Karaelmas Fen ve Mühendislik Dergisi
Journal home page: http://fbd.beun.edu.tr

Research Article

Inequalities via *n*-Times Differentiable *Quasi*-Convex Functions

n-Mertebeden Türevlenebilir Quasi-Konveks Fonksiyonlar Yardımıyla Eşitsizlikler

Merve Avcı Ardıç

Adıyaman University, Faculty of Science and Arts, Department of Mathematics, Adıyaman, Turkey

Abstract

In this paper, some integral inequalities for n-times differentiable quasi-convex functions are established.

Keywords: Hermite-Hadamard inequality, Hölder inequality, Quasi-convex functions, Power-mean inequality

Öz

Bu çalışmada, *n*-mertebeden türevlenebilir *quasi*-konveks fonksiyonlar için yeni bazı integral eşitsizlikler elde edilmiştir.

Anahtar Kelimeler: Hermite-Hadamard eşitsizliği, Hölder eşitsizliği, Quasi-konveks fonksiyonlar, Power-mean eşitsizliği
2010 MSC: 26D10; 26D15

1. Introduction

A function $f:[a,b]\to\mathbb{R}$ is said to be *quasi*-convex function on $[a, b]$ if the inequality $f(\alpha x+(1-\alpha)y)\leq max\{f(x), f(y)\}$ holds for all $x,y\in[a,b]$ and $\alpha\in[0,1]$. For some results about *quasi*-convexity see [1], [5], [7], [8], [13].

Hermite-Hadamard inequality is defined as below:

Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a convex function on an interval I and $a,b\in I$ with $a<b$. Then

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{f(a)+f(b)}{2}$$

holds. Both inequalities hold in the reversed direction if f is concave. For some inequalities, generalizations and applications concerning Hermite-Hadamard inequality see [1], [3], [6], [7], [9]-[12], [16] and [17].

Recently, in the literature there are so many manuscripts about n-times differentiable functions on several kinds of convexities. In references [3], [4], [6], [9], [10] and [14]-[16] readers can find some results about this issue.

The main aim of this paper is to prove new integral inequalities for n-times differentiable *quasi*-convex functions.

*Corresponding Author: mavci@adiyaman.edu.tr

Received / Geliş tarihi : 17.10.2016
Accepted / Kabul tarihi : 25.11.2016

To prove our main results, we use the following lemmas.

Lemma 1.1. [2] Let $f:[a,b]\to\mathbb{R}$ be a mapping such that the derivative $f^{(n-1)}(n\geq1)$ is absolutely continuous on $[a, b]$. Then for any $x\in[a, b]$ one has the equality:

$$\int_a^b f(t)\,dt=\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[\begin{array}{l}(x-a)^{k+1}f^{(k)}(a)+\\(-1)^k(b-x)^{k+1}f^{(k)}(b)\end{array}\right]+\frac{1}{n!}\int_a^b(x-t)^n f^{(n)}(t)\,dt.$$

Lemma 1.2. [6] Let $f:I^o\subseteq\mathbb{R}\to\mathbb{R}, a,b\in I^o$ with $a<b$, $f^{(n)}$ exists on I^o and $f^{(n)}\in L(a,b)$ for $n\geq1$. We have the following identity:

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)$$
$$=\frac{(b-a)^n}{2n!}\int_0^1 t^{n-1}(n-2t)f^{(n)}(ta+(1-t)b)\,dt.$$

2. Inequalities for *n*-Times Differentiable *Quasi*-Convex Functions Via Lemma 1.1.

The first result is given in the following theorem:

Theorem 2.1. Let $f:[a,b]\to\mathbb{R}$ be a mapping such that the derivative $f^{(n-1)}(n\geq1)$ is absolutely continuous on $[a, b]$. If $|f^{(n)}|$ is *quasi*-convex on $[a, b]$, following inequality holds

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[\begin{matrix}(x-a)^{k+1}f^{(k)}(a)+\\(-1)^k(b-x)^{k+1}f^{(k)}(b)\end{matrix}\right]\right|$$
$$\leq\frac{1}{n!}\left\{\begin{matrix}[max\{|f^{(n)}(a)|,|f^{(n)}(x)|\}]\dfrac{(x-a)^{n+1}}{n+1}+\\ [max\{|f^{(n)}(x)|,|f^{(n)}(b)|\}]\dfrac{(b-x)^{n+1}}{n+1}\end{matrix}\right\}$$

for all $x\in[a,b]$.

