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Research Article

Weighted Approximation Properties of Positive Linear Operators Based on q-Integer

Doğrusal Pozitif Operatörlerin q Tamsayılarla Ağırlıklı Yaklaşım Özellikleri

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Abstract

This paper deals with q-analogues of linear positive operators defined in weighted space of continuous functions defined on real axis. We study, using q-calculus, the existence of Korovkin's theorem in the spaces of continuous and unbounded functions defined on unbounded sets.

Keywords: Korovkin theorem, Linear positive operator, Q-calculus, Weighted space

Öz

Bu çalışma reel eksen üzerinde tanımlı sürekli fonksiyonların ağırlıklı uzayında tanımlı doğrusal pozitif operatörlerin q benzerleri ile ilgilidir. q analizi kullanarak, sınırsız kümeler üzerinde tanımlı, sınırsız ve sürekli fonksiyonların uzayında Korovkin teoreminin varlığını araştıracağız.

Anahtar Kelimeler: Korovkin teoremi, Doğrusal pozitif operatörler, Q-analiz, Ağırlıklı uzay

1. Introduction

The history of q-calculus, dates back to the eighteenth century. It can in fact be taken as far back as Euler. Afterwards, many remarkable results were obtained in the nineteenth century. In the second half of the twentieth century there was a significant increase of activity in the area of q-calculus due to applications of q-calculus in mathematics and physics. During the last two decades, the applications of q-calculus emerged as a new area in the field of approximation theory [3]. Several generalizations of well-known positive linear operators based on q-integers were introduced by several authors. Approximation theory has important applications in functional analysis, numerical solutions of differential and integral equations [8], [9], [10]. In the classical Korovkin theorem [7] the uniform convergence in $C([a; b])$, the space of all continuous real-valued functions defined on the compact interval $[a; b]$, is proved for a sequence of positive linear operators, assuming the convergence only on the test functions $1, x, x^2$. Now, we give Gadjiev's results in weighted spaces. We use the same notation as in [1]. We now introduce some notation and basic definitions used in this paper.

For any fixed real number $q > 0$ and non-negative integer r, the q-integers of the number r are defined by

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & if\ q \neq 1 \\ n, & q = 1 \end{cases}$$

Also we have $[0]_q = 0$.

Gadjiev defined the weighted spaces $\rho \geq 1$ (ρ is unbounded function) as the spaces $B\rho: \{f: \| f(x) \| \leq M_f \rho(x): x \in \mathbb{R}\}$ and $C_\rho = (f \in B_\rho: f \text{ continuous})$ of functions which are defined on unbounded regions. B_ρ is a normed space with the norm $\| f \|_\rho = \sup_{x \in \mathbb{R}} \frac{\| f(x) \|}{p(x)}$. Gadjiev gave the following theorem in [4], [5] and [6].

Theorem 1.1 Let $\{L_n\}$ be sequence of linear positive operators taking C_ρ into B_ρ and satisfying the conditions

$$\lim_{n\to\infty} \| L_n(t^\vartheta, x) - x^\vartheta \|_\rho = 0, \vartheta = 0, 1, 2.$$

Then, there exists a function $f^* \in C_\rho$ such that

$$\lim_{n\to\infty} \| L_n(f^*, x) - f^*(x) \|_\rho \neq 0.$$

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Then [1] extend these results of Gadjiev for different ρ_1 and ρ_2, acting from C_{ρ_1} into B_{ρ_2}. We recall some notations defined in [1]. Let positive linear operator L satisfy following properties.

1. Positive linear operators, defined on C_{ρ_1}, acting from C_{ρ_1} to B_{ρ_1} iff the inequality

$$\| L(\rho_1,x)\|_{\rho 2} \le M_1$$

holds.

2. Let L: $C_{\rho_1} \to B_{\rho_1}$ be positive linear operator. Then

$$\| L \|_{C_{\rho 1}\to B_{\rho 2}} = \| L(\rho_1,x)\|_{\rho_2}$$

and therefore for all $f \in C_{\rho_1}$;

$$\| L(f,x)\|_{\rho_2} \le \| L(\rho_1,x)\|_{\rho_2} \| f \|_{\rho_1}.$$

3. Let $A_n: C_{\rho_1} \to B_{\rho_2}$ be positive linear operators. Suppose that there exists $M > 0$ such that for all $x \in \mathbb{R}$, $\rho_1(x) \le M_{\rho_2}(x)$. If

$$\lim_{n\to\infty} \| A_n(\rho_1,x) - \rho_1(x)\|_{\rho_2} = 0$$

then the sequence of norms $\| A_n \|_{C_{\rho 1}\to B_{\rho 2}}$ is uniformly bounded.

In this work we show that a Korovkin's theorem does not hold for a class of positive linear operators acting from C_{ρ_1} into B_{ρ_2} for different ρ_1 and ρ_2 using q-calculus.

