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Generalization of Differential Operators by Using Differential Forms

Diferansiyel Formlar ile Diferansiyel Operatörlerin Genelleştirilmesi

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Abstract

In this study, we derive the mostly used differential operators in physics, such as gradient, divergence, curl and Laplacian in different coordinate systems; Cartesian, cylindrical and spherical coordinate systems by using the differential forms. Also, we finally derive these differential operators for the generalized coordinates.

Keywords: Differential forms, Differential operators, Vector calculus

Öz

Bu çalışmada, diferansiyel formları kullanarak farklı koordinat sistemlerinde; Kartezyen, silindirik ve küresel koordinat sistemlerinde, gradyan, diverjans, rotasyonel ve Laplasyen gibi fizikte en çok kullanılan diferansiyel operatörleri türeteceğiz. Son olarak bu diferansiyel operatörleri genelleştirilmiş koordinatlar için türeceğiz.

Anahtar Kelimeler: Diferansiyel formlar, Diferansiyel operatörler, Vektör kalkülüs

1. Introduction

In physics, differential operators are widely used in order to obtain the dependency of a physical quantity with respect to some coordinates. From classical mechanics to electrodynamics and general relativity to quantum mechanics, the differential operators are all in use in different forms. For instance, the gradient of the potential energy scalar yields as the conservative force in classical mechanics, or the divergence of the electric vector field yields as the scalar charge density in electrodynamics, or so on. With the change of the symmetry of the space in which we investigate the change of the physical quantities, we alter the symmetry of the coordinate, which changes to obtain the differential of the physical quantity, such as, from Cartesian coordinate to spherical coordinates for a spherically symmetric quantity. Therefore, we need to make the coordinate transformation for the differential operator in use.

For this purpose, we derive a simple approach to obtain the differential operators in different coordinates by using the differential forms. Some general definitions and properties are given as follows.

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Let S be a scalar field such that

$$S = S(q_1, q_2, ..., q_n) \tag{1}$$

and F be a vector field which can be defined as

$$F = \sum_{i=1}^{n} F_i e_i = F_1 e_1 + F_2 e_2 + ...F_n e_n \tag{2}$$

where q_i's are the components of the coordinate system and e_i's are the basis one-forms (vectors) for the considered coordinate system. We then define the basis one-forms such as

$$e_i = \left|\frac{dr}{dq_i}\right| dq_i \tag{3}$$

where r is any vector [1]. The Hodge Dual (star) operator $*$ should be defined for the basis one-forms, such that [2-4]

$$*e_i = \varepsilon_{ijk} e_j \wedge e_k \tag{4}$$

$$*(e_i \wedge e_j \wedge e_k) = 1 \tag{5}$$

Generally, we can derive the following derivative operators in different coordinates, on fields as follows

$$\nabla S = dS \tag{6}$$

$$\nabla \cdot F = *d * F \tag{7}$$

$$\nabla \times F = *dF \tag{8}$$

$$\nabla^2 S = *d * dS \tag{9}$$

(6) means the gradient of the scalar S, (7) is the divergence, (8) is for the curl of a vector field F and the (9) is of course for the Laplacian of the scalar S. Here, the symbol d is exterior differentiation.

2. Cartesian Coordinates

The scalar field in Cartesian coordinates is written as

$$S = S(x,y,z) \tag{10}$$

and the vector field is defined to be

$$F = F_x e_x + F_y e_y + F_z e_z. \tag{11}$$

The basis one-forms are also written for vector $r = xe_x + ye_y + ze_z$ from (3), such that

$$e_x = dx,\ e_y = dy,\ e_z = dz \tag{12}$$

2.1. Gradient

From (6) for the scalar S we can see

$$\nabla S = dS = \frac{\partial S}{\partial x}dx + \frac{\partial S}{\partial y}dy + \frac{\partial S}{\partial z}dz \tag{13}$$

and by using (12) to make the gradient a vector change dq_i's into e_i's

$$\nabla S = dS = \frac{\partial S}{\partial x}e_x + \frac{\partial S}{\partial y}e_y + \frac{\partial S}{\partial z}e_z. \tag{14}$$

2.2. Divergence

From equation (7) for the given vector (11) in Cartesian coordinates, we obtain

$$F = F_x e_x + F_y e_y + F_z e_z \tag{15}$$

$$*F = F_x * e_x + F_y * e_y + F_z * e_z \tag{16}$$

$$= F_x(e_y \wedge e_z) + F_y(e_z \wedge e_x) + F_2(e_x \wedge e_y). \tag{17}$$

After this step we will need $d * F$, so we should convert e_i's into dq_i's from (12)

$$*F = F_x(dy \wedge dz) + F_y(dz \wedge dx) + F_z(dx \wedge dy) \tag{18}$$

and to calculate the exterior differentiation of (18), we proceed by

$$\begin{aligned} d * F = &\frac{\partial F_x}{\partial x}(dx \wedge dy \wedge dz) + \frac{\partial F_y}{\partial x}(dx \wedge dz \wedge dx) \\ &+ \frac{\partial F_z}{\partial x}(dx \wedge dx \wedge dy) + \frac{\partial F_x}{\partial y}(dy \wedge dy \wedge dz) \\ &+ \frac{\partial F_y}{\partial y}(dy \wedge dz \wedge dx) + \frac{\partial F_z}{\partial y}(dy \wedge dx \wedge dy) \\ &+ \frac{\partial F_x}{\partial z}(dz \wedge dy \wedge dz) + \frac{\partial F_y}{\partial z}(dz \wedge dz \wedge dx) \\ &+ \frac{\partial F_z}{\partial z}(dz \wedge dx \wedge dy) \end{aligned} \tag{19}$$

After canceling the repeating indices in triple wedge products $dq_i \wedge dq_j \wedge dq_k = \varepsilon_{ijk}$.

