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Research Article

Integral Inequalities for Different Kinds of Convex Functions Involving Riemann-Liouville Fractional Integrals

Konveks Fonksiyonların Farklı Tipleri İçin Riemann-Liouville Kesirli İntegrallerini İçeren İntegral Eşitsizlikleri

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Abstract

In this paper, we obtain some new integral inequalities for different kinds of co-ordinated convex functions by using elemantery analysis and Riemann-Liouville fractional integrals.

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Öz

Bu makalede, Riemann-Liouville kesirli integralleri ve elementer analiz işlemleri kullanılarak coordinatlarda konveks fonksiyonların farklı tipleri için bazı yeni integral eşitsizlikleri elde edilmiştir.

Anahtar Kelimeler: Konveks fonksiyonlar, Ko-ordinatlar, Riemann-Liouville kesirli integralleri

1. Introduction

Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a<b$. The following inequality;

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}$$

is known in the literature as Hadamard's inequality for convex mappings.

In (Dragomir 2001), Dragomir defined convex functions on the co-ordinates as following:

Definition 1.1 *Let us consider the bidimensional interval $\Delta=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$, $c<d$. A function $f:\Delta\to\mathbb{R}$ will be called convex on the co-ordinates if the partial mappings $f_y:[a,b]\to\mathbb{R}$, $f_y:(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}$, $f_x(v)=f(x,v)$ are convex where defined for all $y\in[c,d]$ and $x\in[a,b]$. Recall that the mapping $f:\Delta\to\mathbb{R}$ is convex on Δ if the following inequality holds,*

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w) \leq \lambda f(x,y)+(1-\lambda)f(z,w)$$

for all $(x,y),(z,w)\in\Delta$ and $\lambda\in[0,1]$.

Every convex function is co-ordinated convex but the converse is not generally true.

In (Dragomir 2001), Dragomir established the following inequalities of Hadamard's type for co-ordinated convex functions on a rectangle from the plane $\mathbb{R}^2$.

Theorem 1.1 *Suppose that $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ is convex on the co-ordinates on Δ. Then one has the inequalities;*

$$\begin{aligned} & f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\ & \leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy \\ & \leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned} \tag{1.1}$$

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The above inequalities are sharp.

Similar results can be found in [(Dragomir 2001)-(Özdemir *et al. 2012*)].

In (Özdemir *et al.* 2011), Özdemir *et al.* defined co-ordinated m-convex functions as following:

Definition 1.2 *Let us consider the bidimensional interval* $\Delta=[0,b]\times[0,d]$ *in* $[0,\infty)^2$. *The mapping* $f:\Delta\to\mathrm{R}$ *is m-convex on* Δ *if*

$$f(tx+(1-t)z,ty+m(1-t)w)\le tf(x,y)+m(1-t)f(z,w) \tag{1.2}$$

holds for all $(x,y),(z,w)\in\Delta$ and $t\in[0,1]$, $b,d>0$ and for some fixed $m\in[0,1]$.

In (Akdemir and Özdemir 2010), Akdemir and Özdemir defined Godunova-Levin functions and *P*-functions on the co-ordinates as followings and proved some integral inequalities:

Definition 1.3 *Let us consider the bidimensional interval* $\Delta=[a,b]\times[c,d]$ *in* R^2 *with a<b, c<d. A function* $f:\Delta\to\mathrm{R}$ *is said to belong to the class of Q(I) if it is nonnegative and for all* $(x,y),(z,w)\in\Delta$ *and* $\lambda\in(0,1)$ *satisfies the following inequality;*

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)\le\frac{f(x,y)}{\lambda}+\frac{f(z,w)}{1-\lambda}. \tag{1.3}$$

We denote this class of functions by $QX(f,\Delta)$. If the inequality reversed then *f* is said to be concave on Δ and we denote this class of functions by $QV(f,\Delta)$.

