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On the Quadra Lucas-Jacobsthal Numbers

Quadra Lucas-Jacobsthal Sayıları Üzerine

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Abstract

In this paper, we define the Quadra Lucas-Jacobsthal numbers and then, we give some properties of this sequences. Moreover, we obtain spectral norms of circulant matrices with Quadra Lucas-Jacobsthal numbers.

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Öz

Bu makalede, Quadra Lucas-Jacobsthal sayıları tanımlanmış ve bu dizilerin çeşitli özellikleri verilmiştir. Dahası elamanları Quadra Lucas-Jacobsthal sayıları olan sirkülant matrislerin spektral normları elde edilmiştir.

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Anahtar Kelimeler: Binet formülü, Sirkülant matris, Jacobsthal sayıları, Lucas sayıları

1. Introduction

Let p and q be non-zero integers. The second order linear recurrences of the Fibonacci and Lucas types are defined as follows:

$$U_n = pU_{n-1} + qU_{n-2}, \quad (1)$$

$$V_n = pV_{n-1} + qV_{n-2} \quad (2)$$

where $U_0 = 0$, $U_1 = 1$ and $V_0 = 2$, $V_1 = p$.

Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas numbers can be derived from (1) and (2). For $p = q = 1$, $U_n = F_n$ where F_n is the n-th Fibonacci number. For $p = 2$, $q = 1$, $U_n = P_n$ where P_n is the n-th Pell number. Similarly, $p = 1$ and $q = 2$, $U_n = J_n$ where J_n is the n-th Jacobsthal number. On the other hand, for $p = q = 1$, $V_n = L_n$ where L_n is the n-th Lucas number. For $p = 2$ and $q = 1$, $V_n = Q_n$ where Q_n is the n-th Pell-Lucas number. Similarly, for $p = 1$ and $q = 2$, $V_n = j_n$ where j_n is the n-th Jacobsthal-Lucas number.

The characteristic equation of U_n and V_n is $x^2 - px - q = 0$ and the roots are

$$x_1 = \frac{p+\sqrt{p^2+4q}}{2}, x_2 = \frac{p-\sqrt{p^2+4q}}{2}.$$

The generating functions of U_n and V_n are

$$U(x) = \frac{x}{1-px-qx^2}, V(x) = \frac{2-px}{1-px-qx^2}. \quad (3)$$

Some authors have studied certain integer sequences. For example in [5], Taşcı defined Quadrapell numbers by the following recurrence relation for $n \geq 4$,

$$D_n = D_{n-2} + 2D_{n-3} + D_{n-4}$$

with the initial values $D_0 = D_1 = D_2 = 1$ and $D_3 = 2$. Then the author gave some algebraic identities for the Quadrapell numbers. In [4], Özkoç defined a Quadra Fibona-Pell numbers by the following recurrence relation for $n \geq 4$,

$$W_n = 3W_{n-1} - 3W_{n-3} - W_{n-4}$$

with the initial values $W_0 = W_1 = 0$, $W_2 = 1$, $W_3 = 3$. Then the author gave some properties of Quadra Fibona-Pell sequences. In [6], the authors studied the integer sequence

$$T_n = -5T_{n-1} - 5T_{n-2} + 2T_{n-3} + 2T_{n-4}$$

with the initial values $T_0 = T_1 = 0$, $T_2 = -3$and $T_3 = 12$.

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In this paper, we define a similar sequence related to Lucas and Jacobsthal numbers. Then we give some properties for these numbers.

2. Quadra Lucas-Jacobsthal Numbers

Definition 2.1. The Quadra Lucas-Jacobsthal numbers S_n are defined by the following recurrence relation for $n \geq 4$,

$$S_n = 2S_{n-1} + 2S_{n-2} - 3S_{n-3} - 2S_{n-4} \quad (4)$$

with the initial values $S_0 = S_1 = 2$, $S_2 = 4$ and $S_3 = 7$.

We note that the characteristic equation (4) is $x^4 - 2x^3 - 2x^2 + 3x + 2 = 0$ and thus the roots of it are

$$\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}, \gamma = 2 \text{ and } \delta = -1$$

where α, β are the roots of the characteristic equation of Lucas numbers and γ, δ are the roots of the characteristic equation of Jacobsthal numbers. Now we give the generating function of S_n.

Theorem 2.1. The generating function for S_n is

$$S(x) = \frac{x^3 - 4x^2 - 2x + 2}{2x^4 + 3x^3 - 2x^2 - 2x + 1}.$$

Proof. The generating function $S(x)$ has the form

$$S(x) = \sum_{n=0}^{\infty} S_n x^n = S_0 + S_1 x + S_2 x^2 + \cdots.$$

Since the characteristic equation of (4) is $x^4-2x^3-2x^2+3x+2=0$, we have

$$(2x^4 + 3x^3 - 2x^2 - 2x + 1)S(x) = (2x^4 + 3x^3 - 2x^2 - 2x + 1)(S_0 + S_1 x + S_2 x^2 + \ldots + S_n x^n + \ldots)$$

$$(2x^4 + 3x^3 - 2x^2 - 2x + 1)S(x) = S_0 + (S_1 - 2S_0)x + \ldots + (S_n - 2S_{n-1} - 2S_{n-2} + 3S_{n-3} + 2S_{n-4})x^n + \ldots$$

From the (4), we get

$$(2x^4 + 3x^3 - 2x^2 - 2x + 1)S(x) = x^3 - 4x^2 - 2x + 2.$$

So

$$S(x) = \frac{x^3 - 4x^2 - 2x + 2}{2x^4 + 3x^3 - 2x^2 - 2x + 1}.$$

which is desired result.