Now we will convert dq_i's into e_i's, because we will take their Hodge in the final step,

$$\begin{aligned} d * F = &\frac{\partial F_x}{\partial x}(e_x \wedge e_y \wedge e_z) + \frac{\partial F_y}{\partial y}(e_y \wedge e_z \wedge e_x) \\ &+ \frac{\partial F_z}{\partial z}(e_z \wedge e_x \wedge e_y) \end{aligned} \tag{20}$$

$$*d * F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \tag{21}$$

where we have used (5) and obtained the divergence equation in Cartesian coordinates for a vector field F

$$\nabla \cdot F = *d * F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

2.3. Curl

We begin by writing F from (8) with dq_i's instead of e_i's, so

$$F = F_x dx + F_y dy + F_z dz \tag{22}$$

and we take the exterior differentiation for using in $\nabla \times F = *dF$ as follows,

$$\begin{aligned} dF = &\frac{\partial F_x}{\partial x}dx \wedge dx + \frac{\partial F_y}{\partial x}dx \wedge dy + \frac{\partial F_z}{\partial x}dx \wedge dz \\ &+ \frac{\partial F_x}{\partial y}dy \wedge dx + \frac{\partial F_y}{\partial y}dy \wedge dy + \frac{\partial F_z}{\partial y}dy \wedge dz \\ &+ \frac{\partial F_x}{\partial z}dz \wedge dx + \frac{\partial F_y}{\partial z}dz \wedge dy + \frac{\partial F_z}{\partial z}dz \wedge dz. \end{aligned} \tag{23}$$

We expect to get a vector at the end of the calculation, so we convert the differential one-forms into basis one-forms, but before eliminate the repeating $dq_i \wedge dq_i = 0$ terms, and remaining is

$$\begin{aligned} dF = &\frac{\partial F_x}{\partial y}(e_y \wedge e_x) + \frac{\partial F_x}{\partial z}(e_z \wedge e_x) + \frac{\partial F_y}{\partial x}(e_x \wedge e_y) \\ &+ \frac{\partial F_y}{\partial z}(e_z \wedge e_y) + \frac{\partial F_z}{\partial x}(e_x \wedge e_z) + \frac{\partial F_z}{\partial y}(e_y \wedge e_z). \end{aligned} \tag{24}$$

When we apply Hodge operator on (24) and by using (4), we obtain

$$*dF = -\frac{\partial F_x}{\partial y}e_z + \frac{\partial F_x}{\partial z}e_y + \frac{\partial F_y}{\partial x}e_z - \frac{\partial F_y}{\partial z}e_x - \frac{\partial F_z}{\partial x}e_y + \frac{\partial F_z}{\partial y}e_x. \tag{25}$$

By rearranging it, we finally obtain the curl

$$\begin{aligned} \nabla \times F = *dF = &\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)e_x + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)e_y \\ &+ \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)e_z \end{aligned} \tag{26}$$

2.4. Laplacian

We already obtain the part such given in (14) for

$\nabla^2 S = *d * dS$ in (9). Now we take its star

$$*dS = \frac{\partial S}{\partial x} * e_x + \frac{\partial S}{\partial y} * e_y + \frac{\partial S}{\partial z} * e_z \tag{27}$$

from (4), we are led to

$$*dS = \frac{\partial S}{\partial x}(e_y \wedge e_z) + \frac{\partial S}{\partial y}(e_z \wedge e_x) + \frac{\partial S}{\partial z}(e_x \wedge e_y). \tag{28}$$

We convert the basis one-forms into differential one-forms, because we will take the differentiation

$$*dS = \frac{\partial S}{\partial x}(dy \wedge dz) + \frac{\partial S}{\partial y}(dz \wedge dx) + \frac{\partial S}{\partial z}(dx \wedge dy) \tag{29}$$

and take the differential of it, and get

$$\begin{aligned} d * dS = &\frac{\partial}{\partial x}\left(\frac{\partial S}{\partial x}\right)(dx \wedge dy \wedge dz) + \frac{\partial}{\partial x}\left(\frac{\partial S}{\partial y}\right)(dx \wedge dz \wedge dx) \\ &+ \frac{\partial}{\partial x}\left(\frac{\partial S}{\partial z}\right)(dx \wedge dx \wedge dy) + \frac{\partial}{\partial y}\left(\frac{\partial S}{\partial x}\right)(dy \wedge dy \wedge dz) \\ &+ \frac{\partial}{\partial y}\left(\frac{\partial S}{\partial y}\right)(dy \wedge dz \wedge dx) + \frac{\partial}{\partial y}\left(\frac{\partial S}{\partial z}\right)(dy \wedge dx \wedge dy) \\ &+ \frac{\partial}{\partial z}\left(\frac{\partial S}{\partial x}\right)(dz \wedge dy \wedge dz) + \frac{\partial}{\partial z}\left(\frac{\partial S}{\partial y}\right)(dz \wedge dz \wedge dx) \\ &+ \frac{\partial}{\partial z}\left(\frac{\partial S}{\partial z}\right)(dz \wedge dx \wedge dy) \end{aligned} \tag{30}$$