Definition 1.4 *Let* $f:\Delta[a,b]\times[c,d]\to\mathrm{R}$ *be a P-function with a<b, c<d. If it is nonnegative and for all* $(x,y),(z,w)\in\Delta$ *and* $\lambda\in(0,1)$ *the following inequality holds:*

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)\le f(x,y)+f(z,w) \tag{1.4}$$

We denote this class of functions by $PX(f,\Delta)$.

Theorem 1.2 *Suppose that* $f:\Delta[a,b]\times[c,d]\to\mathrm{R}$ *is said to belong to the class* $QX(f,\Delta)$ *on the co-ordinates on* Δ *with* $f_x\in L_1[c,d]$ *and* $f_y\in L_1[a,b]$, *then one has the inequalities:*

$$\begin{aligned}&\frac{1}{16}\left[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]\\&\le\frac{1}{8}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]\\&\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\end{aligned} \tag{1.5}$$

Theorem 1.3 *Suppose that* $f:\Delta[a,b]\times[c,d]\to\mathrm{R}$ *is said to belong to the class* $PX(f,\Delta)$ *on the co-ordinates on* Δ *with* $f_x\in L_1[c,d]$ *and* $f_y\in L_1[a,b]$, *then one has the inequalities:*

$$\begin{aligned}&F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\le\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\\&\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\\&\le\frac{4}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\\&\le\frac{2}{(b-a)}\left[\int_a^b f(x,c)\,dx+\int_a^b f(x,d)\,dx\right]\\&+\frac{2}{(d-c)}\left[\int_c^d f(a,y)\,dy+\int_c^d f(b,y)\,dy\right].\end{aligned} \tag{1.6}$$

A formal definition for co-ordinated convex functions may be stated as follow (see (Latif and Alomari 2009)):

Definition 1.5 *A function* $f:\Delta\to\mathrm{R}$ *is said to be convex on the co-ordinates on* Δ *if the following inequality:*

$$\begin{aligned}&f(tx+(1-t)y,su+(1-s)w)\\&\le tsf(x,u)+t(1-s)f(x,w)+s(1-t)f(y,u)+(1-t)\\&(1-s)f(y,w)\end{aligned} \tag{1.7}$$

holds for all $t,s\in[0,1]$ and $(x,u),(x,w),(y,u),(y,w)\in\Delta$.

Similar to definition of co-ordinated convex functions Latif and Alomari gave the notion of *h*-convexity of a function *f* on a rectangle from the plane R^2 and *h*-convexity on the co-ordinates on a rectangle from the plane R^2 in (Latif and Alomari 2009), as follows:

Definition 1.6 *Let us consider a bidimensional interval* $\Delta=:[a,b]\times[c,d]$ *in* R^2 *with a<b and c<d. Let* $h:J\subseteq\mathrm{R}\to\mathrm{R}$ *where* $(0,1)\subseteq J$, *be a positive function. A mapping* $f:\Delta=:[a,b]\times[c,d]\to\mathrm{R}$ *is said to be h-convex on* Δ, *if f is non-negative and if the following inequality:*

$$f(\alpha x+(1-\alpha)z,\alpha y+(1-\alpha)w)\le h(\alpha)f(x,y)+h(1-\alpha)f(z,w)$$

holds, for all $(x,y),(z,w)\in\Delta$ and $\alpha\in(0,1)$. Let us denote this class of functions by $SX(h,\Delta)$. The function *f* is said to be *h*-concave if the inequality reversed. We denote this class of functions by $SV(h,\Delta)$.

A formal definition of *h*-convex functions may also be stated as follows (see Latif and Alomari (2009)):

Definition 1.7 *A function* $f:\Delta\to\mathrm{R}$ *is said to be h-convex on the co-ordinates on* Δ, *if the following inequality:*

$$\begin{aligned}&f(tx+(1-t)y,su+(1-s)w)\le h(t)h(s)f(x,u)+\\&h(t)h(1-s)f(x,w)+h(s)h(1-t)f(y,u)+\\&h(1-t)h(1-s)f(y,w)\end{aligned} \tag{1.8}$$

holds for all $t,s\in[0,1]$ and $(x,u),(x,w),(y,u),(y,w)\in\Delta$.

