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Research Article

On The Fundamental Units of Certain Real Quadratic Number Fields

Bazı Reel Kuadratik Sayı Cisimlerinin Temel Birimleri Üzerine

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Abstract

In this paper, we consider the real quadratic fields $\mathbb{Q}(\sqrt{d})$ where *d* is a square free positive integer congruent to 1(*mod*4). We construct the parametrization of *d* which correspond to some types of real quadratic fields including a specific kind of continued fraction expansion. Then, we determine the explicit representation of fundamental unit and obtain some results on Yokoi's invariants. Besides, we give several tables for which satisfy the obtained results. In this paper, the recent results of the paper (Özer 2016a) have also been extended and completed in the case of d≡1(*mod*4).

Keywords: Continued fraction expansion, Fundamental unit, Quadratic fields, Yokoi's invariants

Öz

Bu makalede, *d*, (*mod*4)'e göre 1'e denk olan kare çarpansız bir pozitif tamsayı olmak üzere $\mathbb{Q}(\sqrt{d})$ reel kuadratik cisimleri göz önüne almaktayız. Sürekli kesir açılımının özel bir çeşidini içeren reel kuadratik sayı cisimlerinin bazı tiplerine karşılık gelen *d* nin parametrik ifade edilişini belirlemekteyiz. Daha sonra, temel birimin kesin gösterimini belirlemekte ve Yokoi'nin değişmezleri üzerine bazı sonuçlar elde etmekteyiz. Buna ek olarak, elde edilen sonuçları sağlayan bazı tablolar vermekteyiz. Bu makalede ayrıca d≡1(*mod*4) olması durumunda (Özer 2016a) makalesinde elde edilen sonuçlar tamamlanmakta ve genişletilmektedir.

Anahtar Kelimeler: Sürekli kesir genişlemesi, Temel birim, Kuadratik cisimler, Yokoi'nin invaryantları

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1. Introduction

In 2016, Benamar et all worked on lower bounds of the number of some specific types of monic and non-square free polynomials related with fixed period continued fraction expansion of square root of rational integers. In 2015, Jeongho gave significant results on the solvability of the negative Pell equation and prime ideals by considering real quadratic integers with fixed norm as well as lower bound of regulator of real quadratic fields. In 2016, Badziahin and Shallit considered some real numbers with special continued fraction expansion besides transcendental numbers. In 2008, Tomita and Kawamoto constructed an infinite family of real quadratic fields with large even period of minimal type and class number with Yokoi's invariants. Zhang and Yue (2014) interested in real quadratic fields with odd class number and fundamental unit with positive norm. Also, they gave several congruences relation about the coefficient of fundamental unit in their paper.

In 2002, new lower bound for fundamental unit ϵ_d was obtained by Tomita and Yamamuro and several examples of *d* were given in the terms of Fibonacci sequence for the some types of real quadratic fields. Tomita, in 1995, also described representation of fundamental unit of real quadratic fields for period length equals 3 in the continued fraction expansion of w_d where *d* is square free integer congruent to 1(*mod*4). William and Buck, in 1994, compared with the lengths of the continued fractions of rational integers. Also, many authors obtained significant results for some types of continued fractions, fundamental unit and the real quadratic fields like in the valuable papers (Clemens et all 1995, Elezovic 1997, Friesen 1988, Halter Koch 1991). Sasaki (Sasaki 1986) and Mollin (Mollin 1996) also studied on lower bound of fundamental unit for real quadratic number fields, and they got certain important results. Yokoi

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(Yokoi 1990, 1991, 1993a, 1993b) defined several invariants important for class number problem and solutions of Pell equation by using coefficients of fundamental unit. Besides, the author (Özer 2016a, 2016b, 2017) obtained some types of real quadratic fields and determined their fundamental unit in the case of $d \equiv 2,3(mod 4)$ square free integer. Moreover, we can refer to the readers references (Old 1963, Perron 1950, Sierpinski 1964) for getting more information about the quadratic fields.