The repeating terms in wedge product of basis-one forms or differential one-forms vanishes, and the remaining is

$$\begin{aligned} d * dS = &\frac{\partial}{\partial x}\left(\frac{\partial S}{\partial x}\right)(dx \wedge dy \wedge dz) + \frac{\partial}{\partial y}\left(\frac{\partial S}{\partial y}\right)(dy \wedge dz \wedge dx) \\ &+ \frac{\partial}{\partial z}\left(\frac{\partial S}{\partial z}\right)(dz \wedge dx \wedge dy). \end{aligned} \tag{31}$$

Before we take its star again, we convert the dq_i's into e_i's, and use (5), then get

$$*d * dS = \frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} + \frac{\partial^2 S}{\partial z^2} \tag{32}$$

and this is already the Laplacian itself;

$$\nabla^2 S = *d * dS = \frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} + \frac{\partial^2 S}{\partial z^2}.$$

3. Cylindrical Coordinates

The scalar field in cylindrical coordinates is written as

$$S = S(s, \phi, z) \tag{33}$$

and the vector field is defined to be

$$F = F_s e_s + F_\phi e_\phi + F_z e_z. \tag{34}$$

The basis one-forms are also written for vector $r = xe_x + ye_y + ze_z = s\cos\phi e_x + s\sin\phi e_y + ze_z$ from (3), such that

$$\begin{aligned} e_s &= \left|\frac{\partial r}{\partial s}\right| ds = |\cos\phi e_x + \sin\phi e_y| ds \\ &= (\cos^2\phi + \sin^2\phi)^{1/2} ds = ds \Rightarrow e_s = ds \\ e_\phi &= \left|\frac{\partial r}{\partial \phi}\right| d\phi = |-s\sin\phi e_x + s\cos\phi e_y| d\phi \\ &= (s^2\cos^2\phi + s^2\sin^2\phi)^{1/2} d\phi = sd\phi \Rightarrow e_\phi = sd\phi \\ e_z &= dz \end{aligned} \tag{35}$$

3.1. Gradient

From (6) for the scalar S we can see

$$\nabla S = dS = \frac{\partial S}{\partial s} ds + \frac{\partial S}{\partial \phi} d\phi + \frac{\partial S}{\partial z} dz \tag{36}$$

and by using (35) to make the gradient a vector change dq_i's into e_is

$$\begin{aligned} \nabla S = dS &= \frac{\partial S}{\partial s} e_s + \frac{\partial S}{\partial \phi}\frac{e_\phi}{s} + \frac{\partial S}{\partial z} e_z \\ \nabla S = dS &= \frac{\partial S}{\partial s} e_s + \frac{1}{s}\frac{\partial S}{\partial \phi} e_\phi + \frac{\partial S}{\partial z} e_z. \end{aligned} \tag{37}$$

3.2. Divergence

By equation (7) for the given vector F in (34) in cylindrical coordinates, we obtain

$$F = F_s e_s + F_\phi e_\phi + F_z e_z \tag{38}$$

$$*F = F_s * e_s + F_\phi * e_\phi + F_z * e_z \tag{39}$$

$$= F_s(e_\phi \wedge e_z) + F_\phi(e_z \wedge e_s) + F_z(e_s \wedge e_\phi). \tag{40}$$

After this step we will need $d * F$, so we should convert e_i's into dq_i's from (35)

$$*F = sF_s(d\phi \wedge dz) + F_\phi(dz \wedge ds) + sF_z(ds \wedge d\phi)) \tag{41}$$

$$\begin{aligned} d * F = &\frac{\partial(sF_s)}{\partial s}(ds \wedge d\phi \wedge dz) + \frac{\partial F_\phi}{\partial s}(ds \wedge dz \wedge ds) \\ &+ \frac{\partial(sF_z)}{\partial s}(ds \wedge ds \wedge d\phi) + \frac{\partial(sF_s)}{\partial \phi}(d\phi \wedge d\phi \wedge dz) \\ &+ \frac{\partial F_\phi}{\partial \phi}(d\phi \wedge dz \wedge ds) + \frac{\partial(sF_z)}{\partial \phi}(d\phi \wedge ds \wedge d\phi) \\ &+ \frac{\partial(sF_s)}{\partial z}(dz \wedge d\phi \wedge dz) + \frac{\partial F_\phi}{\partial z}(dz \wedge dz \wedge ds) \\ &+ \frac{\partial(sF_z)}{\partial z}(dz \wedge ds \wedge d\phi) \end{aligned} \tag{42}$$

After canceling the repeating indices in triple wedge products $dq_i \wedge dq_j \wedge dq_k = \varepsilon_{ijk}$

Now we will convert dq_i's into e_i's, because we will take their Hodge in the final step,

$$d*F=\frac{\partial(sF_s)}{\partial s}\frac{(e_s\wedge e_\phi\wedge e_z)}{s}+\frac{\partial F_\phi}{\partial\phi}\frac{(e_\phi\wedge e_z\wedge e_s)}{s}$$
$$+\frac{\partial(sF_z)}{\partial z}\frac{(e_z\wedge e_s\wedge e_\phi)}{s}=\frac{1}{s}\frac{\partial(sF_s)}{\partial s}(e_s\wedge e_\phi\wedge e_z) \quad (43)$$
$$+\frac{1}{s}\frac{\partial F_\phi}{\partial\phi}(e_s\wedge e_\phi\wedge e_z)+\frac{1}{s}\frac{s\partial F_z}{\partial z}(e_s\wedge e_\phi\wedge e_z)$$

$$*d*F=\frac{1}{s}\frac{\partial(sF_s)}{\partial s}+\frac{1}{s}\frac{\partial F_\phi}{\partial\phi}+\frac{\partial F_z}{\partial z} \quad (44)$$

where we have used (5) and obtained the divergence equation in cylindrical coordinates for a vector field F

$$\nabla\cdot F=*d*F=\frac{1}{s}\frac{\partial(sF_s)}{\partial s}+\frac{1}{s}\frac{\partial F_\phi}{\partial\phi}+\frac{\partial F_z}{\partial z}$$