In (Latif and Alomari 2009), Latif and Alomari proved the following inequalities for h-convex functions on the co-ordinates:

Theorem 1.4 *Let* $f:\Delta=[a,b]\times[c,d]\subset \mathbb{R}^2\to\mathbb{R}$ *be an h-convex function on the co-ordinates on Δ and let $f\in L_2(\Delta)$ and $h\in L_1[0,1]$. Then one has the inequalities;*

$$\begin{aligned}&\frac{1}{4h^2\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\&\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \qquad (1.9)\\&\le[f(a,c)+f(a,d)+f(b,c)+f(b,d)]\left(\int_0^1 h(\alpha)\,d\alpha\right)^2.\end{aligned}$$

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 1.8 *Let* $f\in L_1[a,b]$. *The Riemann-Liouville integrals* $J_{a^+}^{\alpha}f$ *and* $J_{b^-}^{\alpha}f$ *of order* $\alpha>0$ *with* $a\geq 0$ *are defined by*

$$J_{a^+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)\,dt,\quad x>a$$

and

$$J_{b^-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)\,dt,\quad x<b$$

where $\Gamma(\alpha)=\int_0^\infty e^{-t}u^{\alpha-1}du$, here is $J_{a^+}^{0}f(x)=J_{b^-}^{0}f(x)=f(x)$.

In the case of $\alpha=1$ the fractional integral reduces to the classical integral. Properties of this operator can be found in the references [(Dahmani 2010)-(Sarkıkaya *et al.* 2013)].

The aim of this paper is to establish some new integral inequalities for different kinds of convex functions via Riemann-Liouville fractional integrals.

2. Main Results

Throughout of this paper, we will use the following notation (See e.g. Sarıkaya 2014):

$$J_{b^-,d^-}^{\alpha,\beta}f(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d(x-a)^{\alpha-1}(y-c)^{\beta-1}f(x,y)dydx$$

$$J_{a^+,d^-}^{\alpha,\beta}f(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d(x-a)^{\alpha-1}(d-y)^{\beta-1}f(x,y)dydx$$

$$J_{b^-,c^+}^{\alpha,\beta}f(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d(b-x)^{\alpha-1}(y-c)^{\beta-1}f(x,y)dydx$$

$$J_{a^+,c^+}^{\alpha,\beta}f(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d(b-x)^{\alpha-1}(d-y)^{\beta-1}f(x,y)dydx.$$

Theorem 2.1 *Let* $f:\Delta=[0,b]\times[0,d]\to\mathbb{R}$ *be an m-convex function on the co-ordinates on Δ with $0\le a<b<\infty$, $0\le c<d<\infty$ and $f_x\in L_1[0,d]$, $f_y\in L_1[0,b]$. Then the following inequalities for fractional integrals with $\alpha>0$ and $m\in(0,1]$ hold:*

$$\begin{aligned}&\frac{\Gamma(\alpha)\Gamma(\beta)}{(b-a)^{\alpha}(d-c)^{\beta}}J_{a^+,c^+}^{\alpha,\beta}f(x,y)\\&\le\frac{1}{(\alpha+1)(\beta+1)}\left[\begin{array}{l}f(a,c)+\frac{1}{\alpha}f(b,c)+\frac{1}{\beta}mf\left(a,\frac{d}{m}\right)+\\ \frac{1}{\alpha\beta}mf\left(b,\frac{d}{m}\right)\end{array}\right],(2.1)\end{aligned}$$

and

$$\begin{aligned}&\frac{\Gamma(\alpha)\Gamma(\beta)}{(b-a)^{\alpha}(d-c)^{\beta}}J_{b^-,d^-}^{\alpha,\beta}f(x,y)\\&\le\frac{1}{(\alpha+1)(\beta+1)}\left[\begin{array}{l}f(b,d)+\frac{1}{\alpha}f(a,d)+\frac{1}{\beta}mf\left(b,\frac{c}{m}\right)+\\ \frac{1}{\alpha\beta}mf\left(a,\frac{c}{m}\right)\end{array}\right].\end{aligned}$$

