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Research Article

New Results on Global Asymptotic Stability of Certain Third Order Nonlinear Vector Differential Equations

Belirli Üçüncü Mertebeden Doğrusal Olmayan Vektörel Diferansiyel Denklemlerin Global Asimptotik Kararlılığı Üzerine Yeni Sonuçlar

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Abstract

The aim of this paper is to give sufficient conditions to guarantee global asymptotic stability of a certain third order nonlinear vector differential equation. The results presented in this work were not published before and upgraded some results in the current literature. A simple example is also given to illustrate our main results.

Keywords: Nonlinear differential equation; Global stability; Lyapunov method

AMS-Mathematical Subject Classification: 34D23; 34C20; 34C23

Öz

Bu çalışmanın amacı, belirli üçüncü mertebeden doğrusal olmayan vektörel diferansiyel denklemlerin global asimptotik kararlılığını garanti etmek için yeterli şartları vermektir. Bu çalışmada sunulan sonuçlar önceden yayınlanmamış ve literatürdeki mevcut bazı sonuçları geliştirmiştir. Ana sonuçlarımızı resimlemek için basit bir örnek de verilmiştir.

Anahtar Kelimeler: Doğrusal olmayan diferansiyel denklemler; Global kararlılık; Lyapunov yöntemi

AMS-Matematik Konular Sınıflaması: 34D23; 34C20; 34C23

1. Introduction

This paper is concerned with the following vector differential equation:

$$\dddot{X} + G(X,\dot{X},\ddot{X}) + F(X,\dot{X})\dot{X} + H(X) = 0 \qquad (1)$$

where, in the real Euclidean space R^n; F, G and H have the following arguments: F is continuous n x n symmetric matrix, G and H are n- continuous vector functions. $J_F(X,Y)$ and $J_H(X)$ are symmetric matrices and denote the Jacobian matrices corresponding to the functions $F(X,Y)$ and $H(X)$ respectively, and have the following relations

$$J_F((X,Y)Y \mid X) = \sum_{k=1}^{n} \frac{\partial f_{ik}}{\partial x_j} y_k, \text{ and } J_H(X) = \left(\frac{\partial h_i}{\partial x_j}\right).$$

The symbol $\langle X,Y \rangle$ stands for the usual scalar product $\sum_{i=1}^{n} x_i y_i$. $\lambda_i(\cdot)$ are the eigenvalues of the corresponding $(\cdot)$ matrix. In addition, let the derivatives $\frac{\partial f_{ik}}{\partial x_j}$ and $\frac{\partial h_i}{\partial x_j}$ exist, $(i,j = 1,\ldots,n)$.

Such equations are closely related to n - dimensional third order ordinary differential systems which arise in the analysis of Jerk equations $(J(x,x',x'',x''') = 0)$ of multi input- multi output dynamical systems, nonlinear oscillations and biological mathematics (Zhang and Yu 2013), the third order nonlinear oscillatory systems (Talukdar et. al. 2012) for a real application.

The most efficient tool for the study of the stability of a given nonlinear dynamical system is provided by Lyapunov theory (Lyapunov 1992, Barbashin 1970, Iggidr and Sallet 2003). One can find a large number of beautiful works in the literature, discussing the qualitative behavior of nonlinear differential equations (see some examples, Korkmaz and Tunc 2014, Korkmaz and Tunc 2015, Korkmaz and Tunc 2016, Tunc 2009, Ates 2013, Ezeilo 1960, Qian 2000, Omeike 2007, Tunc and Ates 2006, Tunc 2004, Tunc 2006).

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Furthermore, we can announce the most recent studies which state the importance of the subject as follows:

Tunc (2009) proved global asymptotic stability of the equation

$$\dddot{X}+\Psi(\dot{X})\ddot{X}+B\dot{X}+cX=P(t).$$

Ates (2013), Zhang and Yu (2013) investigated the global asymptotic stability of the following equations:

$$x'''+\psi(x,x',x'')x''+f(x,x',x'')=p(t,x,x',x''),$$

and

$$x'''+g(x,x',x'')+f(x,x')x'+h(x)=0,$$

respectively.