Let $k=\mathbb{Q}(\sqrt{d})$ be a real quadratic number field where $d>0$ is a positive square-free integer. w_d is integral basis element of $Z[w_d]$ and $l(d)$ is the period length in simple continued fraction expansion of integral basis element. The fundamental unit ϵ_d of real quadratic number fields is also denoted by $\epsilon_d=\frac{t_d+u_d\sqrt{d}}{2}>1$ where $N(\epsilon_d)=(-1)^{l(d)}$. For the set $I(d)$ of all quadratic irrational numbers in $k=\mathbb{Q}(\sqrt{d})$ we say that α in $I(d)$ is reduced if $\alpha>1, -1<\alpha'<0$ (α' is the conjugate of α) and $R(d)$ denotes the set of all reduced quadratic irrational numbers in $I(d)$. Then, it is well known that any number α in $R(d)$ is purely periodic in the continued fraction expansion and the denominator of its modular automorphism is equal to fundamental unit ϵ_d of $\mathbb{Q}(\sqrt{d})$. Yokoi's invariants are defined as $m_d=\left[\!\left[\frac{u_d^2}{t_d}\right]\!\right]$ and $n_d=\left[\!\left[\frac{t_d}{u_d^2}\right]\!\right]$ where $[\![x]\!]$ represents the greatest integer not greater than x.

Present paper deals with the investigating some types of real quadratic fields including specific continued fraction expansions consist of partial quotients elements equal to each others and written as 3*s* (except the last digit of the period) where d is a square free integer congruent to $1(mod 4)$.

Also, we determine the general representation form of fundamental unit ϵ_d and obtain some results on the Yokoi's invariants n_d, m_d determined in the terms of the coefficient of fundamental units for such real quadratic fields. Further, we give several tables satisfy the obtained results.

2. Preliminaries

We need following definition and lemma in the sequel.

Definition 2.1. Let $\{S_i\}$ be a sequence defined by recurrence relation

$$S_i = 3S_{i-1} + S_{i-2}$$

for $i \geq 2$ with seed values $S_0 = 0$ and $S_1 = 1$.

Lemma 2.2. Let d be a square-free positive integer congruent to 1 *modulo* 4. If we put $\omega_d=\frac{1+\sqrt{d}}{2}, a_0=[\![\omega_d]\!]$ into the $\omega_R=a_0-1+\omega_d$, then $\omega_d\notin R(d)$, but $\omega_R\in R(d)$ holds. Moreover, for the period $l = \ell(d)$ of ω_R, we get $\omega_R=[\overline{2a_0-1,a_1,.......,a_{\ell-1}}]$ and $\omega_d=[a_0,\overline{a_1,.......,a_{l-1},2a_0-1}]$. Furthermore, let $\omega_R=\frac{P_l\omega_R+P_{l-1}}{Q_l\omega_R+Q_{l-1}}=[2a_0-1,a_1,.......,a_{\ell-1},\omega_R]$ be a modular automorphism of ω_R, then the fundamental unit ϵ_d of $\mathbb{Q}(\sqrt{d})$ is given by the following formula:

$$\epsilon_d=\frac{t_d+u_d\sqrt{d}}{2}$$

$$t_d=(2a_0-1).Q_{l(d)}+2Q_{l(d)-1}, u_d=Q_{l(d)}.$$

where Q_i is determined by $Q_0 = 0$, $Q_1 = 1$ and $Q_{i+1} = a_i Q_i + Q_{i-1}$, $(i \geq 1)$.

Proof. Proof is in the paper of Tomita (Tomita 1995).

3. Results

The followings are our main theorem and results with the notations of the preliminaries section.

Theorem 3.1. Let d be a square free positive integer and $\ell > 1$ be a positive integer.

(1) If

$$d=(2mS_\ell+3)^2+8mS_{\ell-1}+4$$

for $m > 0$ positive integer, then $d\equiv 1(mod 4)$ and

$$w_d=\left[mS_\ell+2;\overline{\underbrace{3,3,...,3}_{\ell-1},2mS_\ell+3}\right]$$

and $\ell=\ell(d)$. Moreover, in this case it holds

$$t_d=2mS_\ell^2+3S_\ell+2S_{\ell-1}\ and\ u_d=S_\ell$$

for $\epsilon_d=\frac{t_d+u_d\sqrt{d}}{2}$.

