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Research Article

Tan ($F(\xi)/2$)-Expansion Method for Traveling Wave Solutions of the Variant Bussinesq Equations

Variant Bussinesq Denklemlerinin Hareket Eden Dalga Çözümleri için Tan ($F(\xi)/2$) Açılım Metodu

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Abstract

In this paper, we implemented a tan ($F(\xi)/2$)-expansion method for the traveling wave solutions of the variant Bussinesq equations. We have hyperbolic function solution, trigonometric function solution, exponential solution and rational solution for this equation. Recently, this method has been studied for obtaining traveling wave solutions of nonlinear partial differential equations by sciences.

Keywords: Exponential solution, Hyperbolic function solution, Rational solution, Tan ($F(\xi)/2$)-expansion method, The variant Bussinesq equations, Trigonometric function solution

Öz

Bu makalede farklı Bussinesq denklemlerinin hareket eden dalga çözümleri için tan ($F(\xi)/2$) açılım metodu sunulmuştur. Bu denklem için hiperbolik fonksiyon çözümü, trigonometric fonksiyon çözümü, üstel fonksiyon çözümü ve rasyonel çözüm elde edilmiştir. Son zamanlarda, bu metot lineer olmayan kısmi diferensiyel denklemlerin hareket eden dalga çözümlerinin elde edilmesi için bilim adamları tarafından çalışılmaktadır.

Anahtar Kelimeler: Üstel fonksiyon çözüm, Hiperbolik fonksiyon çözüm, Rasyonel çözüm, Tan ($F(\xi)/2$) açılım metodu, Variant Bussinesq denklemleri, Trigonometric fonksiyon çözüm

1. Introduction

Nonlinear partial differential equations (NPDEs) have an important place in applied mathematics and physics. These equations are mathematical models of physical phenomenon that arise in engineering, chemistry, biology, mechanics and physics. It is very important to have information about the solutions of mathematical models. To better understand the mechanisms of the mathematical models, it is necessary to solve these equations. Thus, it has an important place to obtain the analytic solutions of nonlinear differential equations in applied sciences. Recently, it has become attractive solving these equations. Therefore, some methods have been developed by sciences. Some of them are: Hirota method [1], Bäcklund transformation [2], Cole-hopf transformation method [3], Generalized Miura Transformation [4], inverse scattering method [5], Darboux transformation [6], Painleve method [7], homogeneous balance method [8], similarity reduction method [9] and sine cosine method [10].

Besides these methods, there are a lot of methods based on the use of an auxiliary equation. Firstly, the nonlinear partial differential equations are transformed into nonlinear ordinary differential equations by using these methods. Second, the obtained nonlinear ordinary differential equations are solved with the help of the auxiliary equation. These methods can be listed as: tanh function method [11], extended tanh function method [12], modified extended tanh method [13], improved tanh function method [14], Jacobi elliptic function method [15], extended Jacobi elliptic function method [16], generalized Jacobi elliptic function method [17], Jacobi elliptic rational expansion method [18], Weierstrass Jacobi elliptic function expansion method [19], $\frac{G'}{G}$-expansion method [20], extended $\frac{G'}{G}$-expansion method [21], generalized $\frac{G'}{G}$-expansion method [22], $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion method [23].

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In this study, we implemented tan-expansion method [24] for finding the analytic solutions of variant Bussinesq equations [25].

2. An Analysis of the Method

In this section, we present a simple description of tan ($F(\xi)$/2)-expansion method. For doing this, one can consider in a two variables general form of nonlinear PDE

$$Q(u,u_t,u_x,u_{xx},...) = 0, \tag{1}$$

and transform Eq. (1) with

$$u(x,t) = u(\xi), \quad \xi = x-kt,$$

where k is arbitrary constants. After the transformation, we get a nonlinear ODE for $u(\xi)$

$$Q'(u',u'',u''',...) = 0 \tag{2}$$

Then, the solution of the equation (2) we are looking for is expressed in the form as a

$$u(x,t) = u(\xi) = \sum_{i=0}^{m} A_i\left[p+\tan\left(\frac{F(\xi)}{2}\right)\right]^i + \sum_{i=1}^{m} B_i\left[p+\tan\left(\frac{F(\xi)}{2}\right)\right]^{-i}, \quad A_i \neq 0, B_i \neq 0, \tag{3}$$