In system theory, one of the most basic issues is the stability of dynamical systems. The most complete contribution to the stability analysis of nonlinear dynamical systems is due to Lyapunov (Zhang and Yu 2013). Lyapunov's results with the LaSalle invariance principle (LaSalle 1960) provide a powerful framework for analyzing the stability of nonlinear dynamical systems. Today, this method is recognized as an excellent tool not only in the study of differential equations but also in the theory of control systems, dynamical systems, systems with time lag, power system analysis, time-varying nonlinear feedback systems, and so on.

Hence, the importance of the subject may need some extension. Thus, we paid our attention on the very recent study of Zhang and Yu (2013), and then we upgraded this work to n- dimension systems under the related assumptions of Zhang and Yu (2013). Moreover, this study improves and extends the work of Zhang and Yu (2013). Here, we carefully constructed a suitable Lyapunov function for (1). Then, by using the well-known LaSalle's invariance principle (LaSalle 1960), and consequently we establish a new result (which not published before) on the global asymptotic stability of the zero solution of (1).

2. Main Result

Equation (1) is equivalent to the system

$$\dot{X}=Y,\dot{Y}=Z,\dot{Z}=-H(X)-F(X,Y)Y-G(X,Y,Z). \qquad (2)$$

The following algebraic results which will be needed in improving our main results.

Lemma 1 Let A be a real symmetric n x n matrix, then for any $X\in R^n$, we have

$$\delta_d\|X\|^2\le\langle AX,X\rangle\le\|X\|^2\Delta_d,$$

where δ_d and Δ_d are the least and the greatest eigenvalues of A, respectively.

Proof. See (Abou-El-Ela and Sadek 1990).

Lemma 2

1) $\frac{d}{dt}\int_0^1\langle H(\sigma X),X\rangle d\sigma=\langle H(X),Y\rangle;$

2) $\frac{d}{dt}\int_0^1\langle \sigma F(X,\sigma Y)Y,Y\rangle d\sigma=\langle F(X,Y)Z,Y\rangle+\int_0^1\langle\sigma J_F((X,\sigma Y)Y\mid X)Y,Y\rangle d\sigma$

Proof. See (Abou-El-Ela and Sadek 1990)

Theorem

Let there exist two positive numbers $a>0$ and $b>\frac{1}{2}$ such that for all $(X,Y,Z)\in R^n$, and $i=1,\ldots,n$. Then we have the following:

(i) $a<\frac{\|G(X,Y,Z)\|}{\|Z\|}<a+\frac{2}{a}$ *for* $\|Z\|\neq 0$, *and* $G(X,Y,0)=0$;

(ii) $\lambda_i(F(X,Y))\ge b$;

(iii) $J_F((X,Y)Y\mid X)\le 0$;

(iv) $\lambda_i(J_H(X))<a\left(b-\frac{1}{2}\right)$;

(v) $h(x_i)sgnx_i>0$ *for* $x_i\neq 0$, *and* $H(0)=0$;

(vi) $\int_0^{\pm\infty}h(x_i)dx_i=\infty$.

Then, the zero solution of (1) is globally asymptotically stable.

Proof.

The main results will be proved by employing Lyapunov function $V=V(X,Y,Z)$ which is given by:

$$V(X,Y,Z)=a\int_0^1\langle H(\sigma X),X\rangle d\sigma+\langle H(X),Y\rangle$$
$$+\frac{1}{2}\langle aY+Z,aY+Z\rangle+\int_0^1\langle\sigma F(X,\sigma Y)Y,Y\rangle d\sigma.$$

V can be rearranged as follows:

$$V(X,Y,Z)=a\int_0^1\langle H(\sigma X),X\rangle d\sigma-\frac{1}{2b}\langle H(X),H(X)\rangle$$
$$+\frac{1}{2}\langle aY+Z,aY+Z\rangle+\frac{1}{2b}\langle bY+H(X),bY+H(X)\rangle$$
$$+\int_0^1\left\langle\left(\sigma F(X,\sigma Y)-\frac{1}{2}bI\right)Y,Y\right\rangle d\sigma.$$