(2) If ℓ is divided by 3 and

$$d=(mS_\ell+3)^2+4mS_{\ell-1}+4$$

for $m > 0$ positive odd integer, then $d\equiv 1(mod 4)$ and

$$w_d=\left[\frac{m}{2}S_\ell+2;\overline{\underbrace{3,3,...,3}_{\ell-1},mS_\ell+3}\right]$$

and $\ell=\ell(d)$. Moreover, in this case it holds

$$t_d=mS_\ell^2+3S_\ell+2S_{\ell-1}\ and\ u_d=S_\ell$$

for $\epsilon_d=\frac{t_d+u_d\sqrt{d}}{2}$.

Remark 3.2. it is clear that S_ℓ is odd number if ℓ is not divided by 3. If we substitute m odd positive integer into the parametrization of d then we obtain that $\frac{mS_\ell}{2}$ is not integer where ℓ is not divided by 3. So, we have to accept that ℓ is divided by 3. Also, if we choose m is even integer in the case of (2), then the parametrization of d coincides with the case of (1). That's why we assume $l \equiv 0 (mod 3)$ and m is positive odd integer in the case of (2).

Proof. (1) Let the parametrization of d be $d=(2mS_\ell+3)^2+8mS_{\ell-1}+4$. Since $(2mS_\ell+3)^2$ is positive odd integer, we have $d \equiv 1 (mod 4)$. From Lemma 2.2, we know that $\omega_d=\frac{1+\sqrt{d}}{2}, a_0=[\![\omega_d]\!], \omega_R=a_0-1+\omega_d$. By using these equations, we have

$$w_R=(mS_\ell+1)+\left[mS_\ell+2;\overline{\underbrace{3,3,...,3}},2mS_\ell+3\right]$$

so we get

$$\omega_R=(2mS_\ell+3)+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3}}}} \\ \ddots \\ 3+\cfrac{1}{3+\cfrac{1}{\omega_R}}$$

By a straight forward induction argument, we obtain

$$\omega_R^2-(2mS_\ell+3)\omega_R-(1+2mS_{\ell-1})=0$$

This requires that $\omega_R=\frac{(2mS_\ell+3)+\sqrt{d}}{2}$ since $\omega_R>0$. If we consider Lemma 2.2, we get

$$\omega_d=\left[mS_\ell+2;\overline{\underbrace{3,3,...,3}_{\ell-1},2mS_\ell+3}\right]$$

and $\ell=\ell(d)$.

Now, we have to determine ϵ_d using Lemma 2.2. In the paper (Özer 2016a), it was obtained that $Q_i=S_i$ using induction for $\forall i \geq 0$. If we substitute these values of sequence into the $\epsilon_d=\frac{t_d+u_d\sqrt{d}}{2}$ in Lemma 2.2 and rearrange, we get t_d and u_d:

$$t_d=2mS_\ell^2+3S_\ell+2S_{\ell-1}\ and\ u_d=S_\ell$$

for $\epsilon_d=\frac{t_d+u_d\sqrt{d}}{2}$. So, we complete the proof of (1).

(2) If we assume that $\ell \equiv 0 (mod 3)$ and the parametrization of d is

$$d=(mS_\ell+3)^2+4mS_{\ell-1}+4$$

for $m > 0$ positive odd integer, then we have $d \equiv 1 (mod 4)$ since S_ℓ is even integer. By substituting $\frac{m}{2}$ instead of m into the case (1), we get

$$w_d=\left[\frac{m}{2}S_\ell+2;\overline{\underbrace{3,3,...,3}_{\ell-1},mS_\ell+3}\right]$$

and $\ell=\ell(d)$. Furthermore,

$$t_d=mS_\ell^2+3S_\ell+2S_{\ell-1} and\ u_d=S_\ell$$

hold for $\epsilon_d=\frac{t_d+u_d\sqrt{d}}{2}$ which completes the proof.

Remark 3.3. Infinitely many values of d which correspond to new real quadratic fields $\mathbb{Q}(\sqrt{d})$ can be obtained by using our main theorem.