Proof. From Lemma 1.1., property of the modulus and *quasi*-convexity of $|f^{(n)}|$, we can write

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[\begin{matrix}(x-a)^{k+1}f^{(k)}(a)+\\(-1)^k(b-x)^{k+1}f^{(k)}(b)\end{matrix}\right]\right|$$
$$\leq\frac{1}{n!}\int_a^b|x-t|^n|f^{(n)}(t)|dt$$
$$=\frac{1}{n!}\left\{\int_a^x(x-t)^n|f^{(n)}(t)|dt+\int_x^b(t-x)^n|f^{(n)}(t)|dt\right\}$$
$$=\frac{1}{n!}\left\{\int_a^x(x-t)^n\left|f^{(n)}\left(\frac{t-a}{x-a}x+\frac{x-t}{x-a}a\right)\right|dt\right.$$
$$\left.+\int_x^b(t-x)^n\left|f^{(n)}\left(\frac{t-x}{b-x}b+\frac{b-t}{b-x}x\right)\right|dt\right\}$$
$$\leq\frac{1}{n!}\left\{\int_a^x(x-t)^n[max\{|f^{(n)}(a)|,|f^{(n)}(x)|\}]\right.$$
$$\left.+\int_x^b(t-x)^n[max\{|f^{(n)}(x)|,|f^{(n)}(b)|\}]\right\}.$$

If we use the fact that

$$\int_a^x(x-t)^n dt=\frac{(x-a)^{n+1}}{n+1}$$

and

$$\int_x^b(t-x)^n dt=\frac{(b-x)^{n+1}}{n+1}$$

we get the desired result.

Corollary 2.2. In Theorem 2.1., if we choose $x=\frac{a+b}{2}$, following inequality holds:

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{b-a}{2}\right)^{k+1}[f^{(k)}(a)+(-1)^kf^{(k)}(b)]\right|$$
$$\leq\frac{(b-a)^{n+1}}{2^{n+1}(n+1)!}\left\{\begin{matrix}\left[max\left\{|f^{(n)}(a)|,\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|\right\}\right]+\\ \left[max\left\{\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|,|f^{(n)}(b)|\right\}\right]\end{matrix}\right\}.$$

Theorem 2.3. *Let* $f:[a,b]\to\mathbb{R}$ *be a mapping such that the derivative* $f^{(n-1)}$ $(n\geq1)$ *is absolutely continuous on* $[a,b]$. *If* $|f^{(n)}|^q$ *is quasi-convex on* $[a,b]$, *following inequality holds for all* $x\in[a,b]$

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[\begin{matrix}(x-a)^{k+1}f^{(k)}(a)+\\(-1)^k(b-x)^{k+1}f^{(k)}(b)\end{matrix}\right]\right|$$
$$\leq\frac{(b-a)^{1/q}}{n!}\left(\frac{(x-a)^{np+1}+(b-x)^{np+1}}{np+1}\right)^{1/p}\left[max\left\{\begin{matrix}|f^{(n)}(a)|^q,\\|f^{(n)}(b)|^q\end{matrix}\right\}\right]^{\frac{1}{q}}$$

where $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1.1., property of the modulus, well-known Hölder integral inequality and *quasi*-convexity of $|f^{(n)}|^q$, we can write