2. q-Analogues in Different Weighted Spaces

Theorem 2.1 Let q > 0, $n \in \mathbb{N}$ and φ_1, φ_2 be two monotone increasing continuous functions, on real axis such that $\lim_{x\to\pm\infty} \varphi_1(x) = \lim_{x\to\pm\infty} \varphi_2(x) = \pm\infty$ and that $\rho_1(x) \le M\rho_2(x)$ (M>0 is arbitrary constant) for all $x \in \mathbb{R}$ where

$$\rho_k(x) = 1 + \varphi_k^2(x), \quad k = 1,2$$

and

$$\lim_{x\to\infty} \frac{\rho_1(x)}{\rho_2(x)} = a \neq 0.$$

Then there exist a sequence $A_n: C_{p_1} \to B_{\rho_2}$ of positive linear operators satisfying the following three conditions

$$\lim_{n\to\infty} \| A_n(\varphi_1^v,x) - \varphi_1^v(x)\|_{\rho_2} = 0, v = 0,1,2.$$

Then there exist $f^* \in C_{\rho_1}$ *such that*

$$\lim_{n\to\infty} \sup \| A_n(f^*,x) - f^*(x)\|_{\rho_2} \neq 0.$$

Proof. φ_1, φ_2 are two continuous functions that provide given conditions, A_n be a sequence of operators for every $n \in \mathbb{N}$ defined as follows

$$A_n(f,x) := \begin{cases} f(x) + \dfrac{\rho_2(x)}{2\rho_2([n]_q)}\left(\dfrac{\varphi_1^2(x)}{\varphi_1^2(x+1)} f(x+1) - f(x)\right); & 0 \le x \le [n]_q \\ f(x) & ;\textit{if others} \end{cases}$$

We can assume that $\varphi_1(0) = 0$ and $\varphi_2(0) = 0$ since $\overline{\varphi_1}(x) := \varphi_1(x) - \varphi_1(0)$ implies $\overline{\varphi_1}(0) = 0$ whenever $\varphi_1(0) \neq 0$.

It is obvious that for every $n \in \mathbb{N}$; A_n are linear. Now we show that this operator is positive.

Since $[n]_q = 1 + q + q^2 + \ldots + q^{n-1}$, for all $0 \le x \le [n]_q$, $\rho_2(x) \le \rho_2([n]_q)$ and so $1 - \frac{\rho_2(x)}{2\rho_2([n]_q)} \ge \frac{1}{2} > 0$ then we get

$$\begin{aligned} A_n(f,x) &= f(x) + \frac{\rho_2(x)}{2\rho_2([n]_q)}\left(\frac{\varphi_1^2(x)}{\varphi_1^2(x+1)} f(x+1) - f(x)\right) \\ &= f(x)\left(1 - \frac{\rho_2(x)}{2\rho_2([n]_q}\right) + \frac{\rho_2(x)}{2\rho_2([n]_q)}\left(\frac{\varphi_1^2(x)}{\varphi_1^2(x+1)} f(x+1) - f(x)\right) \end{aligned}$$

It is clear that $A_n(f,x) \ge 0$ for each $x \in [0,[n]_q]$ and $f(x) \ge 0$.

Now we show that this sequence of operators $A_n: C_{\rho_1} \to B_{\rho_2}$. Since φ_1 is a monotonic function satisfying $\varphi_1^2(x) \le \varphi_1^2(x+1)$ and using properties 1 we get the following inequality

$$A_n(\rho_1,x) = \rho_1(x) + \frac{\rho_2(x)}{2\rho_2([n]_q)}\left(\begin{matrix}\dfrac{\varphi_1^2(x)}{\varphi_1^2(x+1)}(1 + \varphi_1^2(x+1)) - \\ (1 + \varphi_1^2(x))\end{matrix}\right)$$

$$\le \rho_1(x) \le M\rho_2(x)$$

So $A_n(\rho_1,x) \in B_{\rho_2}$. In this way are positive linear operators $A_n; C_{\rho_1} \to B_{\rho_2}$. Now we show that our operators satisfy three conditions. Since for each $x \in [0,[n]_q]$

$$A_n(1,x) = 1 + \frac{\rho_2(x)}{2\rho_2([n]_q)}\left[\frac{\varphi_1^2(x)}{\varphi_1^2(x+1)} - 1\right]$$

$$\frac{|A_n(1,x) - 1|}{\rho_2(x)} = \frac{1}{2\rho_2([n]_q)}\left|\frac{\varphi_1^2(x)}{\varphi_1^2(x+1)} - 1\right| \le \frac{1}{2\rho_2([n])}$$

For $q > 0$ one has $\lim_{n\to\infty} \frac{1}{[n]_q} = 1 - q$. For this reason, we consider a sequence $q = q_n$ such that $\lim_{n\to\infty} q_n = 1$.

