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**Research Article**

# Biharmonic General Helices in 3-Dimensional Finsler Manifold

*Üç Boyutlu Finsler Manifoldu $F^3$'de Biharmonik Helisler*

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**Abstract**

In the present paper, we study on the 3- dimensional Finsler manifold and obtain some characterizations for helices of this space.

**Keywords:** Biharmonic helices, Finsler manifold, Mean curvature

**Öz**

Bu çalışmada üç boyutlu Finsler manifoldu üzerinde çalışılmıştır ve bu tip uzaylar için bazı helis karakterizasyonları elde edilmiştir.

**Anahtar Kelimeler:** Biharmonik helisler, Finsler manifoldu, Ortalama eğrilik
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## 1. Introduction

As it is well known, the submanifolds harmonic mean curvature vector field called biharmonic. The study of such manifolds is originated of a Chen's conjecture in 1988 which was indicated an Euclidean submanifolds with the following property

$$\Delta H = \lambda H \qquad (1)$$

where $H$ is the mean curvature vector and $\lambda \in$ R. In other words biharmonic submanifolds are special submanifolds for which its mean curvature vector is an eigenvector of $\Delta$.

In (Yaz et al. 2003) the authors studied general helices and submanifolds in indefinite Riemannian manifolds and gave some characterizations for these geometric tools. Also in (Yıldırım Yılmaz et al. 2009, Yıldırım Yılmaz et al. 2011), the authors obtained some results for helices of 3-dimensional Finsler manifold $F^3$. Also Voicu, obtained some characterizations for biharmonic curves in his paper (Voicu, 2014). There are many valuable works and characterizations on helices, Bertrand curves, and biharmonic helices and curves in various spaces (Keleş et al. 2010, Körpınar et al. 2012, Körpınar et al. 2013, Külahcı et al. 2008, Külahcı et al. 2009, Voicu, 2014, Yüksel Perktaş et al. 2012).

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Our study is inspired by the recent studies indicated above. In this paper, after a short description of Finsler manifolds, we give two characterizations for a curve with respect to the Frenet frame of the 3- dimensional Finsler manifold $F^3$.

## 2. Preliminaries

Finsler geometry is the most natural generalization of Riemannian geometry. It started in 1918 when P. Finsler wrote his thesis on curves and surfaces in what he called generalized metric spaces. Due to its importance it has a huge research field from geometry to biology, physics and also engineering and computer sciences, (Brandt 2005, Solange et al. 2001) .The following part of the study is on the basic concepts of the Finsler manifolds

**Definition 2.1.** Let $M$ be a real m-dimensional smooth manifold and $TM$ be the tangent bundle of $M$. Denote by $\Pi$ the canonical projection of $TM$ on $M$.

Let $M'$ be an non-empty open submanifold of $TM$ such that $\Pi\,(M') = M$ and $\theta(M) \cap M' = \phi$ where $\theta$ is the zero section of $TM$.

We now consider a smooth function $F\colon M' \to (0,\infty)$ and take $F^* = F^2$. Then suppose that for any coordinate system $\{(\acute{u},\Phi');x^i,y^i\}$ in $M'$ the following conditions are fulfilled:

$(F_1)$ $F$ is positively homogeneus of degree one with respect to $(y^1,\ldots,y^m)$ i.e. we have

$F(x^1,...x^m, ky^1,..., ky^m) = kF(x^1,..., x^m, y^1,...,y^m)$ (2)

for any $(x,y) \in \Phi'(U')$ and any $k>0$.

$(F_2)$ At any point $(x,y) \in \Phi'(U')$

$$g_{ij}(X,Y) = \frac{1}{2}\frac{\partial^2 F^*}{\partial y^i \partial y^j}(X,Y), i,j \in \{1,...,m\}$$

are the components of a positive definite quadratic form on $R^m$, (Bejancu et al. 2000).

We say that the triple $F^m = (M,M',F)$ with satisfying $(F_1)$ and $(F_2)$ is a Finsler manifold and $F$ is the fundamental function of $F^m$.

**Definition 2.2.** Let $F^{m+1} = (M,M',F)$ be a Finsler manifold and $F' = (C,C',F)$ be a 1-dimensional Finsler submanifold of $F^{m+1}$, where $C$ is a smooth curve in $M$ given locally by the equations

$x^i = x^i(s), \ i \in \{1,...,m+n\}, \ s \in (a, b)$

$s$ being the arclength parameter on $C$. Denote by $(s, v)$ the coordinates on $C'$. Then we have

$$y^i(s,v) = v\frac{dx^i}{ds} \quad i \in \{0,...,m\}$$

Moreover $\left\{\frac{\partial}{\partial s}, \frac{\partial}{\partial v}\right\}$ is a natural field of frames on $C$ where $\frac{\partial}{\partial v}$ is a unit Finsler vector field, (Bejancu et al. 2000).

