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Research Article

New Hermite-Hadamard-Fejér Type Inequalities for Harmonically-P-Functions

Harmonically-P-Fonksiyonlar İçin Yeni Hermite-Hadamard-Fejér Tipli Eşitsizlikler

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Abstract

In this paper, we give new the theorems and results of the right side of Hermite-Hadamard-Fejér type inequalities for harmonically--functions via fractional integrals. Finally we implement our result using Hypergeometric functions and special means.

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Öz

Bu makalede, harmonically-P-fonksiyonlar için Hermite-Hadamard- Fejér tipli kesirli integral eşitsizlikleri için yeni teoremler ve sonuçlar verildi. Son kısmında ise özel ortalamaları ve Hipergeometrik fonksiyonları kullanarak sonuçlara uyguladık.

Anahtar Kelimeler: Kesirli integral, Harmonically-P-fonksiyonlar, Hermite-Hadamard-Fejér tipli eşitsizlikler

1. Introduction

Lots of inequalities have been established for convex functions but the most famous is the Hermite-Hadamard inequality, due to its rich importance and applications, which is stated as follows: Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a,b \in I$ with $a < b$. Then following double inequalities holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The inequalities (1.1) hold in reversed direction if f is concave.

Many researcher have studied on the Hermite-Hadamard inequalities for convex functions. (1.1) have been generalized and enhanced for many classes of convex functions. See therein (Latif et al. 2015, Noor et al. 2015, Mihai et al. 2015, Varosanec 2007, Xi and Qi 2013, Xi et al. 2013).

İşcan have represented harmonically convex function and have proved inequalities related to its as follows (İşcan 2014):

Definition 1. (İşcan 2014) *Let* $I \subset \mathbb{R} \backslash \mathbb{R}\{0\}$ *be a real interval. A function* $f: I \to \mathbb{R}$ *is said to be harmonically convex, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y) + (1-t)f(x) \tag{1.2}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (1.2) is reversed, then f is said to be harmonically concave.

Proposition 1. (İşcan 2014) *Let* $I \subset R\backslash\{0\}$ *be a real interval and* $f: I \to \mathbb{R}$ *is function, then:*

1. if $I \subset (0,\infty)$ and f is convex and nondecreasing function then f is harmonically convex.
2. if $I \subset (0,\infty)$ and f is harmonically convex and nonincreasing function then f is convex.
3. if $I \subset (-\infty,0)$ and f is harmonically convex and nondecreasing function then f is convex.
4. if $I \subset (-\infty,0)$ and f is convex and nonincreasing function then f is harmonically convex.

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The following definitions and mathematical preliminaries of fractional calculus theory are used further in this paper.

Definition 2. (Kilbas et al. 2006) *Let* $f \in L[a,b]$. *The Riemann-Liouville integrals* $J_{a^+}^{\alpha} f$ *and* $J_{b^-}^{\alpha} f$ *of order* $\alpha > 0$ *with* $a \geq 0$ *are defined by*

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt, \quad x > a \tag{1.3}$$

and

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\,dt, \quad x < b \tag{1.4}$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^{\alpha} e^{-t} t^{\alpha-1} dt$ and $J_{a^+}^{0} f(x) = J_{b^-}^{0} f(x) = f(x)$.

Theorem 1. (İşcan 2014) *Let* $I \subset \mathrm{R}\backslash\{0\}$ *be a harmonically convex function on* I^0, $a,b \in I^0$ *with* $a<b$. *If* $f \in L([a,b])$ *then the following inequalities hold*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq \frac{f(a)+f(b)}{2}. \tag{1.5}$$

The above inequalities are sharp.

Latif et al. showed the following definition (Latif et al. 2015):

Definition 3. (Latif et al. 2015) *A function* $g:[a,b] \subseteq \mathrm{R}\backslash\{0\} \to \mathrm{R}$ *is said to be harmonically symmetric with respect to* $2ab/a+b$, *if*

$$g(x) = g\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{1}{x}}\right)$$

holds for all $x \in [a,b]$.