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[\begin{matrix}(x-a)^{k+1}f^{(k)}(a)+(-1)^k\\(b-x)^{k+1}f^{(k)}(b)\end{matrix}\right]\right|$$
$$\leq\frac{1}{n!}\left(\int_a^b|x-t|^{np}dt\right)^{1/p}\left(\int_a^b|f^{(n)}(t)|^q dt\right)^{1/q}$$
$$=\frac{1}{n!}\left(\int_a^x(x-t)^{np}dt+\int_x^b(t-x)^{np}dt\right)^{1/p}\left(\int_a^b\left|f^{(n)}\left(\begin{matrix}\dfrac{b-t}{b-a}a+\\ \dfrac{t-a}{b-a}b\end{matrix}\right)\right|^q dt\right)^{1/q}$$
$$\leq\frac{1}{n!}\left(\frac{(x-a)^{np+1}+(b-x)^{np+1}}{np+1}\right)^{1/p}\left(\int_a^b\left[\begin{matrix}\dfrac{b-t}{b-a}|f^{(n)}(a)|^q+\\ \dfrac{t-a}{b-a}|f^{(n)}(b)|^q\end{matrix}\right]dt\right)^{1/q}$$
$$=\frac{(b-a)^{1/q}}{n!}\left(\frac{(x-a)^{np+1}+(b-x)^{np+1}}{np+1}\right)^{1/p}\left[max\left\{\begin{matrix}|f^{(n)}(a)|^q,\\|f^{(n)}(b)|^q\end{matrix}\right\}\right]^{\frac{1}{q}}.$$

The proof is completed.

3. Inequalities for *n*-Times Differentiable *Quasi*-Convex Functions Via Lemma 1.2.

Theorem 3.1. *Let* $f:[a,b]\to\mathbb{R}$ *be an n-times differentiable function. If* $|f^{(n)}|^q$ *is quasi-convex on* $[a,b]$ *following inequality holds*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right|$$
$$\leq\frac{(b-a)^n(n-1)}{2(n+1)!}[max\{|f^{(n)}(a)|^q,|f^{(n)}(b)|^q\}]^{\frac{1}{q}}$$

for $q\geq1$ and $n\geq2$.

Proof. From Lemma 1.2., property of the modulus, well-known power-mean integral inequality and *quasi*-convexity of $|f^{(n)}|^q$ we can write

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right|$$
$$\leq\frac{(b-a)^n}{2n!}\int_0^1 t^{n-1}(n-2t)|f^{(n)}(ta+(1-t)b)|dt$$
$$\leq\frac{(b-a)^n}{2n!}\left(\int_0^1 t^{n-1}(n-2t)\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{n-1}(n-2t)\left|\begin{matrix}f^{(n)}(ta+\\(1-t)b)\end{matrix}\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq\frac{(b-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{n-1}(n-2t)\left[max\left\{\begin{matrix}|f^{(n)}(a)|^q,\\|f^{(n)}(b)|^q\end{matrix}\right\}\right]dt\right)^{\frac{1}{q}}$$
$$=\frac{(b-a)^n(n-1)}{2(n+1)!}[max\{|f^{(n)}(a)|^q,|f^{(n)}(b)|^q\}]^{\frac{1}{q}}$$

which completes the proof.

Theorem 3.2. *Let* $f:[a,b]\to\mathbb{R}$ *be an n-times differentiable function. If* $|f^{(n)}|^q$ *is quasi-convex on* $[a,b]$ *following inequality holds*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right|$$
$$\leq\frac{(b-a)^n}{2^{1+\frac{1}{q}}n!}\left(\frac{1}{np-p+1}\right)^{\frac{1}{p}}\left(\frac{n^{q+1}-(n-2)^{q+1}}{q+1}\right)^{\frac{1}{q}}$$
$$\times[max\{|f^{(n)}(a)|^q,|f^{(n)}(b)|^q\}]^{\frac{1}{q}}$$

for $n\geq 2$, $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1.2., property of the modulus, well-known Hölder integral inequality and *quasi*-convexity of $|f^{(n)}|^q$ we can write