3.3. Curl

We begin by writing F with dq_i's instead of e_i's, so

$$F=F_s ds+sF_\phi d\phi+F_z dz \quad (45)$$

and we take the exterior differentiation for using in $\nabla\times F=*dF$ as follows,

$$dF=\frac{\partial F_s}{\partial s}ds\wedge ds+\frac{\partial(sF_\phi)}{\partial s}ds\wedge d\phi+\frac{\partial F_z}{\partial s}ds\wedge dz$$
$$+\frac{\partial F_s}{\partial\phi}d\phi\wedge ds+\frac{\partial(sF_\phi)}{\partial\phi}d\phi\wedge d\phi+\frac{\partial F_z}{\partial\phi}d\phi\wedge dz \quad (46)$$
$$+\frac{\partial F_s}{\partial z}dz\wedge ds+\frac{\partial(sF_\phi)}{\partial z}dz\wedge d\phi+\frac{\partial F_z}{\partial z}dz\wedge dz.$$

We expect to get a vector at the end of the calculation, so we convert the differential one-forms into basis one-forms, but before eliminate the repeating $dq_i\wedge dq_i=0$ terms, and remaining is

$$dF=\frac{\partial(sF_\phi)}{\partial s}\frac{e_s\wedge e_\phi}{s}+\frac{\partial F_z}{\partial s}e_s\wedge e_z+\frac{\partial F_s}{\partial\phi}\frac{e_\phi\wedge e_s}{s}$$
$$+\frac{\partial F_z}{\partial\phi}\frac{e_\phi\wedge e_z}{s}+\frac{\partial F_s}{\partial z}e_z\wedge e_s+\frac{\partial(sF_\phi)}{\partial z}\frac{e_z\wedge e_\phi}{s} \quad (47)$$

When we apply Hodge operator on (47) and by using (4), we obtain

$$*dF=-\frac{1}{s}\frac{\partial F_s}{\partial\phi}e_z+\frac{\partial F_s}{\partial z}e_\phi+\frac{1}{s}\frac{\partial(sF_\phi)}{\partial s}e_z$$
$$-\frac{1}{s}\frac{s\partial F_\phi}{\partial z}e_s-\frac{\partial F_z}{\partial s}e_\phi+\frac{1}{s}\frac{\partial F_z}{\partial\phi}e_s \quad (48)$$

By rearranging it, we finally obtain the curl

$$\nabla\times F=*dF=\left(\frac{1}{s}\frac{\partial F_z}{\partial\phi}-\frac{\partial F_\phi}{\partial z}\right)e_s+\left(\frac{\partial F_s}{\partial z}-\frac{\partial F_z}{\partial s}\right)e_\phi$$
$$+\frac{1}{s}\left(\frac{\partial F_\phi}{\partial s}-\frac{\partial F_s}{\partial\phi}\right)e_z \quad (49)$$

3.4. Laplacian

We already obtain the part such given in (37) for $\nabla^2 S=*d*dS$. Now we take its star

$$*dS=\frac{\partial S}{\partial s}*e_s+\frac{1}{s}\frac{\partial S}{\partial\phi}*e_\phi+\frac{\partial S}{\partial z}*e_z \quad (50)$$

from (4), we are led to

$$*dS=\frac{\partial S}{\partial s}(e_\phi\wedge e_z)+\frac{1}{s}\frac{\partial S}{\partial\phi}(e_z\wedge e_s)+\frac{\partial S}{\partial z}(e_s\wedge e_\phi). \quad (51)$$

We convert the basis one-forms into differential one-forms, because we will take the differentiation

$$*dS=s\frac{\partial S}{\partial s}(d\phi\wedge dz)+\frac{1}{s}\frac{\partial S}{\partial\phi}(dz\wedge ds)+\frac{\partial S}{\partial z}s(ds\wedge d\phi) \quad (52)$$

and take the differential of it, and get

$$d*dS=\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right)(ds\wedge d\phi\wedge dz)+\frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial S}{\partial\phi}\right)(ds\wedge dz\wedge ds)$$
$$+\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial z}\right)(ds\wedge ds\wedge d\phi)+\frac{\partial}{\partial\phi}\left(s\frac{\partial S}{\partial s}\right)(d\phi\wedge d\phi\wedge dz)$$
$$+\frac{\partial}{\partial\phi}\left(\frac{1}{s}\frac{\partial S}{\partial\phi}\right)(d\phi\wedge dz\wedge ds)+\frac{\partial}{\partial\phi}\left(s\frac{\partial S}{\partial z}\right)(d\phi\wedge ds\wedge d\phi)$$
$$+\frac{\partial}{\partial z}\left(s\frac{\partial S}{\partial s}\right)(dz\wedge d\phi\wedge dz)+\frac{\partial}{\partial z}\left(\frac{1}{s}\frac{\partial S}{\partial\phi}\right)(dz\wedge dz\wedge ds)$$
$$+\frac{\partial}{\partial z}\left(s\frac{\partial S}{\partial z}\right)(dz\wedge ds\wedge d\phi) \quad (53)$$