İşcan and Wu have revealed Hermite-Hadamard's inequalities for harmonically convex function via fractional integrals as follow (İşcan and Wu 2014):

Theorem 2. (İşcan and Wu 2014) *Let* $f: I \subseteq (0,\infty) \to \mathrm{R}$ *be a function such that* $f \in L[a,b]$, *where* $a,b \in I$ *with* $a<b$. *If f is a harmonically convex function on [a,b], then the following inequalities for fractional integrals hold:*

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\leq \frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha} \{J_{1/a^-}^{\alpha}(f \circ g)(1/b) \\ &+ J_{1/b^+}^{\alpha}(f \circ g)(1/a)\} \leq \frac{f(a)+f(b)}{2} \end{aligned} \tag{1.6}$$

with $\alpha > 0$.

Chen and Wu represented Hermite-Hadamard-Fejér inequality for harmonically convex functions as follow (Chen and Wu 2014):

Theorem 3. (Chen and Wu 2014) *Let* $f: I \subseteq \mathrm{R}\backslash\{0\} \to \mathrm{R}$ *be a harmonically convex function and* $a,b \in I$ *with* $a<b$. *If* $f \in L[a,b]$ *and* $g:[a,b] \subseteq \mathrm{R}\backslash\{0\} \to \mathrm{R}$ *is nonnegative integrable and harmonically symmetric with respect to* $2ab/a+b$ *then*

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) \int_a^b \frac{g(x)}{x^2} dx &\leq \int_a^b \frac{f(x)g(x)}{x^2} dx \\ &\leq \frac{f(a)+f(b)}{2} \int_a^b \frac{g(x)}{x^2} dx \end{aligned} \tag{1.7}$$

İşcan and Kunt showed Hermite-Hadamard-Fejér type inequality for harmonically convex functions in fractional integral forms and established following identity as follow (İşcan and Kunt 2015):

Theorem 4. (İşcan and Kunt 2015) *Let* $f:[a,b] \to \mathrm{R}$ *be harmonically convex function with* $a<b$ *and* $f \in L[a,b]$. *If* $g:[a,b] \to \mathrm{R}$ *is nonnegative, integrable and harmonically symmetric with respect to* $2ab/a+b$ *then the following inequalities for fractional integrals holds:*

$$\begin{aligned} & f\left(\frac{2ab}{a+b}\right)\left[J_{1/b^+}^{\alpha}(g \circ h)(1/a) + J_{1/a^-}^{\alpha}(g \circ h)(1/b)\right] \\ & \leq \left[J_{1/b^+}^{\alpha}(fg \circ h)(1/a) + J_{1/a^-}^{\alpha}(fg \circ h)(1/b)\right] \\ & \leq \frac{f(a)+f(b)}{2}\left[J_{1/b^+}^{\alpha}(g \circ h)(1/a) + J_{1/a^-}^{\alpha}(g \circ h)(1/b)\right] \end{aligned} \tag{1.8}$$

with $\alpha > 0$ and $h(x) = 1/x, x \in \left[\frac{1}{b}, \frac{1}{a}\right]$.

Definition 4. (Noor et al. 2015) *Let* $h:[0,1] \subseteq J \to \mathrm{R}$ *be a non-negative function. A function* $f: I \subset (0,\infty) \to \mathrm{R}$ *is said to be harmonically h-convex function, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq h(1-t)f(x) + h(y)f(y) \tag{1.9}$$

$\forall x,y \in I, t \in (0,1)$. Note that for $t = \frac{1}{2}$, we have Jensen's type harmonically *h*-convex function or harmonically-arithmetically (HA) *h*-convex functions

$$f\left(\frac{2xy}{x+y}\right) \leq h\left(\frac{1}{2}\right)[f(x)+f(y)]. \tag{1.10}$$

Remark 1. (Noor et al. 2015) It is obvious that for $h(t) = t, h(t) = t^s, h(t) = 1, h(t) = \frac{1}{t}$ and $h(t) = \frac{1}{t^s}$ in Definition 4, we have the definitions of harmonically convex functions, harmonically *s*-convex functions of second kind, harmonically *P*-functions, harmonically Godunova-Levin functions and harmonically *s*-Godunova-Levin functions of second kind respectively.