Corollary 3.4. Let d be a square free positive integer congruent to 1 modulo 4. If d satisfies the conditions in the Theorem 3.1, then it always hold $m_d = 0$ (i.e. $n_d \neq 0$.)

Proof. In the case of (1) in the Theorem 3.1, we have

$$n_d=\left[\!\left[\frac{t_d}{u_d^2}\right]\!\right]=\left[\!\left[\frac{2mS_\ell^2+3S_\ell+2S_{\ell-1}}{S_\ell^2}\right]\!\right]=2m+\left[\!\left[\frac{3S_\ell+2S_{\ell-1}}{S_\ell^2}\right]\!\right]$$

Since $m > 0$ is positive integer and (S_ℓ) is increasing sequence, we get $n_d \neq 0$ for $\ell > 1$. In a similar way, for the case of (2) in the Theorem 3.1, we get

$$n_d=\left[\!\left[\frac{t_d}{u_d^2}\right]\!\right]=m+\left[\!\left[\frac{3S_\ell+2S_{\ell-1}}{S_\ell^2}\right]\!\right]$$

We obtain $n_d \neq 0$ since $m > 0$ is positive odd integer and $t_d > u_d^2$. This shows that $m_d=\left[\!\left[\frac{u_d^2}{t_d}\right]\!\right]=0$.

Corollary 3.5. Let d be the square free positive integer corresponding to $\mathbb{Q}(\sqrt{d})$ holding (1) in the Theorem 3.1. Table 1 is valid where fundamental unit is ϵ_d, integral basis element is ω_d and Yokois invariant is n_d for m = 1 or 2 and $2 \leq \ell(d) \leq 11$. (In this table, we rule out $\ell(d)=10,11$ for m = 1 and $\ell(d)=2,9,10,11$ in the case of m = 2 since d is not a square free positive integer.)

Proof. This Table is obtained if we substitute m = 1 or 2 into (1) in the Theorem 3.1. Now, we have to determine the values of Yokoi invariant n_d *as follows:*

$$n_d=\begin{cases}3, & if\ \ell=2\\ 2, & if\ \ell>2\end{cases}$$

for m = 1. We know that $n_d=\left[\!\left[\frac{t_d}{u_d^2}\right]\!\right]$ from Yokoi's references (Yokoi 1990, 1991, 1993a, 1993b). If we substitute u_d and t_d into the n_d, then we get

$$n_d=\left[\!\left[\frac{t_d}{u_d^2}\right]\!\right]=\left[\!\left[\frac{2mS_\ell^2+3S_\ell+2S_{\ell-1}}{S_\ell^2}\right]\!\right]$$

By using the above equality, we have n_d = 3 in the case of $\ell=2$ while m = 1. Since S_ℓ is increasing sequence, we get

$$2,36 \geq \left(2+\frac{3}{S_\ell}+\frac{2S_{\ell-1}}{S_\ell^2}\right)>2$$

for $\ell>2$, while m = 1. In a similar way, we obtain n_d = 4 in the case of $\ell>2$ and m = 2 since following inequality satisfies

$$4,36 \geq \left(4+\frac{3}{S_\ell}+\frac{2S_{\ell-1}}{S_\ell^2}\right) > 4$$

Corollary 3.6. Let d be a square free positive integer concerning the case of (2) in the Theorem 3.1. Table 2 is valid where fundamental unit is ϵ_d, integral basis element is ω_d, Yokoi's d-invariants n_d and m_d for m = 1,3 with $3 \leq \ell(d) \leq 12$. (In this table, we rule out $\ell(d) = 9, 12$ for m = 3 since d is not a square free positive integer.)