where m is a positive integer that can be determined by balancing the highest order derivative and with the highest nonlinear terms in equation, the coefficients A_i ($0\le i\le m$), B_i ($1\le i\le m$) are constant to be determined and $F = F(\xi)$ satisfies the first order nonlinear ODE:

$$F'(\xi) = a\sin(F(\xi)) + b\cos F(\xi)) + c. \tag{4}$$

Substituting solution (3) into Eq. (2) yields a set of algebraic equations for $\tan\left(\frac{F(\xi)}{2}\right)^i$, $\cot\left(\frac{F(\xi)}{2}\right)^i$,then, all coefficients of $\tan\left(\frac{F(\xi)}{2}\right)^i$,$\cot\left(\frac{F(\xi)}{2}\right)^i$ have to vanish. After this separated algebraic equation, we can found k, p, $A_0, A_1, B_1 ..., A_m, B_m$ constants.

In this work, we aim to obtain solution of variant Bussinesq and the (2+1)-dimensional Burgers equations by using tan ($F(\xi)$/2)-expansion method. The solutions of Eq.(4) are given in [24].

3. Applications

In this section, we consider the variant Bussinesq equations,

$$\begin{aligned} &u_t + u_x v + v_x u + v_{xxx} = 0.\\ &v_t + u_x + vv_x = 0. \end{aligned} \tag{5}$$

As a model for water waves, v is the velocity and u is the total depth, and the subscripts denote partial derivatives. Many scientists have also considered the variant Bussinesq equations. Yuan [26] presented meromorphic solutions of Eq. (5). Guo [27] obtained multiple soliton solutions and multiple singular soliton solutions of Eq. (5). Khan [28] obtained the soliton solutions of Eq. (5). Let us consider the traveling wave solutions $u(x,t) = u(\xi)$, $u(\xi) = x+y-kt$ then Eq. (5) becomes

$$\begin{aligned} &-ku'+u'v+v'u+v''' = 0,\\ &-kv'+u'+vv' = 0, \end{aligned} \tag{6}$$

when balancing v''' with vu' and u' with vv' then $m_1 = 2$ and $m_2 = 1$ gives. Therefore, we may choose

$$\begin{aligned} &u(\xi) = A_0 + A_1\left[p+\tan\left(\frac{F(\xi)}{2}\right)\right] + A_2\left[p+\tan\left(\frac{F(\xi)}{2}\right)\right]^2 +\\ &B_1\left[p+\tan\left(\frac{F(\xi)}{2}\right)\right]^{-1} + B_2\left[p+\tan\left(\frac{F(\xi)}{2}\right)\right]^{-2}, \ A_i \neq 0, B_i \neq 0.\\ &v(\xi) = C_0 + C_1\left[p+\tan\left(\frac{F(\xi)}{2}\right)\right] + D_1\left[p+\tan\left(\frac{F(\xi)}{2}\right)\right]^{-1},\\ &C_i \neq 0, D_i \neq 0. \end{aligned} \tag{7}$$

Substituting (7) into Eq. (6) yields a set of algebraic equations for $k, p, A_0, A_1, A_2, B_1, B_2, C_0, C_1, D_1$. These algebraic equations system are obtained as

$$\begin{aligned} &6A_2 - 6B_2 + 3C_1^2 - 3D_1^2 + 3A_1p - 3B_1p + 3C_0C_1p - 3C_0D_1p -\\ &3C_1kp + 3D_1kp + 12A_2p^2 + 6C_1^2p^2 + 3A_1p^3 + 3C_0C_1p^3 - 3C_1kp^3 +\\ &6A_2p^4 + 3C_1^2p^4 = 0,\\ &-8A_2 - 8B_2 - 4C_1^2 - 4D_1^2 - 4B_1p - 4C_0D_1p + 4D_1kp + 4A_1p^3 +\\ &4C_0C_1p^3 - 4C_1kp^3 + 8A_2p^4 + 4C_1^2p^4 = 0,\\ &2A_2 - 2B_2 + 2C_1^2 - D_1^2 - 3A_1p - B_1p + 3C_0C_1p - C_0D_1p +\\ &3C_1kp + D_1kp - 12A_2p^2 - 6C_1^2p^2 + A_1p^3 + C_0C_1p^3 - C_1kp^3 +\\ &2A_2p^4 + C_1^2p^4 = 0,\\ &\vdots \end{aligned} \tag{8}$$