M. Aslam Noor et al. gave theorem for harmonically *h*-convex function as follow (Noor et al. 2015):

Theorem 5. (Noor et al. 2015) *Let $f:I \to \mathrm{R}$ be harmonically h-convex function where $a,b \in I$ with $a<b$. If $f \in L[a,b]$, then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq [f(a)+f(b)]\int_0^1 h(t)dt \tag{1.11}$$

Corollary 1. (Noor et al. 2015) *Let $f:I \to \mathrm{R}$ be harmonically P-function where $a,b \in I$ with $a<b$. If $f \in L[a,b]$, then*

$$\frac{1}{2} f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq f(a)+f(b) \tag{1.12}$$

Theorem 6. (Noor et al. 2015) *Let $f:I \to \mathrm{R}$ be differentiable function on I^{o} where $a,b \in I$ with $a<b$ and $f' \in L[a,b]$. If $|f'|^q, q \geq 0$ is harmonically h-convex function, then we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{ab(b-a)}{2} C_1^{1-\frac{1}{q}} \begin{pmatrix} C_2|f'(a)|^q + \\ C_3|f'(b)|^q \end{pmatrix}^{\frac{1}{q}} \tag{1.13}$$

where

$$C_1 = \frac{1}{ab} - \frac{2}{(b-a)^2} ln\left(\frac{(a+b)^2}{4ab}\right),$$

$$C_2 = \int_0^1 \frac{|1-2t|h(t)}{(tb+(1-t)a)^2 dt},$$

$$C_3 = \int_0^1 \frac{|1-2t|(1-t)}{(tb+(1-t)a)^2}dt$$

respectively.

İşcan et al. identities for harmonically convex function as follow (İşcan et al. 2016):

Lemma 1. (İşcan et al. 2016) *$f:I \subseteq \mathrm{R}\backslash\{0\} \to \mathrm{R}$ be a differentiable function on I^0, $h:[a,b] \to [0,\infty)$ be differentiable function on I, $a,b \in I$ and $a<b$. If $f' \in L[a,b]$ then the following equality holds:*

$$[h(b)-2h(a)]\frac{f(a)}{2} + h(b)\frac{f(b)}{2} - \int_a^b f(x)h'(x)dx$$

$$= \frac{b-a}{4ab}\left\{\int_0^1 [2h(L(t))-h(b)]f'(L(t))(L(t))^2 dt \right. \tag{1.14}$$

$$\left. + \int_0^1 [2h(U(t))-h(b)]f'(U(t))(U(t))^2 dt\right\}$$

where

$$L(t) = \frac{aH}{tH+(1-t)a}, U(t) = \frac{bH}{tH+(1-t)b}, \forall t \in [0,1]$$

and $H := H(a,b) = \dfrac{2ab}{a+b}$.

In this paper, we study both Fejér and Fejér fractional of new Hermite-Hadamard's inequalities related to both right and left of the inequalities for harmonically quasi convex functions.

2. Material and Methods

Throughout in this section, we will use the notations $L(t) = \dfrac{aH}{tH+(1-t)a}, U(t) = \dfrac{bH}{tH+(1-t)b}$ and $H = H(a,b) := \dfrac{2ab}{a+b}$.

Theorem 7. *Let $f:I \subseteq (0,\infty) \to \mathrm{R}$ be differentiable mapping on I^0, $a,b \in I$ with $a<b$. If $h:[a,b] \to [0,\infty)$ is a differentiable function and $|f'|^q$ is harmonically-P-function on $[a,b]$ for $q \geq 1$, the following inequality holds:*

$$\left|[h(b)-2h(a)]\frac{f(a)}{2} + h(b)\frac{f(b)}{2} - \int_a^b f(x)h'(x)dx\right|$$

$$\leq \frac{b-a}{4ab}\left\{\begin{array}{c} \left(\int_0^1 |2h(L(t))-h(b)|dt\right)^{1-\frac{1}{q}} \\ \left(\int_0^1 |2h(L(t))-h(b)|L^{2q}(t)(|f'(a)|^q + |f'(H)|^q)dt\right)^{\frac{1}{q}} \\ +\left(\int_0^1 |2h(U(t))-h(b)|dt\right)^{1-\frac{1}{q}} \\ \left(\int_0^1 |2h(U(t))-h(b)|U^{2q}(t)(|f'(b)|^q + |f'(H)|^q)dt\right)^{\frac{1}{q}} \end{array}\right\} \tag{2.1}$$