**Definition 2.3.** Let $F^3 = (M,M',F)$ be a 3-dimensional Finsler manifold and $C$ be a smooth curve in $M$ given locally by the parametric equations

$$x^i = x^i(s); \ (x'^1(s), x'^2, x'^3(s)) \neq (0,0,0)$$

where $s$ is the arclength parameter on $C$.

Then we have

$$\nabla^*_{\frac{\partial}{\partial s}}\frac{\partial}{\partial v} = \kappa n, \qquad \nabla^*_{\frac{\partial}{\partial s}} n = -\kappa\frac{\partial}{\partial v} + \tau b, \qquad \nabla^*_{\frac{\partial}{\partial s}} b = \tau n. \tag{3}$$

where $n$ and $b$ called principal normal Finsler vector field and binormal Finsler vector field on $C$, respectively. We are entiteled to call $\left\{\frac{\partial}{\partial v}, n, b\right\}$ be the Frenet frame for the curve $C$ in $F^3$. As in the Riemannian case we call $\kappa$ the curvature and $\tau$ the torsion of $C$ respectively, (Bejancu et al. 2000).

## 3. Biharmonic General Helices in $F^3$

**Definition 3.1.** Let $C$ be a curve of a Finsler manifold $F^3$ and $\left\{\frac{\partial}{\partial v}, n, b\right\}$ be the Frenet frame on $F^3$ along $C$. If $k$ and $\tau$ are positive constants along $C$ then $C$ is called a circular helix with respect to the Frenet frame.

**Definition 3.2** Let $C$ be a curve of a Finsler manifold $F^3$. $C$ is a general helix with respect to Frenet frame $\left\{\frac{\partial}{\partial v}, n, b\right\}$ on $F^3$ along $C$. A curve $C$ such that satisfied

$$\frac{\kappa}{\tau} = const$$

is called a general helix with respect to Frenet frame $\left\{\frac{\partial}{\partial v}, n, b\right\}$.

**Definition 3.3.** Let $\sigma$ be the second fundamental form associated to the curve $C$. Then the mean curvature vector field is defined by

$$H = tr(\sigma) = \sigma(X,X) = \nabla^*_X \frac{\partial}{\partial v} = \kappa n \tag{4}$$

**Theorem 3.1.** Let $C$ be a unit speed curve of a Finsler manifold $F^3$. Then $C$ satisfies $\Delta H = \lambda H$ if and only if

$$\kappa\kappa' = 0, \ \frac{\kappa''}{\kappa} - \kappa^2 + \tau^2 = \lambda, \quad 2\kappa'\tau + \tau'\kappa = 0 \tag{5}$$

**Proof.** Suppose that $C$ satisfies $\Delta H = \lambda H$. Then

$$\Delta H = \lambda H$$
$$\Delta H = \nabla^{*2}_X H$$
$$= \nabla^*(\nabla^*_X(\kappa n))$$

One can calculate

$$\Delta H = \nabla^{*2}_X H = (-3\kappa\kappa')\frac{\partial}{\partial v} + (\kappa'' - \kappa^3 + \kappa\tau^2)n + (2\kappa'\tau + \tau'\kappa)b \tag{6}$$

Writing

$$n = \frac{1}{\kappa}H$$

in the above equation we get

$$\Delta H = \nabla^{*2}_X H = (-3\kappa\kappa')\frac{\partial}{\partial v} + \left(\frac{\kappa''}{\kappa} - \kappa^2 + \tau^2\right)H + (2\kappa'\tau + \tau'\kappa)b \tag{7}$$

Now from hypothesis we obtain

$$\kappa\kappa' = 0, \ \frac{\kappa}{\kappa} - \kappa^2 + \tau^2 = \lambda, \quad 2\kappa'\tau + \tau'\kappa = 0 \tag{8}$$

Conversely, supposing

$$\kappa\kappa' = 0, \ \frac{\kappa''}{\kappa} - \kappa^2 + \tau^2 = \lambda, \quad 2\kappa'\tau + \tau'\kappa = 0 \tag{9}$$

and taking into account of the equations above in (7), we conclude that $C$ satisfies $\nabla H = \lambda H$.

**Theorem 3.2.** Let $C$ be a unit speed curve of Finsler manifold $F^3$. $C$ is a general helix with respect to the Frenet frame $\left\{\frac{\partial}{\partial v}, n, b\right\}$, if and only if

$$\Delta H = \lambda \nabla_X H + \mu H. \tag{10}$$

Then

$$\lambda = -3\frac{\varkappa}{\varkappa'}, \qquad \mu = \frac{\varkappa''}{\varkappa} + 3\left(\frac{\varkappa'}{\varkappa}\right)^2 - \varkappa^2 + \tau^2 \tag{11}$$