from the solutions of the system, we obtain

Case 1

$$\begin{aligned} &a = -bp + cp, A_0 = b^2 - c^2 + b^2p^2 - 2bcp^2 + c^2p^2, A_1 = 0,\\ &A_2 = \frac{1}{2}(-b^2 + 2bc - c^2), B_1 = 0, C_1 = \mp\sqrt{b^2 - 2bc + c^2},\\ &B_2 = \frac{1}{2}(-b^2 - 2bc - c^2 - 2b^2p^2 + 2c^2p^2 - b^2p^4 + 2bcp^4 - c^2p^4),\\ &C_1 \neq 0, D_1 = \frac{b^2 - c^2 + b^2p^2 - 2bcp^2 + c^2p^2}{C_1}, k = C_0, b^2 - c^2 +\\ &b^2p^2 - 2bcp^2 + c^2p^2 \neq 0. \end{aligned} \tag{9}$$

with the aid of Mathematica. Substituting (9) into (7) we have the following solutions of Eq. (5),

Solution 1.1 For $a^2+b^2-c^2<0$ and $b-c\neq 0$, we have trigonometric solution of Eq. (5) as

$$u_{11}(x,t)=2(b-c)(b+c+bp^2-cp^2)Csc^2\left(\sqrt{-(b-c)(b+c+bp^2-cp^2)}(x-C_0t)\right),$$
$$v_{11}(x,t)=\frac{C_0\sqrt{-(b-c)(b+c+bp^2-cp^2)}}{\sqrt{-(b-c)(b+c+bp^2-cp^2)}}$$
$$-\frac{2\sqrt{(b-c)^2(b+c+bp^2-cp^2)}Cot\left(\sqrt{-(b-c)(b+c+bp^2-cp^2)}(x-C_0t)\right)}{\sqrt{-(b-c)(b+c+bp^2-cp^2)}}$$

(10)

Solution 1.2 For $a^2+b^2-c^2>0$ and $b-c\neq 0$, we have hyperbolic solution of Eq. (5) as

$$u_{12}(x,t)=-2(b^2-c^2+(bp-cp)^2)Csch^2\left(\sqrt{b^2-c^2+(bp-cp)^2}(x-C_0t)\right)$$
$$v_{12}(x,t)=\frac{C_0\sqrt{(b-c)(b+c+bp^2-cp^2)}}{\sqrt{b^2-c^2+(bp-cp)^2}}$$
$$+\frac{2\sqrt{(b-c)^2(b+c+bp^2-cp^2)}Coth\left(\sqrt{b^2-c^2+(bp-cp)^2}(x-C_0t)\right)}{\sqrt{b^2-c^2(bp-cp)}}$$

(11)

Solution 1.3 For $a^2+b^2-c^2>0$, $b\neq 0$ and $c=0$,

$$u_{13}(x,t)=-2b^2(1+p^2)Csch^2\left(\sqrt{b^2(1+p^2)}(x-C_0t)\right)$$
$$v_{13}(x,t)=\frac{C_0\sqrt{b^2(1+p^2)}+2b\sqrt{b^2(1+p^2)}Coth\left(\sqrt{b^2(1+p^2)}(x-C_0t)\right.}{\sqrt{b^2(1+p^2)}}$$

(12)

Solution 1.4 For $a^2+b^2-c^2<0$, $c\neq 0$ and $b=0$, we have trigonometric solution of Eq. (5) as

$$u_{14}(x,t)=2c^2(-1+p^2)Csc^2\left(\sqrt{-c^2(-1+p^2)}(x-C_0t)\right)$$
$$v_{14}(x,t)=C_0-\sqrt{-c^2(-1+p^2)}Cot\left(\frac{1}{2}\sqrt{-c^2(-1+p^2)}(x-C_0t)\right)$$
$$+\sqrt{-c^2(-1+p^2)}Tan\left(\frac{1}{2}\sqrt{-c^2(-1+p^2)}(x-C_0t)\right)$$