The repeating terms in wedge product of basis-one forms or differential one-forms vanishes, and the remaining is

$$d*dS=\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right)(ds\wedge d\phi\wedge dz)$$
$$+\frac{\partial}{\partial\phi}\left(\frac{1}{s}\frac{\partial S}{\partial\phi}\right)(d\phi\wedge dz\wedge ds)+\frac{\partial}{\partial z}\left(s\frac{\partial S}{\partial z}\right)(ds\wedge ds\wedge d\phi). \quad (54)$$

Before we take its star again, we convert the dq_i's into e_i's, and use (5), then get

$$d*dS=\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right)+\frac{1}{s}\frac{\partial}{\partial\phi}\left(\frac{1}{s}\frac{\partial S}{\partial\phi}\right)+\frac{1}{s}\frac{\partial}{\partial z}\left(s\frac{\partial S}{\partial z}\right)$$
$$*d*dS=\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right)+\frac{1}{s^2}\frac{\partial}{\partial\phi}\left(\frac{\partial S}{\partial\phi}\right)+\frac{\partial}{\partial z}\left(\frac{\partial S}{\partial z}\right) \quad (55)$$

and this is already the Laplacian itself;

$$\nabla^2 S=*d*dS=\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right)+\frac{1}{s^2}\frac{\partial}{\partial\phi}\left(\frac{\partial S}{\partial\phi}\right)+\frac{\partial}{\partial z}\left(\frac{\partial S}{\partial z}\right).$$

4. Spherical Coordinates

The scalar field in spherical coordinates is written as

$$S=S(r,\theta,\phi) \quad (56)$$

and the vector field is defined to be

$$F = F_r e_r + F_\theta e_\theta + F_\phi e_\phi \tag{57}$$

The basis one-forms are also written for vector

$r = xe_x + ye_y + ze_z = r\sin\theta\cos\phi e_x + r\sin\theta\sin\phi e_y + r\cos\theta e_z$ from (3), such that

$$\begin{aligned} e_r &= \left|\frac{\partial r}{\partial r}\right| dr = |\sin\theta\cos\theta e_x + \sin\theta\sin\theta e_y + \cos\theta e_z| dr \\ &= (\sin^2\theta + \cos^2\theta)^{1/2} dr = dr \Rightarrow e_r = dr \\ e_\theta &= \left|\frac{\partial r}{\partial \theta}\right| d\theta = |r\cos\theta\cos\phi e_x + r\cos\theta\sin\phi e_y + r\sin\theta e_z| d\theta \\ &= (r^2)^{1/2} d\theta = r d\theta \Rightarrow e_\theta = r d\theta \\ e_\phi &= \left|\frac{\partial r}{\partial \phi}\right| d\phi = |-r\sin\theta\sin\phi e_x + r\sin\theta\cos\phi e_y| d\theta \\ &= (r^2\sin^2\theta)^{1/2} d\theta \Rightarrow e_\phi = r\sin\theta d\phi \end{aligned} \tag{58}$$

4.1. Gradient

From (6) for the scalar S we can see

$$\nabla S = dS = \frac{\partial S}{\partial r} dr + \frac{\partial S}{\partial \theta} + \frac{\partial S}{\partial \phi} d\phi \tag{59}$$

and by using (58) to make the gradient a vector change dq_i's into e_i's

$$\begin{aligned} \nabla S = dS &= \frac{\partial S}{\partial r} e_r + \frac{\partial S}{\partial \theta}\frac{e_\theta}{r} + \frac{\partial S}{\partial \phi}\frac{e_\phi}{r\sin\theta} \\ \nabla S = dS &= \frac{\partial S}{\partial r} e_r + \frac{1}{r}\frac{\partial S}{\partial \theta} e_\theta + \frac{1}{r\sin\theta}\frac{\partial S}{\partial \phi} e_\phi . \end{aligned} \tag{60}$$

4.2. Divergence

By equation (7) for the given vector F in (57) in spherical coordinates, we obtain

$$F = F_r e_r + F_\theta e_\theta + F_\phi e_\phi \tag{61}$$

$$*F = F_r * e_r + F_\theta * e_\theta + F_\phi * e_\phi \tag{62}$$

$$= F_r (e_\theta \wedge e_\phi) + F_\theta (e_\phi \wedge e_r) + F_\phi (e_r \wedge e_0). \tag{63}$$

After this step we will need $d * F$, so we should convert e_i's into dq_is from (58)

$$*F = r^2 \sin\theta F_r (d\theta \wedge d\phi) + r\sin\theta F_\theta (d\phi \Lambda dr) + rF_\phi (dr \wedge d\theta) \tag{64}$$

$$\begin{aligned} d * F &= \frac{\partial}{\partial r}(r^2\sin\theta F_r)(dr\wedge d\theta\wedge d\phi) \\ &+ \frac{\partial}{\partial r}(r\sin\theta F_\theta)(dr\wedge d\phi\wedge dr) + \frac{\partial}{\partial r}(rF_\phi)(dr\wedge dr\wedge d\theta) \\ &\frac{\partial}{\partial \theta}(r^2\sin\theta F_r)(d\theta\wedge d\theta\wedge d\phi) \\ &+ \frac{\partial}{\partial \theta}(r\sin\theta F_\theta)(d\theta\wedge d\phi\wedge dr) + \frac{\partial}{\partial \theta}(rF_\phi)(d\theta\wedge dr\wedge d\theta) \\ &\frac{\partial}{\partial \phi}(r^2\sin\theta F_r)(d\phi\wedge d\theta\wedge d\phi) \\ &+ \frac{\partial}{\partial \phi}(r\sin\theta F_\theta)(d\phi\wedge d\phi\wedge dr) + \frac{\partial}{\partial \phi}(rF_\phi)(d\phi\wedge dr\wedge d\theta) \end{aligned} \tag{65}$$