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right|$$
$$\leq\frac{(b-a)^n}{2n!}\int_0^1 t^{n-1}(n-2t)|f^{(n)}(ta+(1-t)b)|dt$$
$$\leq\frac{(b-a)^n}{2n!}\left(\int_0^1 t^{(n-1)p}dt\right)^{\frac{1}{p}}\left(\int_0^1 (n-2t)^q\left|\begin{matrix}f^{(n)}(ta+\\(1-t)b)\end{matrix}\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq\frac{(b-a)^n}{2n!}\left(\frac{1}{np-p+1}\right)^{\frac{1}{p}}\left(\int_0^1 (n-2t)^q\left[max\left\{\begin{matrix}|f^{(n)}(a)|^q,\\|f^{(n)}(b)|^q\end{matrix}\right\}\right]dt\right)^{\frac{1}{q}}$$
$$=\frac{(b-a)^n}{2^{1+\frac{1}{q}}n!}\left(\frac{1}{np-p+1}\right)^{\frac{1}{p}}\left(\frac{n^{q+1}-(n-2)^{q+1}}{q+1}\right)^{\frac{1}{q}}$$
$$\times[max\{|f^{(n)}(a)|^q,|f^{(n)}(b)|^q\}]^{\frac{1}{q}}.$$

The proof is completed.

4. References

Alomari, M., Darus, M., Kırmacı U.S. 2010. Refinements of Hadamard-type inequalities for quasi-convex functions with applications to trapezoidal formula and to special means, *Comput. Math. Applic.*, 59: 225-232.

Barnett, N. S., Dragomir, S. S. 2002. Applications of Ostrowski's version of the Grüss inequality for trapezoid type rules, *RGMIA Res. Rep. Coll.,* 5.

Bai, S.-P., Wang, S.-H., Qi, F. 2012. Some Hermite-Hadamard type inequalities for n-time differentiable (α,m)-convex functions, *J. Inequal. Appl.*, 2012:267.

Cerone, P., Dragomir, S. S., Roumelotis, J. 1999. Some Ostrowski type inequalities for n-time differentiable mappings and applications, *Demonstr. Math.*, 32: 697-712.

Dragomir, S. S., Pearce C. E. M. 2012. Jensen's inequality for quasi-convex functions, *NACO,* 2: 279-291.

Hwang, D.-Y. 2003. Some inequalities for n-time differentiable mappings and applications, *Kyungpook Math. J.*, 43: 335-343.

Hussain, S., Qaisar, S. 2013. New Integral Inequalities of the Type of Hermite-Hadamard Through Quasi Convexity, *Punjab University J. of Math.*, 45.

Ion, D. A. 2007. Some estimates on the Hermite-Hadamard inequality through quasi-convex functions, *An. Univ. Craiova, Ser. Mat. Inf.*, 34: 83-88.

Jiang, W.-D., Niu, D.-W., Hua, Y., Qi, F. 2012. Generalizations of Hermite-Hadamard inequality to n-time differentiable functions which are quasi-convex in the second sense, *Analysis (Munich),* 32: 209–220.

Kechriniotis, A. I., Theodorou, Y. A. 2008. New integral inequalities for n-time differentiable functions with applications for pdfs, *Appl. Math. Sciences*, 2: 353 – 362.

Kavurmacı, H., Avcı, M., Özdemir, M. E. 2011. New inequalities of Hermite-Hadamard type for convex functions with applications, *J. Inequal. Appl.*, 2011:86.

Özdemir, M. E., Kavurmacı, H., Akdemir, A. O., Avcı, M. 2012. Inequalities for convex and s-convex functions on Δ=[a,b] x[c,d], *J. Inequal. Appl.*, 2012:20.

Özdemir, M. E., Yıldız, Ç., Akdemir, A. O. 2012. On some new Hadamard-type inequalities for co-ordinated quasi-convex functions, *Hacettepe J. Math. Statistics*, 41: 697 – 707.

Pachpatte, B. G., 2004. New inequalities of Ostrowski and Trapezoid type for n-time differentiable functions, *Bull. Korean Math. Soc.* 41: 633-639.

Sofo, A. 2002. Integral inequalities for n-times differentiable mappings, with multiple branches, on the L_{p} norm, *Soochow J. Math.,* 28: 179-221

Wang, S.-H., Xi, B.-Y., Qi, F. 2012. Some new inequalities of Hermite-Hadamard type for n-time differentiable functions which are m-convex, *Analysis,* 32: 247-262.

Xi, B.-Y., Qi, F. 2013. Hermite-Hadamard type inequalities for functions whose derivatives are of convexities, *Nonlinear Funct. Anal. Appl.*, 18: 163-176.