(13)

Solution 1.5 For $a=c=\beta a$, $b=-\beta a$, we obtain exponential solution of Eq. (5) as

$$u_{15}(x,t)=-\frac{8\beta^2a^2e^{2a(x-C_0t)\beta}}{(e^{2a(x-C_0t)\beta}-1)^2}$$
$$v_{15}(x,t)=\frac{C_0(e^{2a(x-C_0t)\beta}-1)-2\sqrt{a^2\beta^2}(e^{2a(x-C_0t)\beta}+1)}{(e^{2a(x-C_0t)\beta}-1)} \tag{14}$$

Solution 1.6 For $c=a$,

$$u_{16}(x,t)=-\frac{8a^4e^{\frac{2a(-1+p)(-x+C_0t)}{p}}(-1+p)^2}{\left(a^2-e^{\frac{2a(-1+p)(-x+C_0t)}{p}}p^2\right)^2}$$
$$v_{16}(x,t)=\frac{a^2\left(C_0-2(p-1)\sqrt{\frac{a^2}{p^2}}\right)-e^{\frac{2a(p-1)(-x+C_0t)}{p}}\left(C_0+2(p-1)\sqrt{\frac{a^2}{p^2}}\right)p^2}{a^2-e^{\frac{2a(p-1)(-x+C_0t)}{p}}p^2}$$

(15)

Solution 1.7 For $a=c$,

$$u_{17}(x,t)=-\frac{8c^4e^{\frac{2c(p-1)(-x+C_0t)}{p}}(p-1)^2}{\left(c^2-e^{\frac{2c(p-1)(-x+C_0t)}{p}}p^2\right)^2}.$$
$$v_{17}(x,t)=\frac{c^2\left(C_0-2(p-1)\sqrt{\frac{c^2}{p^2}}\right)-e^{\frac{2c(p-1)(-x+C_0t)}{p}}\left(C_0+2(p-1)\sqrt{\frac{c^2}{p^2}}\right)p^2}{c^2-e^{\frac{2c(p-1)(-x+C_0t)}{p}}p^2}$$

(16)

Solution 1.8 For $c=-a$,

$$u_{18}(x,t)=-\frac{8c^4e^{\frac{2a(1+p)(x-C_0t)}{p}}(1+p)^2}{\left(-a^2e^{\frac{2a(1+p)(x-C_0t)}{p}}+p^2\right)^2}$$
$$v_{18}(x,t)=-\frac{a^2\left(-C_0+2(p+1)\sqrt{\frac{a^2}{p^2}}\right)e^{\frac{2a(p+1)(x-C_0t)}{p}}+\left(C_0+2(p+1)\sqrt{\frac{a^2}{p^2}}\right)p^2}{a^2e^{\frac{2a(p+1)(x-C_0t)}{p}}-p^2}$$

(17)

Solution 1.9 For $b=-c$,

$$u_{19}(x,t)=-\frac{32c^4e^{4cp(x-C_0t)}p^2}{(c^2e^{4cp(x-C_0t)}-1)^2}.$$
$$v_{19}(x,t)=\frac{C_0(c^2e^{4cp(x-C_0t)}-1)-4\sqrt{c^2}(c^2e^{4cp(x-C_0t)}+1)p}{(c^2e^{4cp(x-C_0t)}-1)} \tag{18}$$

Solution 1.10 For $b=0$ and $a=c$, we obtain rational solution of Eq. (5) as

$$u_{110}(x,t)=-\frac{2}{(x-C_0t)^2}.$$
$$v_{110}(x,t)=\frac{2\sqrt{a^2}+aC_0(C_0t-x)}{a(C_0t-x)} \tag{19}$$

4. Explanations and Graphical Representations of the Obtained Solutions

The graphical demonstrations of some obtained solutions are shown in Figures 1–4. These figures have the following physical explanations: The variant Bussinesq equations: The shapes of Eqs. (10), (11), (14), and (19) are represented in Figures 1–4, respectively, with wave speed $C_0=1$ within the interval $-3\leq x\leq 3$, $0\leq t\leq 5$. Figure 1 represents periodic wave for $u(x,t)$ and singular periodic antikink wave for $v(x,t)$. Figures 2-4 represent singular solutions for $u(x,t)$. Figures 2-4 represents singular antikink wave for $v(x,t)$.