Proof. Since f is a m-convex function on Δ, we know that for any $t,s\in[0,1]$

$$\begin{aligned}&f(ta+(1-t)b,sc+(1-s)d)\\&\le tsf(a,c)+mt(1-s)f\left(a,\frac{d}{m}\right)+s(1-t)f(b,c)+ \qquad (2.2)\\&m(1-t)(1-s)f\left(b,\frac{d}{m}\right)\end{aligned}$$

and

$$\begin{aligned}&f(tb+(1-t)a,sd+(1-s)c)\\&\le tsf(b,d)+mt(1-s)f\left(b,\frac{c}{m}\right)+s(1-t)f(a,d)+ \qquad (2.3)\\&m(1-t)(1-s)f\left(a,\frac{c}{m}\right)\end{aligned}$$

By multiplying both sides of (2.2) by $t^{\alpha-1}s^{\beta-1}$, then by integrating the resulting inequality with respect to t, s over [0,1]x[0,1], we obtain

$$\begin{aligned}&\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}f(ta+(1-t)b,sc+(1-s)d)\,dsdt\\&\le f(a,c)\int_0^1\int_0^1 t^{\alpha}s^{\beta}dsdt+f(b,c)\int_0^1\int_0^1 t^{\alpha-1}s^{\beta}(1-t)\,dsdt\\&+mf\left(a,\frac{d}{m}\right)\int_0^1\int_0^1 t^{\alpha}s^{\beta-1}(1-s)\,dsdt\\&+mf\left(b,\frac{d}{m}\right)\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}(1-t)(1-s)\,dsdt.\end{aligned}$$

It is easy to see that

$$\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}f(ta+(1-t)b,sc+(1-s)d)\,dsdt=\frac{\Gamma(\alpha)\Gamma(\beta)}{(b-a)^{\alpha}(d-c)^{\beta}}J_{a^+,c^+}^{\alpha,\beta}f(x,y)$$

and by computing the above integrals, we deduce

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{(b-a)^{\alpha}(d-c)^{\beta}}J^{\alpha,\beta}_{a^+,c^+}f(x,y)$$
$$\leq \frac{1}{(\alpha+1)(\beta+1)}\left[\begin{array}{l} f(a,c)+\frac{1}{\alpha}f(b,c)+\frac{1}{\beta}mf\left(a,\frac{d}{m}\right)+ \\ \frac{1}{\alpha\beta}mf\left(b,\frac{d}{m}\right)\end{array}\right]$$

which completes the proof of the first inequality.

For the proof of the second inequality in (2.1), we multiply both sides of (2.3) by $t^{\alpha-1}s^{\beta-1}$, then integrate the resulting inequality with respect to *t,s* over [0,1]x[0,1].

Theorem 2.2 *Let* $f:\Delta=[a,b]\times[c,d]\to \mathrm{R}$ *be Godunova-Levin function on the co-ordinates on* Δ *with* $f_x\in L_1[c,d],\ f_y\in L_1[a,b]$. *Then the following inequality for fractional integrals with* $\alpha>0$ *and* $t,s\in(0,1)$ *holds:*

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$
$$\leq \frac{4\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{\alpha}(d-c)^{\beta}} \tag{2.4}$$
$$\times\{J^{\alpha,\beta}_{b^-,d^-}f(x,y)+J^{\alpha,\beta}_{a^+,d^-}f(x,y)+J^{\alpha,\beta}_{b^-,c^+}f(x,y)+J^{\alpha,\beta}_{a^+,c^+}f(x,y)\}$$