Proof. Firstly, we use power mean inequality in (1.14), we get

$$\left|[h(b)-2h(a)]\frac{f(a)}{2} + h(b)\frac{f(b)}{2} - \int_a^b f(x)h'(x)dx\right| \leq \frac{b-a}{4ab} \times$$

$$\left\{\left(\int_0^1 |2h(L(t))-h(b)|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |2h(L(t))-h(b)||f'(L(t))|^q L^{2q}(t)dt\right)^{\frac{1}{q}}\right.$$

$$\left. +\left(\int_0^1 |2h(U(t))-h(b)|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |2h(U(t))-h(b)||f'(U(t))|^q U^{2q}(t)dt\right)^{\frac{1}{q}}\right\}.$$

Since $|f'|^q, q \geq 1$, is harmonically-*P*-function, it is obtained

$$\left|[h(b)-2h(a)]\frac{f(a)}{2} + h(b)\frac{f(b)}{2} - \int_a^b f(x)h'(x)dx\right| \leq \frac{b-a}{4ab} \times$$

$$\leq \frac{b-a}{4ab}\left\{\begin{array}{c} \left(\int_0^1 |2h(L(t))-h(b)|dt\right)^{1-\frac{1}{q}} \\ \left(\int_0^1 |2h(L(t))-h(b)|L^{2q}(t)(|f'(a)|^q + |f'(H)|^q)dt\right)^{\frac{1}{q}} \\ +\left(\int_0^1 |2h(U(t))-h(b)|dt\right)^{1-\frac{1}{q}} \\ \left(\int_0^1 |2h(U(t))-h(b)|U^{2q}(t)(|f'(b)|^q + |f'(H)|^q)dt\right)^{\frac{1}{q}} \end{array}\right\} \tag{2.2}$$

So the proof is complete.

Corollary 2. *Let* $g:[a,b]\to[0,\infty)$ *be a positive continuous mapping and harmonically symmetric with respect to* $\frac{2ab}{a+b}, a<b$. *If* $h(t)=\int_{1/t}^{1/a}\psi(x)(g\circ\varphi)(x)dx$, $\psi(x)=\left[\left(x-\frac{1}{b}\right)^{\alpha-1}+\left(\frac{1}{a}-x\right)^{\alpha-1}\right]$ *for all* $t\in a,b]$, $\alpha>0$ *in Teorem 7, we obtain*

$$\left|\left(\frac{f(a)+f(b)}{2}\right)[J^{\alpha}_{1/b^+}(g\circ\varphi)(1/a)+J^{\alpha}_{1/a^-}(g\circ\varphi)(1/b)]\right.$$
$$\left.-[J^{\alpha}_{1/b^+}(fg\circ\varphi)(1/a)+J^{\alpha}_{1/a^-}(fg\circ\varphi)(1/b)]\right|$$
$$\leqslant\left(\frac{b-a}{2ab}\right)^{\alpha+1}\frac{\|g\|_\infty}{\Gamma(\alpha+1)}\left(\frac{2^{\alpha+2}-4}{\alpha+1}\right)^{1-\frac{1}{q}}\left(A_1(t,\alpha;q)\begin{pmatrix}|f'(a)|^q+\\|f'(H)|^q\end{pmatrix}\right.$$
$$\left.+A_2(t,\alpha;q)(|f'(b)|^q+|f'(H)|^q)\right)^{\frac{1}{q}}$$

(2.3)

where

$$A_1(t,\alpha;q)=\int_0^1[(1+t)^\alpha-(1-t)^\alpha]L^{2q}(t)dt,$$

$$A_2(t,\alpha;q)=\int_0^1[(1+t)^\alpha-(1-t)^\alpha]U^{2q}(t)dt.$$

Proof. If we use $h(t)=\int_{1/t}^{1/a}\psi(x)(g\circ\varphi)(x)dx, \varphi(x)=\frac{1}{x}$, in (2.1), we get