5. Conclusion

In this paper, we present tan ($F(\xi)/2$)-expansion method by using Eq. (4) and with aid of *Mathematica* 7.0, implement it in a computer algebraic system. An implementation of the method is given by variant Bussinesq equations. The

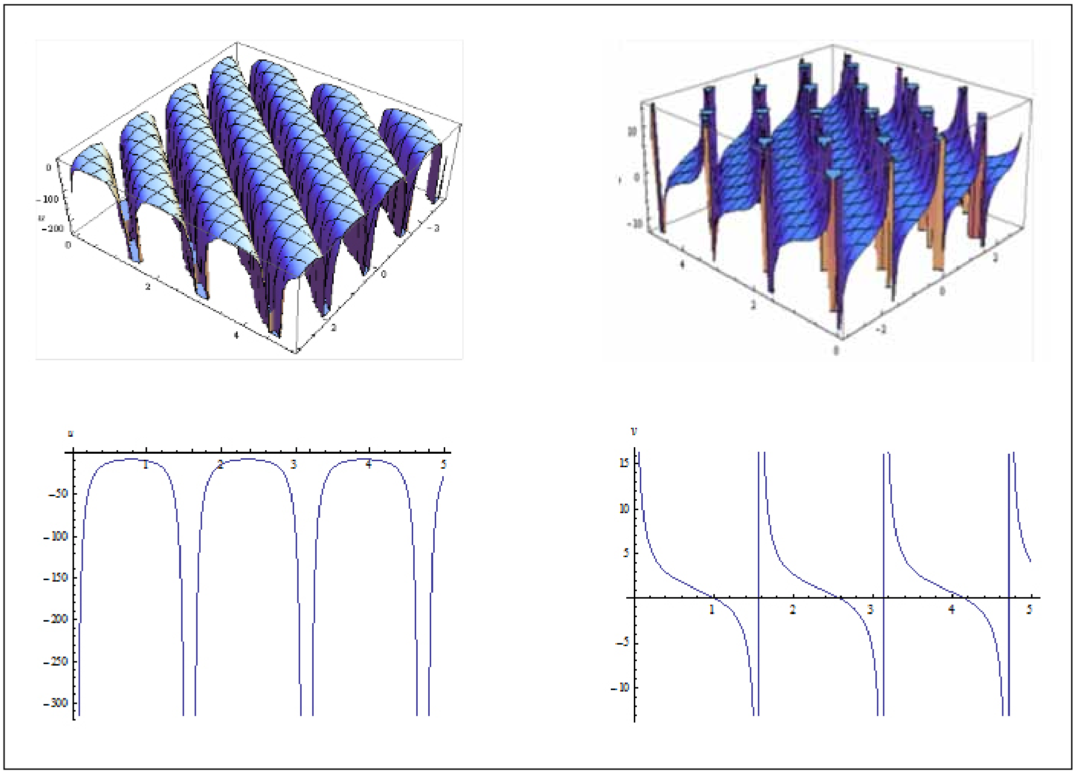

Figure 1. Profiles of u and v solutions (10) for Eq. (5) with $a = 1$, $b = 2$, $c = 3$, $p = 1$, $C_0 = 1$, $x = 0$.