**Proof.** Let us suppose that $C$ is a general helix with respect to the Frenet frame $\left\{\frac{\partial}{\partial v}, n, b\right\}$. From the Definition 3.2 $\frac{\varkappa}{\tau}$ is constant. After a simple calculation we may write this fact as follows

$$\varkappa'\tau = \varkappa\tau'. \tag{12}$$

Taking the secoond derivative of $H$ we get,

$$\Delta H = \nabla_X^{*2} H = (-3\varkappa\varkappa')\frac{\partial}{\partial v} + (\varkappa'' - \varkappa^3 + \varkappa\tau^2)n + (3\varkappa'\tau)b. \tag{13}$$

This equation can be written as

$$\Delta H = \nabla_X^{*2} H = -3\varkappa'(-k\frac{\partial}{\partial v} + \tau b) + (\varkappa'' - \varkappa^3 + \varkappa\tau^2)n. \tag{14}$$

Note that

$$\nabla_{X^n}^* = -\varkappa\frac{\partial}{\partial v} + \tau b \tag{15}$$

and

$$n = \frac{1}{\varkappa}H \tag{16}$$

we obtain

$$\nabla_{X^n}^* = \nabla_X^*\left(\frac{1}{\varkappa}H\right) = -\frac{\varkappa}{\varkappa^2}H + \frac{1}{\varkappa}\nabla_X^* H \tag{17}$$

Using these equations we get

$$\Delta H = \nabla_X^{*2} H = -3\frac{\varkappa'}{\varkappa}\nabla_{\frac{\partial}{\partial s}}^* H + \left(\frac{\varkappa''}{\varkappa} + 3\left(\frac{\varkappa'}{\varkappa}\right)^2 - \varkappa^2 + \tau^2\right)H. \tag{18}$$

Conversely, assuming (18) holds. We see that $C$ is a general helix. Differentiating (17), we get

$$\nabla_X^*\nabla_X^* n = \left(-\frac{\varkappa'}{\varkappa^2}\right)' H - 2\frac{\varkappa'}{\varkappa^2}\nabla_X H + \frac{1}{\varkappa}\nabla_X^{*2} H \tag{19}$$

From the hypothesis we can write

$$-\nabla_X^{2*} H = \lambda\nabla_X H + \mu H \tag{20}$$

$$\nabla_X^* H = -\varkappa^2\frac{\partial}{\partial v} + \varkappa' n + \varkappa b \tag{21}$$

Hence

$$\nabla_X^*\nabla_X^* n = \left(-\frac{\varkappa'}{\varkappa^2}\right)' H - 2\frac{\varkappa'}{\varkappa^2}\nabla_X H + \frac{1}{\varkappa}\nabla_X^{*2} H \tag{22}$$

$$\begin{aligned}\nabla_X^*\nabla_X^* n = &\left(\frac{2\varkappa'}{\varkappa^2} + \frac{\varkappa\lambda}{\varkappa^2}\right)\varkappa^2\frac{\partial}{\partial v} + \left(-2\frac{\varkappa'}{\varkappa^2} - \frac{1}{\varkappa}\lambda\right)\varkappa' n + \\ &\left(-2\frac{\varkappa'}{\varkappa^2} - \frac{1}{\varkappa}\lambda\right)\varkappa b + \left(\left(\frac{\varkappa'}{\varkappa^2}\right)' - \frac{\mu}{\varkappa^2}\right)\nabla_X^* X\end{aligned} \tag{23}$$

According to the Frenet frame, it is known that

$$\nabla_X^*\nabla_X^* n = -\varkappa'\frac{\partial}{\partial v} - k\nabla_X^* X + \tau' b + \tau^2 n \tag{24}$$

Comparing (23) and (24) and taking into account of the value of the $\lambda$ we obtain

$$\tau' = \left(\frac{3\varkappa'}{\varkappa^2} - \frac{2\varkappa'}{\varkappa^2}\right)\varkappa\tau \tag{25}$$

and then,

$$\frac{\tau'}{\tau} = \frac{\varkappa'}{\varkappa} \tag{26}$$

which means

$$\left(\frac{\varkappa}{\tau}\right)' = 0. \tag{27}$$

Hence

$$\frac{\varkappa}{\tau} = constant \tag{28}$$

This means that $C$ is a general helix.

According to the theorem above we have the following corollary.

**Corollary 3.1.** Let $C$ be a unit speed curve of Finsler manifold $F^3$. $C$ is a helix with respect to the Frenet frame $\left\{\frac{\partial}{\partial v}, n, b\right\}$ if and only if

$$\Delta H = \lambda H \tag{29}$$

For this case

$$\lambda = \tau^2 - \varkappa^2 \tag{30}$$

**Proof.** From the hypothesis of Corollary 3.1 and since $C$ is a circular helix, we can show (30) easily.

**Corollary 3.2.** Let $C$ be a unit speed curve of Finsler manifold $F^3$. $C$ is a biharmonic general helix with respect to the Frenet frame $\left\{\frac{\partial}{\partial v}, n, b\right\}$ if and only if $\varkappa$ is constant.

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