After canceling the repeating indices in triple wedge products $dq_i \wedge dq_j \wedge dq_k = \varepsilon_{ijk}$

Now we will convert dq_i's into e_i's, because we will take their Hodge in the final step,

$$\begin{aligned} d * F &= \frac{\partial}{\partial r}(r^2\sin\theta F_r)\frac{(e_r\wedge e_\theta\wedge e_\phi)}{r^2\sin\theta} \\ &+ \frac{\partial}{\partial \theta}(r\sin\theta F_\theta)\frac{(e_\theta\wedge e_\phi\wedge e_r)}{r^2\sin\theta} + \frac{\partial}{\partial \phi}(rF_\phi)\frac{(e_\phi\wedge e_r\wedge e_\theta)}{r^2\sin\theta} \\ &= \frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}(r^2\sin\theta F_r)(e_r\wedge e_\theta\wedge e_\phi) \\ &+ \frac{\partial}{\partial \theta}(r\sin\theta F_\theta)(e_r\wedge e_\theta\wedge e_\phi) + \frac{\partial}{\partial \phi}(rF_\phi)(e_r\wedge e_\theta\wedge e_\phi) \end{aligned} \tag{66}$$

$$*d * F = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi} \tag{67}$$

where we have used (5) and obtained the divergence equation in cylindrical coordinates for a vector field F

$$\nabla\cdot F = *d*F = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi}$$

4.3. Curl

We begin by writing F with dq_i's instead of e_i's, so

$$F = F_r dr + rF_\theta d\theta + r\sin\theta F_\phi d\phi \tag{68}$$

and we take the exterior differentiation for using in $\nabla\times F = *dF$ in (8) as follows,

$$dF = \frac{\partial F_r}{\partial r} dr \wedge dr + \frac{\partial (rF_\theta)}{\partial r} dr \wedge d\theta + \frac{\partial (r \sin\theta F_\phi)}{\partial r} dr \wedge d\phi$$
$$+ \frac{\partial F_r}{\partial \theta} d\theta \wedge dr + \frac{\partial (rF_\theta)}{\partial \theta} d\theta \wedge d\theta + \frac{\partial (r \sin\theta F_\phi)}{\partial \theta} d\theta \wedge d\phi$$
$$+ \frac{\partial F_r}{\partial \phi} d\phi \wedge dr + \frac{\partial (rF_\theta)}{\partial r\phi} d\phi \wedge d\theta + \frac{\partial (r \sin\theta F_\phi)}{\partial \phi} d\phi \wedge d\phi. \tag{69}$$

We expect to get a vector at the end of the calculation, so we convert the differential one-forms into basis one-forms, but before eliminate the repeating $dq_i \wedge dq_i = 0$ terms, and remaining is

$$dF = \frac{\partial (rF_\theta)}{\partial r} \frac{e_r \wedge e_\theta}{r} + \frac{\partial (r \sin\theta F_\phi)}{\partial r} \frac{e_r \wedge e_\phi}{r \sin\theta} + \frac{\partial F_r}{\partial \theta} \frac{e_\theta \wedge e_r}{r}$$
$$+ \frac{\partial (r \sin\theta F_\phi)}{\partial \theta} \frac{e_\theta \wedge e_\phi}{r^2 \sin\theta} + \frac{\partial F_r}{\partial \phi} \frac{e_\phi \wedge e_r}{r \sin\theta} + \frac{\partial (rF_\theta)}{\partial r\phi} \frac{e_\phi \wedge e_\theta}{r^2 \sin\theta} \tag{70}$$

When we apply Hodge operator on (70) and by using (4), we obtain

$$*dF = \frac{1}{r} \frac{\partial (rF_\theta)}{\partial r} e_\phi - \frac{1}{r \sin\theta} \frac{\partial (r \sin\theta F_\phi)}{\partial r} e_\theta - \frac{1}{r} \frac{\partial F_r}{\partial \theta} e_\phi$$
$$+ \frac{1}{r^2 \sin\theta} \frac{\partial (r \sin\theta F_\phi)}{\partial \theta} e_r + \frac{1}{r \sin\theta} \frac{\partial F_r}{\partial \phi} e_\theta - \frac{1}{r^2 \sin\theta} \frac{\partial (rF_\theta)}{\partial r\phi} e_r \tag{71}$$

By rearranging it, we finally obtain the curl

$$\nabla \times F = *dF = \frac{1}{r \sin\theta} \frac{\partial (\sin\theta F_\phi)}{\partial \theta} - \frac{\partial F_\theta}{\partial r\phi} e_r + \frac{1}{r} \frac{1}{\sin\theta} \frac{\partial F_r}{\partial \phi}$$
$$- \frac{\partial (rF_\phi)}{\partial r} e_\theta + \frac{1}{r} \frac{\partial (rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta} e_\phi \tag{72}$$