It is clear that

$$H(0)=0,\frac{\partial}{\partial\sigma}H(\sigma X)=J_H(\sigma X)X.$$

Then, we have

$$H(X)=\int_0^1 J_H(\sigma X)Xd\sigma.$$

By using the assumption

$$\lambda_i(J_H(X)) \le a\left(b-\frac{1}{2}\right),$$

we have

$$\int_0^1 \langle H(\sigma X),X\rangle d\sigma = \int_0^1\int_0^1 \langle \sigma_1 J_H(\sigma_1\sigma_2 X)X,X\rangle d\sigma_2 d\sigma_1$$
$$\le \int_0^1\int_0^1 \left\langle \sigma_1 a\left(b-\frac{1}{2}\right)X,X\right\rangle d\sigma_2 d\sigma_1 = \frac{1}{2}a\left(b-\frac{1}{2}\right)\langle X,X\rangle =$$
$$\frac{1}{2}a\left(b-\frac{1}{2}\right)\|X\|^2$$

and

$$\langle H(X),H(X)\rangle = \int_0^1 \langle J_H(\sigma X)X, J_H(\sigma X)X\rangle d\sigma$$
$$\le a^2\left(b-\frac{1}{2}\right)^2\int_0^1 \langle X,X\rangle d\sigma = a^2\left(b-\frac{1}{2}\right)^2\|X\|^2.$$

In view of the theorem and the above discussion, we obtain

$$V \ge \frac{a^2}{4b}\left(b-\frac{1}{2}\right)\|X\|^2.$$

Consequently, V is a non-negative function Observe that if $V = 0$, then we necessarily have

$X = 0$, $Y = 0$, $Z = 0$.

Thus, we deduce that $V(X,Y,Z)$ is a positive definite function.

Next, we show that the derivative of V along system (2) is negative semi definite.

Let, $(X(t), Y(t), Z(t))$ be any solution of system (2). Differentiating the function $V(X,Y,Z)$

with respect to t along system (2), we obtain

$$\dot V = \langle \dot H(X)Y,Y\rangle + \langle aY, aZ - G(X,Y,Z)\rangle$$
$$-\langle aY, F(X,Y)Y\rangle + \langle aZ,Z\rangle - \langle Z, G(X,Y,Z)\rangle$$
$$+\int_0^1 \langle \sigma J(F(X,\sigma Y)Y\mid X)Y,Y\rangle d\sigma.$$

In view of $xy \le \frac{1}{2}(x^2+y^2)$, and the theorem, we have

$$\dot V \le J_H(X)\|Y\|^2 + \frac{a}{2}\left[\|Y\|^2 + \left\langle \begin{matrix} aZ-G(X,Y,Z), aZ- \\ G(X,Y,Z)\end{matrix}\right\rangle\right]$$
$$-ab\|Y\|^2 + a\langle Z,Z\rangle - \langle Z, G(X,Y,Z)\rangle$$
$$= \frac{a}{2}[aZ-G(X,Y,Z)]\left[\left(a+\frac{2}{a}\right)Z - G(X,Y,Z)\right]$$
$$+\left[J_H(X) + \frac{a}{2} - ab\right]\|Y\|^2$$
$$\le \frac{a}{2}\|aZ-G(X,Y,Z)\|\left\|\left(a+\frac{2}{a}\right)Z - G(X,Y,Z)\right\| +$$
$$\left[J_H(X) + a\left(\frac{1}{2}-b\right)\right]\|Y\|^2$$
$$\le \frac{a}{2}[\|G(X,Y,Z)\| - a\|Z\|]\left[\left(a+\frac{2}{a}\right)\|Z\| - \|G(X,Y,Z)\|\right] +$$
$$\left[J_H(X) + a\left(\frac{1}{2}-b\right)\right]\|Y\|^2.$$

By using the results of Lemma 2, the theorem and the above discussion it is clear that

$$\dot V(X,Y,Z) \le 0, \forall (X,Y,Z) \in R^n.$$

Assume that

$$K = \{(X,Y,Z) \mid \dot V_{(2)}(X,Y,Z) = 0\}.$$

Let K contains no complete solutions of system (2) other than the zero solution. In fact, if $(X(t),Y(t),Z(t))$ is a complete solution of system (2) that is contained in K. Then, by

$\dot V_{(2)}(X,Y,Z) = 0$, that is, $Y(t) \equiv 0 \equiv Z(t)$.