So we get

$$\lim_{n\to\infty} \| A_n(1,x) - 1 \|_{\rho_2} = 0.$$

Similary since

$$\frac{|A_n(\varphi_1,x) - \varphi_1(x)|}{\rho_2(x)} = \frac{|\varphi_1(x)|}{2\rho_2([n]_q)}\left|\frac{\varphi_1(x)}{\varphi_1(x+1)} - 1\right|$$

and by using monotonicity of φ_1, $\left|\frac{\varphi_1(x)}{\varphi_1(x+1)} - 1\right| < 1$. Then we get

$$\| A_n(\varphi_1,x) - \varphi_1(x) \|_{\rho_2} \le \sup_{[0,[n]_q]} \frac{1}{2\rho_2([n]_q)} |\varphi_1(x)| \le \frac{\varphi_1([n]_q)}{2\rho_2([n]_q)}$$

Therefore

$$\lim_{n\to\infty} \| A_n(\varphi_1,x) - \varphi_1(x) \|_{\rho_2} = 0.$$

Lastly, we have

$$A_n(\varphi_1^2,x)-\varphi_1^2(x)=\frac{\rho_2(x)}{2\rho_2([n]_q)}\left(\frac{\varphi_1^2(x)}{\varphi_1^2(x+1)}\begin{pmatrix}1+\varphi_1^2(x+1))-\\(1+\varphi_1^2(x)\end{pmatrix}\right)=0$$

So

$$\lim_{n\to\infty}\| A_n(\varphi_1^2,x)-\varphi_1^2(x)\|_{\rho_2}=0.$$

Finally we show that for a $f^*\in C_{\rho_1}$

$$\lim_{n\to\infty}\sup\| A_n(f^*,x)-f^*(x)\|_{\rho_2}\neq 0.$$

Let $g(x)$ be a function defined on the interval [-1,1] given as follows

$$g(x):=\begin{cases}2(1+x);-1\le x\le 0\\ 2(1-x);0<x<1\end{cases}$$

and let us extend $g(x)$ to a function $h(x)$ on $\mathbb{R}$ with period 2. If f^* is defined by

$$f^*(x):=\varphi_1^2(x).h(x)$$

for all $x\in\mathbb{R}$; then we have the following equality for $x\in[0,[n]_q]$ and $n\in\mathbb{N}$,

$$A_n(f^*,x)=f^*(x)+\frac{\rho_2(x)}{2\rho_2([n]_q}\left[\frac{\varphi_1^2(x)}{\varphi_1^2(x+1)}f^*(x+1)-f^*(x)\right]$$
$$=f^*(x)+\frac{\rho_2(x)}{2\rho_2([n]_q}\left[\frac{\varphi_1^2(x)}{\varphi_1^2(x+1)}\varphi_1^2(x+1)h(x+1)-\varphi_1^2(x)h(x)\right]$$
$$=f^*(x)+\frac{\rho_2(x)}{2\rho_2([n]_q}\varphi_1^2(x)(h(x+1)-h(x)).$$
$$A_n(f^*,x)-f^*(x)=\frac{\rho_2(x)}{2\rho_2([n]_q}\varphi_1^2(x)[h(x+1)-h(x)].$$

Therefore we have the following inequality,

$$\sup_{x\in[0,[n]_q]}\frac{|A_n(f^*,x)-f^*(x)|}{\rho_2(x)}\ge\frac{\varphi_1^2([n]_q)}{2\rho_2([n]_q)}|h([n]_q+1)-h([n]_q)|$$
$$=\frac{\varphi_1^2([n]_q)}{2\rho_2([n]_q)}2=\frac{\varphi_1^2([n]_q)}{1+\varphi_2^2([n]_q)}.$$

Then we get $\lim_{n\to\infty}\frac{1}{1+\varphi_2^2([n]_q)}=0$ and

$$\lim_{n\to\infty}\frac{1+\varphi_1^2([n]_q)}{1+\varphi_2^2([n]_q)}=a\neq 0\ ,$$

which proves the theorem.

3. Results

Theorem 2.1 shows that there exists no theorem of Korovkin type for the class of positive linear operators from C_{ρ_1} to B_{ρ_2} means of *q*-calculus. Note that a more general statement can be proved for a class of positive linear operators acting from C_{ρ_1} to $C_{\rho 2}$. We get following theorems.

Theorem 2.2 Let q > 0, ρ_1 and ρ_2 be as in Theorem 2.1. Then there exist a sequence $A_n:C_{\rho_1}\to B_{\rho_2}$ of positive linear operators such that

$$\lim_{n\to\infty}\| A_n(\varphi_1^v,x)-\varphi_1^v(x)\|_{\rho_2}=0, v=0,1,2.$$

Then there exist $f^*\in C_{\rho_1}$ *such that*

$$\lim_{n\to\infty}\sup\| A_n(f^*,x)-f^*(x)\|_{\rho_2}\neq 0.$$

The positive statements of Korovkin type theorems for linear positive operators $A_n:C_{\rho_1}\to B_{\rho_2}$ may be proved as in [2],[4] and [5] in some subspace of C_{ρ_2}.

We study using the concepts of q-analysis the existence of Korovkin's theorem in the spaces of continuous and unbounded functions defined on unbounded sets.

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