4.4. Laplacian

We already obtain the part dS such given in (59) for $\nabla^2 S = *d*dS$ and now we take its star as

$$*dS = \frac{\partial S}{\partial r} * e_r + \frac{1}{r} \frac{\partial S}{\partial \theta} * e_\theta + \frac{1}{r \sin\theta} \frac{\partial S}{\partial \phi} * e_\phi \tag{73}$$

from (4), we are led to

$$*dS = \frac{\partial S}{\partial r} (e_\theta \wedge e_\phi) + \frac{1}{r} \frac{\partial S}{\partial \theta} (e_\phi \wedge e_r) + \frac{1}{r \sin\theta} \frac{\partial S}{\partial \phi} (e_r \wedge e_\theta). \tag{74}$$

We convert the basis one-forms into differential one-forms, because we will take the differentiation

$$*dS = r^2 \sin\theta \frac{\partial S}{\partial r} (d\theta \wedge d\phi) + \sin\theta \frac{\partial S}{\partial \theta} (d\phi \wedge dr)$$
$$+ \frac{1}{\sin\theta} \frac{\partial S}{\partial \phi} (dr \wedge d\theta) \tag{75}$$

and take the differential of it, and get

$$d*dS = \frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial S}{\partial r} \right)(dr \wedge d\theta \wedge d\phi)$$
$$+ \frac{\partial}{\partial r}\left(\sin\theta \frac{\partial S}{\partial \theta} \right)(dr \wedge d\phi \wedge dr)$$
$$+ \frac{\partial}{\partial r}\left(\frac{1}{\sin\theta} \frac{\partial S}{\partial \phi} \right)(dr \wedge dr \wedge d\theta)$$
$$+ \frac{\partial}{\partial \theta}\left(r^2 \sin\theta \frac{\partial S}{\partial r} \right)(d\theta \wedge d\theta \wedge d\phi)$$
$$+ \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial S}{\partial \theta} \right)(d\theta \wedge d\phi \wedge dr)$$
$$+ \frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta} \frac{\partial S}{\partial \phi} \right)(d\theta \wedge dr \wedge d\theta)$$
$$+ \frac{\partial}{\partial \phi}\left(r^2 \sin\theta \frac{\partial S}{\partial r} \right)(d\phi \wedge d\theta \wedge d\phi)$$
$$+ \frac{\partial}{\partial \phi}\left(\sin\theta \frac{\partial S}{\partial \theta} \right)(d\phi \wedge d\phi \wedge dr)$$
$$+ \frac{\partial}{\partial \phi}\left(\frac{1}{\sin\theta} \frac{\partial S}{\partial \phi} \right)(d\phi \wedge dr \wedge d\theta) \tag{76}$$

The repeating terms in wedge product of basis-one forms or differential one-forms vanishes, and the remaining is

$$d*dS = \frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial S}{\partial r} \right)(dr \wedge d\theta \wedge d\phi)$$
$$+ \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial S}{\partial \theta} \right)(d\theta \wedge d\phi \wedge dr)$$
$$+ \frac{\partial}{\partial \phi}\left(\frac{1}{\sin\theta} \frac{\partial S}{\partial \phi} \right)(d\phi \wedge dr \wedge d\theta) \tag{77}$$

Before we take its star again, we convert the dq_i's into e_i's, and use (5), then get

$$d*dS = \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial S}{\partial r} \right)$$
$$+ \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial S}{\partial \theta} \right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\sin\theta} \frac{\partial S}{\partial \phi} \right)(e_r \wedge e_\theta \wedge e_\phi)$$
$$*d*dS = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial S}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial S}{\partial \theta} \right)$$
$$+ \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 S}{\partial \phi^2}) \tag{78}$$

and this is already the Laplacian itself;

$$\nabla^2 S = *d*dS = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial S}{\partial r} \right)$$
$$+ \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial S}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 S}{\partial \phi^2}$$

5. Generalized Coordinates

The scalar field in generalized coordinates is written as

$$S = S(q_i), \quad i = 1,2,3 \tag{79}$$

and the vector field is defined to be

$$F = \sum_{i=1}^{3} F_i e_i . \qquad (80)$$

The basis one-forms are also written for vector $r = \sum_{j=1}^{3} q_j e_j$ from (3), such that

$$e_i = \left| \frac{\partial r}{\partial q_i} \right| dq_i = h_i dq_i \qquad (81)$$

5.1. Gradient

From (6) for the scalar S we can see

$$\nabla S = dS = \sum_i \frac{\partial S}{\partial q_i} dq_i \qquad (82)$$

and by using (35) to make the gradient a vector change dq_i's into e_i's

$$\nabla S = dS = \sum_i \frac{1}{h_i} \frac{\partial S}{\partial q_i} e_i . (83)$$

5.2. Divergence

By equation (7) for the given vector F in (80) in spherical coordinates, we obtain

$$F = \sum_{i=1}^{3} F_i e_i \qquad (84)$$

then,

$$*F = \sum_{i=1}^{3} F_i * e_i = \sum_{i,j,k} F_i \left(\varepsilon_{ijk} e_j \wedge e_k \right) \qquad (85)$$

After this step we will need $d * F$, so we should convert e_i's into dq_i's from (80)

$$*F = \sum_{i,j,k} h_k h_j F_i \varepsilon_{i,j,k} \left(dq_j \wedge dq_k \right) \qquad (86)$$

$$d * F = \sum_{i,j,k} \frac{\partial}{\partial q_i} \left(h_k h_j F_i \varepsilon_{ijk} \right) \left(dq_i \wedge dq_j \wedge dq_k \right) \qquad (87)$$

