w_j = \sum_{j=0}^{3k}(D_j + D_{j+1}\boldsymbol{i} + D_{j+2}\boldsymbol{j} + D_{j+3}\boldsymbol{k})$$

$$= \sum_{j=0}^{3k} D_j + \left(\sum_{j=0}^{3k+1} D_j - 1\right)\boldsymbol{i} + \left(\sum_{j=0}^{3k+2} D_j - 2\right)\boldsymbol{j} + \left(\sum_{j=0}^{3k+3} D_j - 3\right)\boldsymbol{k}.$$

If we use the Theorem 2.5 and Theorem 2.6 in [7], we get

$$\sum_{j=0}^{3k} w_j = \sum_{j=0}^{3k} D_j + \left(\sum_{j=0}^{3k+1} D_j - 1\right)\boldsymbol{i} + \left(\sum_{j=0}^{3k+2} D_j - 2\right)\boldsymbol{j} + \left(\sum_{j=0}^{3k+3} D_j - 3\right)\boldsymbol{k}$$

$$= D_{3k+2} + (D_{3k+3} - 1)i + (D_{3k+4} - 3)\boldsymbol{j} + (D_{3k+5} - 3)\boldsymbol{k}$$

$$= w_{3k+2} - (\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}).$$

3. Binomial Transform of Quadrapell Quaternions

In this section, we give the binomial transform of quadrapell quaternion sequence and obtain some certain identities related to this binomial transform.

Definition 3.1. Let w_n be the nth quadrapell quaternions. Then the binomial transform of quadrapell quaternion is
$b_n = \sum_{i=0}^{n} \binom{n}{i} w_i$.

From the above definition, it is clear that

$$b_{n+1} = \sum_{i=0}^{n} \binom{n}{i} (w_i + w_{i+1})$$

and

$$b_{n+1} = b_n + \sum_{i=0}^{n} \binom{n}{i} w_{i+1}.$$

Also, we can see that for even n

$$b_n = w_{2n}.$$

Theorem 3.1. For $n \geq 0$, $\{b_n\}$ satisfies the following recurrence relation:

$$b_{n+4} = 4b_{n+3} - 5b_{n+2} + 4b_{n+1} - b_n$$

where $b_0 = 1 + \boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}, b_1 = 2 + 2\boldsymbol{i} + 3\boldsymbol{j} + 6\boldsymbol{k},$ $b_2 = 4 + 5\boldsymbol{i} + 9\boldsymbol{j} + 15\boldsymbol{k},$ and $b_3 = 9 + 14\boldsymbol{i} + 24\boldsymbol{j} + 38\boldsymbol{k}$.

Proof. Firstly, consider the following recurrence relation:

$$b_{n+4} = xb_{n+3} + yb_{n+2} + zb_{n+1} + tb_n.$$

If we find the equivalents of the $b_4, b_5, b_6,$ and b_7 from the above equation and solve the system by Cramer rule, we get $x = 4, y = -5, z = 4,$ and $t = -1$.

Theorem 3.2. The generating function of the sequence $\{b_n\}$ is

$$b(x) = \frac{w_0 + (-3w_0 + w_1)x + (w_2 - 2w_1 + 2w_0)x^2 + (w_3 - w_2 - 2w_0)x^3}{1 - 4x + 5x^2 - 4x^3 + x^4}.$$

Proof. If we use the transformation given by Gould in [1], we obtain

$$\begin{aligned} b(x) &= \frac{1}{1-x} W\left(\frac{x}{1-x}\right) \\ &= \frac{1}{1-x} \frac{w_0 + w_1\left(\frac{x}{1-x}\right) + (w_2 - w_0)\left(\frac{x}{1-x}\right)^2 + (w_3 - w_1 - 2w_0)\left(\frac{x}{1-x}\right)^3}{1 - \left(\frac{x}{1-x}\right)^2 - 2\left(\frac{x}{1-x}\right)^3 - \left(\frac{x}{1-x}\right)^4} \\ &= \frac{w_0 + (-3w_0 + w_1)x + (w_2 - 2w_1 + 2w_0)x^2 + (w_3 - w_2 - 2w_0)x^3}{1 - 4x + 5x^2 - 4x^3 + x^4} \end{aligned}$$

as desired.

Lastly, we give a Binet-like formula for the sequence $\{b_n\}$.

Theorem 3.3. For $n \geq 0$, we have

$$b_n = E(-1)^n \frac{\gamma^{2n} - \delta^{2n}}{\gamma - \delta} + F(-1)^{n+1} \frac{\gamma^{2n+2} - \delta^{2n+2}}{\gamma - \delta} + G\frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} + H\frac{\alpha^{2n+2} - \beta^{2n+2}}{\alpha - \beta}$$

where $E = \frac{1 - \boldsymbol{i} + \boldsymbol{k}}{2}, F = \frac{\boldsymbol{i} - \boldsymbol{j}}{2}, G = \frac{-3 + \boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}}{2}, H = \frac{2 + \boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}}{2}$.

Proof. Recall that the generating function of the sequence $\{b_n\}$ is

$$b(x) = \frac{w_0 + (-3w_0 + w_1)x + (w_2 - 2w_1 + 2w_0)x^2 + (w_3 - w_2 - 2w_0)x^3}{1 - 4x + 5x^2 - 4x^3 + x^4}.$$

Thus, we obtain

$$b(x)=\frac{\left(\frac{1-\boldsymbol{i}+\boldsymbol{k}}{2}\right)x+\frac{\boldsymbol{i}-\boldsymbol{j}}{2}}{1-x+x^2}+\frac{\left(\frac{-3+\boldsymbol{i}-2\boldsymbol{j}-\boldsymbol{k}}{2}\right)x+\frac{2+\boldsymbol{i}+3\boldsymbol{j}+4\boldsymbol{k}}{2}}{1-3x+x^2}$$
$$=\sum_{n=0}^{\infty}\left[E(-1)^n\frac{\gamma^{2n}-\delta^{2n}}{\gamma-\delta}+F(-1)^{n+1}\frac{\gamma^{2n+2}-\delta^{2n+2}}{\gamma-\delta}+G\frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta}+H\frac{\alpha^{2n+2}-\beta^{2n+2}}{\alpha-\beta}\right]x^n.$$

By the equality of the generating function we have

$$b_n=E(-1)^n\frac{\gamma^{2n}-\delta^{2n}}{\gamma-\delta}+F(-1)^{n+1}\frac{\gamma^{2n+2}-\delta^{2n+2}}{\gamma-\delta}+G\frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta}+H\frac{\alpha^{2n+2}-\beta^{2n+2}}{\alpha-\beta}.$$

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