$$\Gamma(\alpha)\left|\left(\frac{f(a)+f(b)}{2}\right)[J^{\alpha}_{1/b^+}(g\circ\varphi)(1/a)+J^{\alpha}_{1/a^-}(g\circ\varphi)(1/b)]\right.$$
$$\left.-[J^{\alpha}_{1/b^+}(fg\circ\varphi)(1/a)+J^{\alpha}_{1/a^-}(fg\circ\varphi)(1/b)]\right|$$
$$\leq\frac{b-a}{4ab}\left\{\begin{array}{l}\left(\int_0^1\left|\begin{array}{l}2\int_{1/L(t)}^{1/a}\psi(x)(g\circ\varphi)(x)dx\\-\int_{1/b}^{1/a}\psi(x)(g\circ\varphi)(x)dx\end{array}\right|dt\right)^{1-\frac{1}{q}}\times\\\left(\int_0^1\left|2\int_{1/L(t)}^{1/a}\psi(x)(g\circ\varphi)(x)dx-\int_{1/b}^{1/a}\psi(x)(g\circ\varphi)(x)dx\right|\times L^{2q}(|f'(a)|^q+|f'(H)|^q)dt\right)^{\frac{1}{q}}\end{array}\right.$$
$$\left.\begin{array}{l}+\left(\int_0^1\left|\begin{array}{l}2\int_{1/U(t)}^{1/a}\psi(x)(g\circ\varphi)(x)dx\\-\int_{1/b}^{1/a}\psi(x)(g\circ\varphi)(x)dx\end{array}\right|dt\right)^{1-\frac{1}{q}}\times\\\left(\int_0^1\left|2\int_{1/L(t)}^{1/a}\psi(x)(g\circ\varphi)(x)dx-\int_{1/b}^{1/a}\psi(x)(g\circ\varphi)(x)dx\right|\times U^{2q}(|f'(b)|^q+|f'(H)|^q)dt\right)^{\frac{1}{q}}\end{array}\right\}.$$

(2.4)

If we use g function that be harmonically symmetric (i.e $\frac{2ab}{a+b}$) in the simple calculation, we get

$$\left|2\int_{1/L(t)}^{1/a}\psi(x)(g\circ\varphi)(x)dx-\int_{1/b}^{1/a}\psi(x)(g\circ\varphi)(x)dx\right|$$
$$=\left|\int_{1/L(t)}^{1/U(t)}\psi(x)(g\circ\varphi)(x)dx\right| \qquad (2.5)$$

and

$$\left|2\int_{1/U(t)}^{1/a}\psi(x)(g\circ\varphi)(x)dx-\int_{1/b}^{1/a}\psi(x)(g\circ\varphi)(x)dx\right|$$
$$=\left|\int_{1/L(t)}^{1/U(t)}\psi(x)(g\circ\varphi)(x)dx\right|. \qquad (2.6)$$

By using (2.5) and (2.6) in (2.4)

$$\left|\left(\frac{f(a)+f(b)}{2}\right)[J^{\alpha}_{1/b^+}(g\circ\varphi)(1/a)+J^{\alpha}_{1/a^-}(g\circ\varphi)(1/b)]\right.$$
$$\left.-[J^{\alpha}_{1/b^+}(fg\circ\varphi)(1/a)+J^{\alpha}_{1/a^-}(fg\circ\varphi)(1/b)]\right|$$
$$\leq\frac{(b-a)\|g\|_\infty}{4ab\Gamma(\alpha)}\left\{\begin{array}{l}\left(\int_0^1\left|\int_{1/L(t)}^{1/U(t)}\psi(x)dx\right|dt\right)^{1-\frac{1}{q}}\times\\\left(\int_0^1\left|\int_{1/L(t)}^{1/U(t)}\psi(x)dx\right|\times L^{2q}(x)(|f'(a)|^q+|f'(H)|^q)\right)^{\frac{1}{q}}\end{array}\right.$$
$$\left.+\left(\int_0^1\left|\int_{1/L(t)}^{1/U(t)}\psi(x)dx\right|dt\right)^{1-\frac{1}{q}}\times\left(\int_0^1\left|\int_{1/L(t)}^{1/U(t)}\psi(x)dx\right|\times U^{2q}(|f'(b)|^q+|f'(H)|^q)dt\right)^{\frac{1}{q}}\right\}.$$

(2.7)

If we write the following integral in (2.7)

$$\left|\int_{1/L(t)}^{1/U(t)}\psi(x)dx\right|=\frac{2^{1-\alpha}}{\alpha}\left(\frac{b-a}{ab}\right)^\alpha[(1+t)^\alpha-(1-t)^\alpha].$$