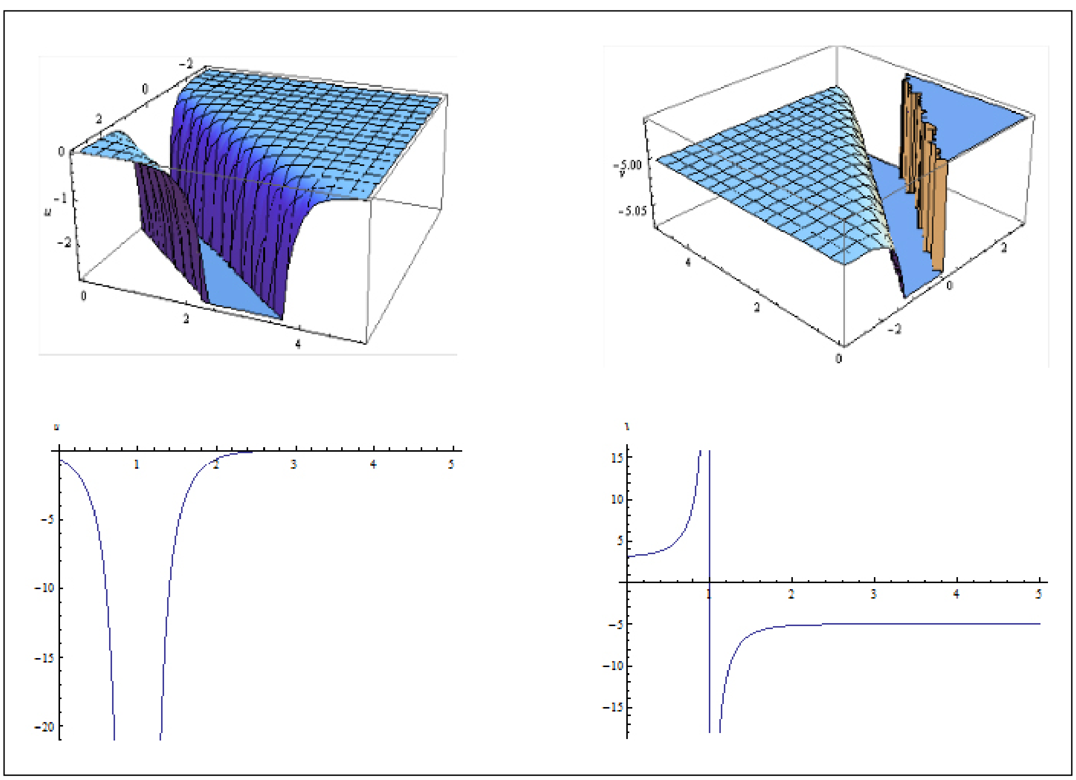

Figure 2. Profiles of u and v solutions (11) for Eq. (5) with $a = 3$, $b = 2$, $c = 1$, $p = 1$, $C_0 = 1$, $x = 1$.