Proof. From the definition of Godunova-Levin function which is given in (1.3), we can write

$$f(tx+(1-t)z,sy+(1-s)w)$$
$$\leq \frac{f(x,y)}{ts}+\frac{f(x,w)}{t(1-s)}+\frac{f(z,y)}{(1-t)s}+\frac{f(z,w)}{(1-t)(1-s)}. \tag{2.5}$$

If we choose $t=s=\frac{1}{2}$ in (2.5), we have

$$f\left(\frac{x+z}{2},\frac{y+w}{2}\right)\leq 4[f(x,y)+f(x,w)+f(z,y)+f(z,w)].$$

By changing of the variables $x = ta+(1-t)b$, $z = (1-t)a+tb$, $y = sc+(1-s)d$ and $w = (1-s)c+sd$, we get

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$
$$\leq 4[f(ta+(1-t)b,sc+(1-s)d)+f(ta+(1-t)b,$$
$$(1-s)c+sd)+f((1-t)a+tb,sc+(1-s)d)+$$
$$f((1-t)a+tb,(1-s)c+sd)].$$

By multiplying both sides of the resulting inequality by $t^{\alpha-1},s^{\beta-1}$, then by integrating with respect to *t,s* over [0,1] x[0,1], we obtain

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}dsdt$$
$$\leq 4\Big[\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}f(ta+(1-t)b,sc+(1-s)d)dsdt$$
$$+\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}f(ta+(1-t)b,(1-s)c+sd)dsdt$$
$$+\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}f((1-t)a+tb,sc+(1-s)d)dsdt$$
$$+\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}f((1-t)a+tb,(1-s)c+sd)dsdt\Big].$$

By computing the above integrals, we deduce

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$
$$\leq \frac{4\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{\alpha}(d-c)^{\beta}}$$
$$\times\{J^{\alpha,\beta}_{b^-,d^-}f(x,y)+J^{\alpha,\beta}_{a^+,d^-}f(x,y)+J^{\alpha,\beta}_{b^-,c^+}f(x,y)+J^{\alpha,\beta}_{a^+,c^+}f(x,y)\}$$

which completes the proof.

Remark 2.1 *In Theorem 2.2, if we choose* α =1 *we obtain the inequality (1.5).*

Theorem 2.3 *Let* $f:\Delta=[a,b]\times[c,d]\to \mathrm{R}$ *be P-function on the co-ordinates on* Δ *with* $f_x\in L_1[c,d],\ f_y\in L_1[a,b]$. *Then the following inequality for fractional integrals with* $\alpha>0$ *and* $t,s\in[0,1]$ *holds:*

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$
$$\leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{\alpha}(d-c)^{\beta}} \tag{2.6}$$
$$\times[J^{\alpha,\beta}_{b^-,d^-}f(x,y)+J^{\alpha,\beta}_{a^+,d^-}f(x,y)+J^{\alpha,\beta}_{b^-,c^+}f(x,y)+J^{\alpha,\beta}_{a^+,c^+}f(x,y)].$$

Proof. By a similar argument to the proof of Theorem 2.2, by using the definition of *P*-function which is given in (1.4), the proof is completed.

Remark 2.2 *In Theorem 2.3, if we choose* α =1, *we obtain the inequality (1.6).*

Theorem 2.4 *Let* $f:\Delta=[a,b]\times[c,d]\to \mathrm{R}$ *be an h-convex function on the co-ordinates on* Δ *with* $f_x\in L_1[c,d],\ f_y\in L_1[a,b],\ h\in L_1[0,1]$ *where h is a positive functions defined on J such that* $(0,1)\subseteq J\subseteq \mathrm{R}$. *Then the following inequalities for fractional integrals with* $\alpha>0$ *and* $t,s\in[0,1]$ *hold:*

$$\frac{1}{\alpha\beta h^2\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$
$$\leq \frac{\Gamma(\alpha)\Gamma(\beta)}{(b-a)^{\alpha}(d-c)^{\beta}}$$
$$\times[J^{\alpha,\beta}_{b^-,d^-}f(x,y)+J^{\alpha,\beta}_{a^+,d^-}f(x,y)+J^{\alpha,\beta}_{b^-,c^+}f(x,y)+J^{\alpha,\beta}_{a^+,c^+}f(x,y)]$$
$$\leq (f(a,c)+f(b,d))\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}\left[\begin{array}{l}h(t)h(s)+h(1-t)\\ h(1-s)\end{array}\right]dsdt$$
$$+(f(b,c)+f(a,d))\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}\left[\begin{array}{l}h(t)h(1-s)+h(s)\\ h(1-t)\end{array}\right]dsdt. \tag{2.7}$$