We have

$$\left|\left(\frac{f(a)+f(b)}{2}\right)[J^{\alpha}_{1/b^+}(g\circ\varphi)(1/a)+J^{\alpha}_{1/a-}(g\circ\varphi)(1/b)]\right.$$
$$\left.-[J^{\alpha}_{1/b^+}(fg\circ\varphi)(1/a)]+J^{\alpha}_{1/a^-}(fg\circ\varphi)(1/b)\right|$$
$$\leq\frac{(b-a)^{\alpha+1}\|g\|_\infty}{(2ab)^{\alpha+1}\Gamma(\alpha+1)}\left(\frac{2^{\alpha+1}-2}{\alpha+1}\right)^{1-\frac{1}{q}}\left\{\begin{array}{l}\left(\int_0^1[(1+t)^\alpha-(1-t)^\alpha]\times L^{2q}(|f'(b)|^q+|f'(H)|^q)dt\right)^{\frac{1}{q}}\\+\left(\int_0^1[(1+t)^\alpha-(1-t)^\alpha]\times U^{2q}(|f'(b)|^q+|f'(H)|^q)dt\right)^{\frac{1}{q}}\end{array}\right\}.$$

(2.8)

If we use $a^r+b^r\leq 2^{1-r}(a+b)^r, a,b>0, r\leq 1$ inequality in (2.8) the proof is completed.

Corollary 3.

1. If we take q = 1, α = 1 in (2.3), we get

$$\left|\left[\frac{f(a)+f(b)}{2}\right]\int_a^b \frac{g(x)}{x^2}dx-\int_a^b f(x)\frac{g(x)}{x^2}dx\right| \le \frac{(b-a)^2\|g\|_\infty}{8(ab)^2}\begin{pmatrix}A_1(t,1;1)(|f'(a)|+|f'(H)|)+\\A_2(t,1;1)(|f'(b)|+|f'(H)|)\end{pmatrix} \tag{2.9}$$

where

$$A_1(t,1;1)=\int_0^1 2tL^2(t)dt,$$

$$A_2(t,1;1)=\int_0^1 2tU^2(t)dt.$$

2. If we take q = 1, $g(x)$ = 1, in (2.3), we get

$$\left|\left(\frac{f(a)+f(b)}{2}\right)-\frac{(ab)^\alpha\Gamma(\alpha+1)}{2(b-a)^\alpha}\begin{bmatrix}J^\alpha_{1/b^+}(f\circ\varphi)(1/a)+\\J^\alpha_{1/a-}(f\circ\varphi)(1/b)\end{bmatrix}\right| \le \frac{b-a}{2^{\alpha+2}ab}\begin{bmatrix}A_1(t,\alpha;1)(|f'(a)|+|f'(H)|)+\\A_2(t,\alpha;1)(|f'(b)|+|f'(H)|)\end{bmatrix} \tag{2.10}$$

3. If we take α = 1, $g(x)$ = 1 in (2.3), we get

$$\left|\left(\frac{f(a)+f(b)}{2}\right)-\frac{ab}{(b-a)}\int_a^b\frac{f(x)}{x^2}dx\right| \le \frac{b-a}{2^{2+\frac{1}{q}}ab}\begin{bmatrix}A_1(t,1;q)(|f'(a)|^q+|f'(H)|^q)+\\A_2(t,1;q)(|f'(b)|^q+|f'(H)|^q)\end{bmatrix}^{1/q} \tag{2.11}$$

where

$$A_1(t,1;q)=\int_0^1 2tL^{2q}(t)dt,$$

$$A_2(t,1;q)=\int_0^1 2tU^{2q}(t)dt.$$

3. Applications

In this section we apply some of the above established inequalities of Hermite-Hadamard type involving the product of a quasi geometrically convex function and geometrically symetric function to construct inequalities for special means.

For positive numbers $a,b > 0$ with $a \neq b$

$$A(a,b)=\frac{a+b}{2}, L(a,b)=\frac{b-a}{ln\,b-ln\,a}, G(a,b)=\sqrt{ab},$$

$$H(a,b)=\frac{2ab}{a+b}$$

are the arithmetic mean, the logarithmic mean, geometric mean, harmonic mean respectively.