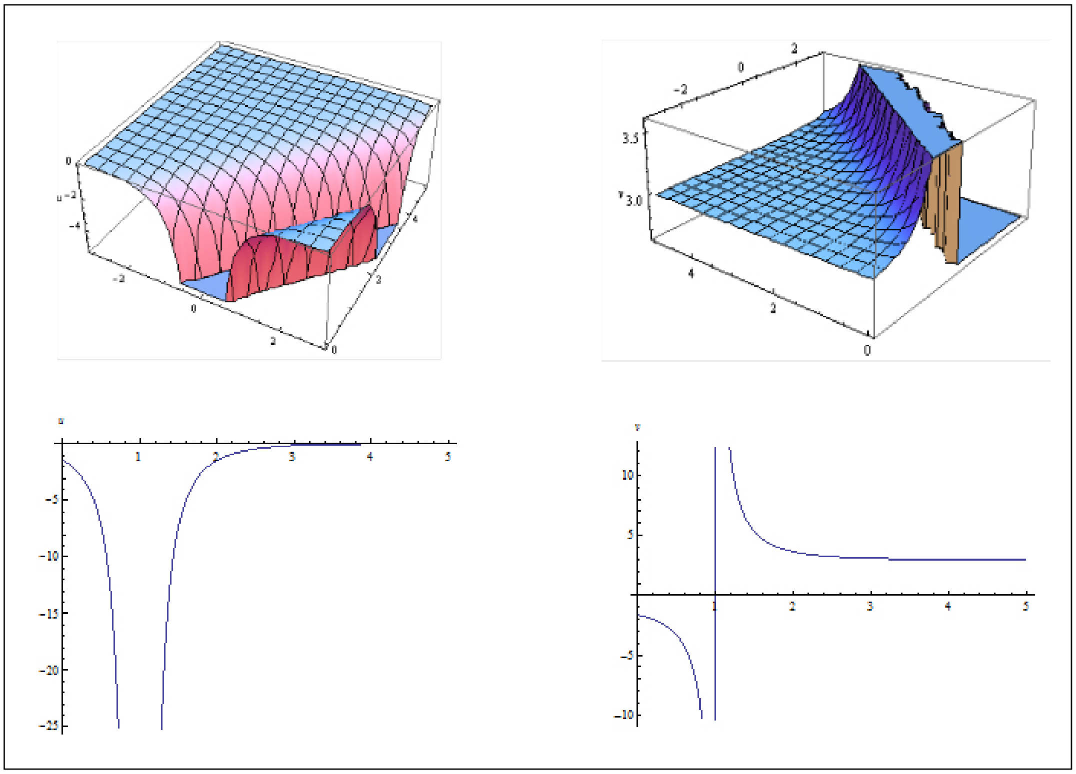

Figure 3. Profiles of u and v solutions (14) for Eq. (5) with a = 1, β = 1, C_0 = 1, x = 1.

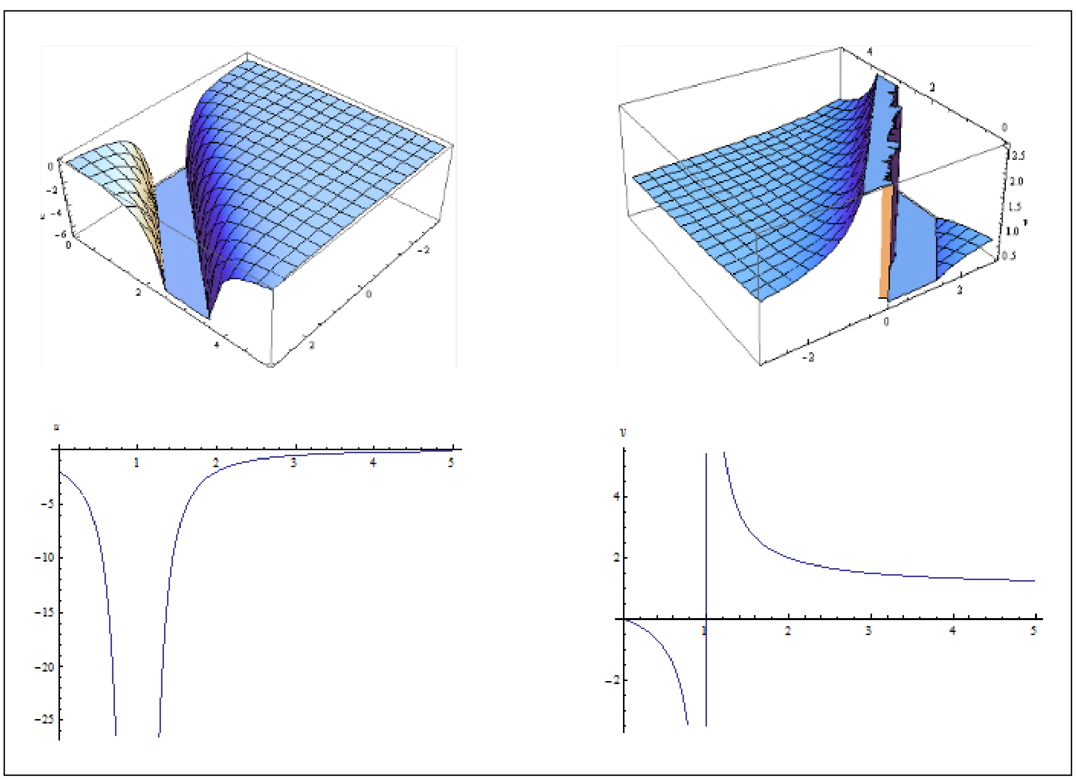

Figure 4. Profiles of u and v solutions (19) for Eq. (5) with a = 4, C_0 = 1, x = 1.

method can be used to many other nonlinear evolution equations or coupled ones. In addition, this method is also computerizable, which allows us to perform complicated and tedious algebraic calculation on a computer by the help of symbolic programs such as Mathematica 7.0, Maple, Matlab, and so on.

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