Proof. According to the definition which is given in (1.8) with $t=s=\frac{1}{2}$ and then if we set $x = ta+(1-t)b$, $y = (1-t)$

$a+tb$, $u = sc+(1-s)d$ and $w = (1-s)c+sd$, we have

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \leq h^2\left(\frac{1}{2}\right)\begin{bmatrix} f(ta+(1-t)b, sc+(1-s)d)+f(ta+(1-t)b, \\ (1-s)c+sd)+f(1-t)a+tb, sc+(1-s)d)+ \\ f((1-t)a+tb,(1-s)c+sd) \end{bmatrix}.$$

By multiplying both sides of the resulting inequality by $t^{\alpha-1}s^{\beta-1}$, then by integrating with respect to t,s over [0,1] x[0,1], we obtain

$$\begin{aligned} &f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}dsdt \\ &\leq h^2\left(\frac{1}{2}\right)\Big[\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}f(ta+(1-t)b, sc+(1-s)d)\,dsdt \\ &+\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}f(ta+(1-t)b,(1-s)c+sd)\,dsdt \\ &+\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}f((1-t)a+tb, sc+(1-s)d)\,dsdt \\ &+\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}f((1-t)a+tb,(1-s)c+sd)\,dsdt\Big]. \end{aligned}$$

By a simple computation, we get the first inequality of (2.7).

Since f is a h-convex function on Δ we can write

$$\begin{aligned} &f(ta+(1-t)b, sc+(1-s)d) \leq h(t)h(s)f(a,c)+ \\ &h(t)h(1-s)f(a,d)+h(s)h(1-t)f(b,c)+ \\ &h(1-t)h(1-s)f(b,d) \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} &f(tb+(1-t)a, sd+(1-s)c) \leq h(t)h(s)f(b,d)+ \\ &h(t)h(1-s)f(b,c)+h(s)h(1-t)f(a,d)+ \\ &h(1-t)h(1-s)f(a,c). \end{aligned} \tag{2.9}$$

By adding the inequalities (2.8) and (2.9), we have

$$\begin{aligned} &f(ta+(1-t)b, sc+(1-s)d)+f(tb+(1-t)a, sd+(1-s)c) \\ &\leq [h(t)h(s)+h(1-t)h(1-s)](f(a,c)+f(b,d)) \\ &+[h(t)h(1-s)+h(s)h(1-t)](f(b,c)+f(a,d)). \end{aligned}$$

By multiplying both sides of the resulting inequality by $t^{\alpha-1}s^{\beta-1}$, then by integrating with respect to t,s over [0,1] x[0,1], we obtain

$$\begin{aligned} &\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}f(ta+(1-t)b, sc+(1-s)d)\,dsdt \\ &+\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}f(tb+(1-t)a, sd+(1-s)c)\,dsdt \\ &\leq (f(a,c)+f(b,d))\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}\begin{bmatrix} h(t)h(s)+h(1-t) \\ h(1-s) \end{bmatrix}dsdt \\ &+(f(b,c)+f(a,d))\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}\begin{bmatrix} h(t)h(1-s)+h(s) \\ h(1-t) \end{bmatrix}dsdt \end{aligned}$$

which completes the proof.

Remark 2.3 *In Theorem 2.4, if we choose h(t) = t and* α =1, *we obtain the inequality (1.1).*

Remark 2.4 *In Theorem 2.4, if we choose* α =1 *we obtain the inequality (1.9).*

3. References

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