Now let $f(x)=x^{p-1}$ for $x>0, p>4$. Then

$$f'(x)=(p-1)x^{p-2},$$
$$f''(x)=(p-1)(p-2)x^{p-3}>0,$$
$$f'''(x)=(p-1)(p-2)(p-3)x^{p-4}>0.$$

As you can see, since f' is increasing and convex then f' is harmonically-P-function. Because f' is harmonically convex, f' is harmonically-P-function.

We use the following special functions in other applications (Kilbas et al. 2006):

1. The Beta function:

$$\beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=\int_0^1 t^{x,1}(1-t)^{y-1}dt,\ x,y>0.$$

2. The hypergeometric function:

$${}_2F_1(a,b;c;z) = \frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt, c>b>0, |z|<1$$

Corollary 4 $p>4$, *I be an interval,* $f:I\subseteq(0,\infty)\to\mathrm{R}$, *be a differentiable on* I^0, $a,b\in I$ *with* $a<b$. *If we take* $f(x) = x^{p-1}$, $p>4$ *that is harmonically-P-function in Corollary 3 (2), we get*

$$\left|A(a^{p-1},b^{p-1})-\frac{a^p b\alpha}{2(b-a)}\left[{}_2F_1\left(p-1,\alpha;\alpha+1;1-\frac{a}{b}\right)\beta(\alpha,1) + {}_2F_1\left(p-1,1;\alpha+1;1-\frac{a}{b}\right)\beta(1,\alpha)\right]\right| \le \frac{(b-a)(p-1)}{2^{\alpha+1}}\begin{bmatrix}A_1(t,\alpha;1)A(a^{p-2},H^{p-2})+\\A_2(t,\alpha;1)A(b^{p-2},H^{p-2})\end{bmatrix} \tag{3.1}$$

where

$$A_1(t,\alpha;1)=\int_0^1[(1+t)^\alpha-(1-t)^\alpha]L^{2q}(t)dt,$$

$$A_2(t,\alpha;1)=\int_0^1[(1+t)^\alpha-(1-t)^\alpha]U^{2q}(t)dt.$$

Proof. We get

$$J^\alpha_{\frac{1}{b}^+}(f\circ\varphi)(1/a)=\frac{1}{\Gamma(\alpha)}\int_{1/b}^{1/a}\left(\frac{1}{a}-t\right)^{\alpha-1}\frac{1}{t^{p-1}}dt.$$

If we apply variable changes $t=\frac{1}{a}-\left(\frac{1}{a}-\frac{1}{b}\right)u, dt=-\left(\frac{1}{a}-\frac{1}{b}\right)du$ on the last integral, we have for $0<a<b$ and $\alpha>0$

$$J^\alpha_{\frac{1}{b}^+}(f\circ\varphi)(1/b)=\frac{(b-a)^{\alpha-1}a^{p-1}}{\Gamma(\alpha)(ab)^{\alpha-1}}\int_0^1\frac{u^{\alpha-1}du}{\left[1-\left(1-\frac{a}{b}\right)u\right]^{p-1}} = \frac{(b-a)^{\alpha-1}a^{p-1}}{\Gamma(\alpha)(ab)^{\alpha-1}}{}_2F_1\left(p-1,\alpha;\alpha+1;1-\frac{a}{b}\right)\beta(\alpha,1). \tag{3.2}$$

If we use the same method, we get

$$J^{\alpha}_{\frac{1}{a}-}(f\circ\varphi)(1/b)=\frac{1}{\Gamma(\alpha)}\int_{1/b}^{1/a}\left(t-\frac{1}{b}\right)^{\alpha-1}\frac{1}{t^{p-1}}dt.$$

$$J^{\alpha}_{\frac{1}{a}-}(f\circ\varphi)(1/b)=\frac{(b-a)^{\alpha-1}a^{p-1}}{\Gamma(\alpha)(ab)^{\alpha-1}}\int_0^1\frac{(1-u)^{\alpha-1}du}{\left[1-\left(1-\frac{a}{b}\right)u\right]^{p-1}} \tag{3.3}$$

$$=\frac{(b-a)^{\alpha-1}a^{p-1}}{\Gamma(\alpha)(ab)^{\alpha-1}}{}_2F_1\left(p-1,1;1;\alpha+1-\frac{a}{b}\right)\beta(1,\alpha).$$

From the combination of (3.2) and (3.3), the proof is completed.

4. Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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