After canceling the repeating indices in triple wedge products $dq_i \wedge dq_j \wedge dq_k = \varepsilon_{ijk}$

Now we will convert dq_i's into e_i's, because we will take their Hodge in the final step,

$$\begin{aligned} d * F &= \sum_{i,j,k} \frac{\partial}{\partial q_i} \left(h_k h_j F_i \varepsilon_{ijk} \right) \frac{\left(e_i \wedge e_j \wedge e_k \right)}{h_i h_j h_k} \\ d * F &= \sum_{i,j,k} \frac{1}{h_i h_j h_k} \frac{\partial}{\partial q_i} \left(h_k h_j F_i \varepsilon_{ijk} \right) \left(e_i \wedge e_j \wedge e_k \right) \end{aligned} \qquad (88)$$

$$*d * F = \sum_{i,j,k} \frac{\varepsilon_{ijk}}{h_i h_j h_k} \frac{\partial}{\partial q_i} \left(h_k h_j F_i \right) \qquad (89)$$

where we have used (5) and obtained the divergence equation in cylindrical coordinates for a vector field F

$$\nabla \cdot F = *d * F = \sum_{i,j,k} \frac{\varepsilon_{ijk}}{h_i h_j h_k} \frac{\partial}{\partial q_i} \left(h_k h_j F_i \right)$$

5.3. Curl

We begin by using (8) and writing F with dq_i's instead of e_i's, so

$$F = \sum_{i=1}^{3} F_i h_i dq_i \qquad (90)$$

and we take the exterior differentiation for using in $\nabla \times F = *dF$ as follows,

$$dF = \sum_{j,i} \frac{\partial \left(h_i F_i \right)}{\partial q_j} \left(dq_j \wedge dq_i \right) \qquad (91)$$

We expect to get a vector at the end of the calculation, so we convert the differential one-forms into basis one-forms, but before eliminate the repeating $dq_i \wedge dq_i = 0$ terms, and remaining is

$$dF = \sum_{j,i} \frac{1}{h_j h_i} \frac{\partial \left(h_i F_i \right)}{\partial q_j} \left(e_j \wedge e_i \right) \qquad (92)$$

When we apply Hodge operator on (92) and by using (4), we obtain

$$*dF = \sum_{i,j,k} \frac{1}{h_j h_i} \frac{\partial \left(h_i F_i \right)}{\partial q_j} \varepsilon_{ijk} e_k \qquad (93)$$

By rearranging it, we finally obtain the curl

$$\nabla \times F = *dF = \sum_{i,j,k} \frac{\varepsilon_{ijk}}{h_i h_j} \frac{\partial \left(h_i F_i \right)}{\partial q_j} e_k \qquad (94)$$

5.4. Laplacian

We already obtain the part such given in (83) for $\nabla^2 S = *d * dS$ and its star is

$$*dS = \sum_i \frac{1}{h_i} \frac{\partial S}{\partial q_i} * e_i (95)$$

from (4), we are led to

$$*dS = \sum_{i,j,k} \frac{1}{h_i} \frac{\partial S}{\partial q_i} \left(\varepsilon_{ijk} e_j \wedge e_k \right). \qquad (96)$$

We convert the basis one-forms into differential one-forms, because we will take the differentiation

$$*dS = \sum_{i,j,k} \varepsilon_{ijk} \frac{h_j h_k}{h_i} \frac{\partial S}{\partial q_i} \left(dq_j \wedge dq_k \right) \qquad (97)$$

and take the differential of it, and get

$$d * dS = \sum_{i,j,k} \varepsilon_{ijk} \frac{\partial}{\partial q_i} \frac{h_j h_k}{h_i} \frac{\partial S}{\partial q_i} \left(dq_i \wedge dq_j \wedge dq_k \right) \qquad (98)$$

The repeating terms in wedge product of basis-one forms or differential one-forms vanishes, and the remaining is

$$d * dS = \sum_{i,j,k} \varepsilon_{ijk} \frac{\partial}{\partial q_i} \frac{h_j h_k}{h_i} \frac{\partial S}{\partial q_i} \left(dq_i \wedge dq_j \wedge dq_k \right) \qquad (99)$$

Before we take its star again, we convert the dq_i's into e_i's, and use (5), then get

$$*d*dS = \sum_{i,j,k} \frac{\varepsilon_{ijk}}{h_i h_j h_k} \frac{\partial}{\partial q_i} \frac{h_j h_k}{h_i} \frac{\partial S}{\partial q_i} \quad (100)$$

and this is already the Laplacian itself;

$$\nabla^2 S = *d*dS = \sum_{i,j,k} \frac{\varepsilon_{ijk}}{h_i h_j h_k} \frac{\partial}{\partial q_i} \frac{h_j h_k}{h_i} \frac{\partial S}{\partial q_i}.$$

6. Conclusion

In this study, we derive the various derivative operators on functions of fields in various coordinate systems by using the differential one-forms, Hodge star operators and the exterior derivative operators. For a classical inner product we use $*A*B$ to imply $A \cdot B$. Also for a classical nabla operator, we use the exterior differentiation *d*. If we use the Hodge operator on a function, the one-forms in this function must be in terms of e_i's. If we use differential (or exterior derivative) operator on a function, the one-forms in this function must be in terms of dq_i's. With the considerations just mentioned, we ensure that, we can obtain the gradient, divergence, curl and Laplacian relations for any functions in any kind of generalized coordinates.

7. References

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