Then, by the first equation in (2), we see that $X(t) \equiv const$. Because the zero solution (0,0,0) is the unique constant solution of system (2), we deduce that $X(t) \equiv 0$. This proves our assumption on K.

Finally, there remains to show the boundedness solutions of system (2). Assume that $(X(t),Y(t),Z(t))$ be a solution of system (2) with initial value

$$X(0) = X_0, Y(0) = Y_0, Z(0) = Z_0,$$

where $(X_0,Y_0,Z_0) \in R^3$. Let take a positive number $\beta>0$ such that

$$(X_0,Y_0,Z_0) \in D := \{(X,Y,Z) \mid V(X,Y,Z) \le \beta\}.$$

By (vi), there exists a positive constant $M>0$ such that

$R(X)>\beta$, if $\|X\| > M$,

where

$$R(X) = a\int_0^1 \langle H(\sigma X),X\rangle d\sigma - \frac{1}{2b}\langle H(X),H(X)\rangle.$$

Let $(X,Y,Z) \in D$. Then, we have

$$V(X,Y,Z) = a\int_0^1 \langle H(\sigma X),X\rangle d\sigma - \frac{1}{2b}\langle H(X),H(X)\rangle$$
$$+\frac{1}{2}\langle aY+Z, aY+Z\rangle + \frac{1}{2b}\langle bY+H(X), bY+H(X)\rangle$$
$$+\int_0^1 \left\langle\left(\sigma F(X,\sigma Y) - \frac{1}{2}bI\right)Y,Y\right\rangle d\sigma \le \beta$$

Then, we necessarily have the following

$$R(X) < \beta,\ \frac{1}{2}\langle aY+Z, aY+Z\rangle < \beta,$$
$$\frac{1}{2b}\langle bY+H(X), bY+H(X)\rangle < \beta \tag{3}$$

Now, Assume that $\|X\| \le M$. Then, by the third inequality in (3), we deduce that

$$\|Y\| \le \sqrt{\frac{2\beta}{b}} + \frac{1}{b}\max_{\|X\|\le M}\|H(X)\| := N.$$

Consequently, the second inequality in (3) implies that

$\| Z \| < \sqrt{2\beta} + aN.$

Hence, D is a bounded set.

Therefore

$(X(t), Y(t), Z(t)) \in D,$ for all $t \geq 0$.

3. Example

If we take in (1), for $n = 2$,

$$G(X,Y,Z) = \begin{bmatrix} (2+cos(x+y))z_1 \\ (2+cos(x+y))z_2 \end{bmatrix},$$

$$F(X,Y) = \begin{bmatrix} 2+\frac{1}{1+x_1y_1} & 0 \\ 0 & 2+\frac{1}{1+x_2y_2} \end{bmatrix}, H(X) = \begin{bmatrix} \frac{1}{4}x_1 \\ \frac{1}{4}x_2 \end{bmatrix},$$

with $a = \frac{1}{2}, b = 1.$

Then, we have the following:

$$\frac{\| G \|}{\| Z \|} = 2 + cos(x+y).$$

Hence

$$\frac{1}{2} < 2 + cos(x+y) < \frac{9}{2};$$

$$\lambda_1(F(X,Y)) = 2 + \frac{1}{1+x_1y_1} \geq 1, x_1y_1 > 0;$$

$$\lambda_2(F(X,Y)) = 2 + \frac{1}{1+x_2y_2} \geq 1, x_2y_2 > 0,$$

and

$$J_F((X,Y)Y \mid X) = -\left[\frac{y_1^2}{(1+x_1y_1)^2} + \frac{y_2^2}{(1+x_2y_2)^2}\right] \leq 0;$$

$$J_H(X) = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}, \lambda_1(J_H) = \frac{1}{4} < \frac{1}{2}, \lambda_2(J_H) = \frac{1}{4} < \frac{1}{2}.$$

Hence, our example verifies all the hypotheses of the theorem.

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