Proof. This Table is got if we substitute m = 1 or 3 into the (2) in the Theorem 3.1. Now, we have to show that n_d = 1 in the case of m = 1. If we put t_d and u_d into the n_d and rearrange, then we obtain

$$n_d = \left[\!\left[\frac{t_d}{u_d^2}\right]\!\right] = \left[\!\left[\frac{mS_\ell^2+3S_\ell+2S_{\ell-1}}{S_\ell^2}\right]\!\right]$$

From the assumption (also since S_ℓ is increasing sequence), we have

Table 1. Square-free positive integers d with $2 \leq \ell(d) \leq 9$

d	m	$\ell(d)$	n_d	m_d	ω_d	ϵ_d
93	1	2	3	0	$[5;\overline{3,9}]$	$(29+3\sqrt{93})/2$
557	1	3	2	0	$[12;\overline{3,3,23}]$	$(236+10\sqrt{557})/2$
4845	1	4	2	0	$[35;\overline{3,3,3,69}]$	$(2297+33\sqrt{4845})/2$
49109	1	5	2	0	$[111;\overline{3,3,3,3,221}]$	$(24115+109\sqrt{49109})/2$
523605	1	6	2	0	$[362;\overline{3,3,...,3,723}]$	$(260498+360\sqrt{523605})/2$
5672045	1	7	2	0	$[1191;\overline{3,3,...,3,2381}]$	$(2831729+1189\sqrt{5672045})/2$
61741965	1	8	2	0	$[3929;\overline{3,3,...,3,7857}]$	$(30856817+3927\sqrt{61741965})/2$
673070669	1	9	2	0	$[12972;\overline{3,3,...,3,25943}]$	$(336488564+12970\sqrt{673070669})/2$
1901	2	3	4	0	$[22;\overline{3,3,43}]$	$(436+10\sqrt{1901})/2$
18389	2	4	4	0	$[68;\overline{3,3,3,135}]$	$(4475+33\sqrt{18389})/2$
193253	2	5	4	0	$[220;\overline{3,3,3,3,439}]$	$(47917+109\sqrt{193253})/2$
2083997	2	6	4	0	$[722;\overline{3,3,...,3,1443}]$	$(519698+360\sqrt{2083997})/2$
22653845	2	7	4	0	$[2380;\overline{3,3,...,3,4759}]$	$(5659171+1189\sqrt{22653845})/2$
246854549	2	8	4	0	$[7856;\overline{3,3,...,3,15711}]$	$(61699475+3927\sqrt{246854549})/2$

Table 2. Square-free positive integers d with $3 \leq \ell(d) \leq 12$.

d	m	$\ell(d)$	n_d	m_d	ω_d	ϵ_d
185	1	3	1	0	$[7;\overline{3,3,13}]$	$\frac{136+10\sqrt{185}}{2}$
132209	1	6	1	0	$[182;\overline{3,...,3,363}]$	$\frac{130898+360\sqrt{132209}}{2}$
168314441	1	9	1	0	$[6487;\overline{3,3,...,3,12973}]$	$\frac{168267664+12970\sqrt{168314441}}{2}$
218353968017	1	12	1	0	$[233642;\overline{3,3,...,3,467283}]$	$\frac{218352283202+467280\sqrt{218353968017}}{2}$
1129	3	3	3	0	$[17;\overline{3,3,33}]$	$\frac{336+10\sqrt{1129}}{2}$
1174201	3	6	3	0	$[542;\overline{3,3,...,3,1083}]$	$\frac{390098+360\sqrt{1174201}}{2}$

$$1,36 \geq \left(1 + \frac{3}{S_\ell} + \frac{2S_{\ell-1}}{S_\ell^2}\right) > 1$$

in the case of $\ell \geq 3$ which completes the first part of the proof. In a similar way, we obtain n_d =3 for $\ell > 2$ since

$$3,36 \geq \left(3 + \frac{3}{S_\ell} + \frac{2S_{\ell-1}}{S_\ell^2}\right) > 3$$

in the case of m = 3.

4. Conclusion

It is well known that the fundamental unit, continued fraction expansion and Yokoi's invariants play an important role in the studying on real quadratic fields.

The focal point in this paper was to investigate some types of real quadratic fields and determine their infrastructure such as fundamental unit, Yokoi's invariants, continued fraction expansions, etc. Also, the present paper extended and completed the paper of the author (Özer 2016a) in the case of d congruent to $1(mod4)$.

The results provide us a practical method so as to rapidly determine continued fraction expansion of ω_d, fundamental unit ε_d, Yokoi's invariants n_d and m_d for such real quadratic number fields.

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