Theorem 2.2. Let S_n be n-th Quadra Lucas-Jacobsthal numbers. For $n \geq 0$, the Binet formula for S_n is

$$S_n = (\alpha^n + \beta^n) + \left(\frac{\gamma^n - \delta^n}{\gamma - \delta}\right).$$

Proof. We can write $S(x)$ as

$$S(x) = \frac{2-x}{1-x-x^2} + \frac{x}{1-x-2x^2}. \quad (5)$$

From (3), the generating function of Lucas and Jacobsthal numbers are

$$L(x) = \frac{2-x}{1-x-x^2}, J(x) = \frac{x}{1-x-2x^2} \quad (6)$$

respectively. From (5) and (6), we have

$$S(x) = L(x) + J(x).$$

Hence we have

$$S_n = (\alpha^n + \beta^n) + \left(\frac{\gamma^n - \delta^n}{\gamma - \delta}\right).$$

Theorem 2.3. Let S_n denote the n-th numbers. Then the sum of first non-zero terms of S_n is

$$\sum_{i=1}^{n} S_i = \frac{S_n + 3S_{n-1} + 5S_{n-2} + 2S_{n-3} - 7}{2}.$$

Proof. From the recurrence relation of S_n, we get

$$2S_{n-1} + 2S_{n-2} = 3S_{n-3} + 2S_{n-4} + S_n. \quad (7)$$

Applying (7), we deduce that

$$\begin{aligned} 2S_3 + 2S_2 &= 3S_1 + 2S_0 + S_4, \\ 2S_4 + 2S_3 &= 3S_2 + 2S_1 + S_5, \\ 2S_5 + 2S_4 &= 3S_3 + 2S_2 + S_6, \\ &\vdots \\ 2S_{n-1} + 2S_{n-2} &= 3S_{n-3} + 2S_{n-4} + S_n. \end{aligned} \quad (8)$$

If we sum both sides of (8), then we have

$$2\sum_{i=1}^{n} S_i = S_n + 3S_{n-1} + 5S_{n-2} + 2S_{n-3} - 2S_0 - 3S_1 - S_2 + S_3.$$

Thus we get

$$\sum_{i=1}^{n} S_i = \frac{S_n + 3S_{n-1} + 5S_{n-2} + 2S_{n-3} - 7}{2}. \quad (9)$$

3. An Application of Quadra Lucas-Jacobsthal Numbers in Matrices

In this section, we will give some applications on matrix norms of Quadra Lucas-Jacobsthal numbers. Let $A = (a_{ij})$ be any m x n complex matrix. The spectral norm of the matrix A are

$$\| A \|_2 = \sqrt{\max_{1 \leq i \leq n} |\lambda_i (A^H A)|},$$

where $\lambda_i(A^H A)$ is an eigenvalue of $A^H A$ and A^H is the conjugate transpose of the matrix A.

By a circulant matrix of order n is meant a square matrix of the form

$$C = Circ(c_0, c_1, c_2, ..., c_{n-1}) = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_0 \end{bmatrix}.$$

The eigenvalues of C are

$$\lambda_{\substack{j \\ 0 \le j \le n-1}} = \sum_{k=0}^{n-1} c_k w^{-jk}, \tag{10}$$

where $w = e^{\frac{2\pi i}{n}}$ and $i = \sqrt{-1}$.

Lemma 3.1. ([1]) Let A be an n x n matrix with eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$. Then, A is a normal matrix if and only if the eigenvalues of $A^H A$ are $|\lambda_1|^2, |\lambda_2|^2, ..., |\lambda_n|^2$.

Theorem 3.1. The spectral norm of $C = Circ\,(S_0, S_1, S_2, ... S_{n-1})$ is

$$\| C \|_2 = \frac{S_{n-1} + 3S_{n-2} + 5S_{n-3} + 2S_{n-4} - 3}{2}.$$

Proof. Since C is circulant matrix, from (10), for all t = 0,1,2,...,n-1,

$$\lambda_t(C) = \sum_{i=0}^{n-1} S_i (w^{-t})^i.$$

Then for t = 0,

$$\lambda_0(C) = \sum_{i=0}^{n-1} S_i. \tag{11}$$

Hence, for $1 \le m \le n-1$, we have

$$\lambda_m = \left| \sum_{i=0}^{n-1} S_i (w^{-m}) \right| \le \left| \sum_{i=0}^{n-1} S_i \right| |(w^{-m})^i| \le \left| \sum_{i=0}^{n-1} S_i \right|. \tag{12}$$

By using the Lemma 3.1 and the fact that the matrix C is a normal matrix, we have

$$\| C \|_2 = \max_{0 \le m \le n-1} |\lambda_m| = max\left(|\lambda_0|, \max_{1 \le m \le n-1} |\lambda_m|\right). \tag{13}$$

From (11), (12) and (13), we have

$$\| C \|_2 = \sum_{i=0}^{n-1} S_i = \frac{S_{n-1} + 3S_{n-2} + 5S_{n-3} + 2S_{n-4} - 3}{2}.$$

Definition 3.1. The companion matrix of the Quadra Lucas-Jacobsthal recurrence relation is defined by

$$A = \begin{bmatrix} 2 & 2 & -3 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

We note that det(A) = 2 and the eigenvalues of A are

$\lambda_1 = \alpha, \lambda_2 = \beta, \lambda_3 = \delta$ and $\lambda_4 = \gamma$.

Definition 3.2. ([1]) Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the eigenvalues of a matrix A. Then its spectral radius $p(A)$ is defined as follows:

$$p(A) = \max_{1 \le i \le n}(|\lambda_i|).$$

Corollory 3.1. Let A be the matrix of the Quadra Lucas-Jacobsthal recurrence relation. Then $p(A) = 2$.

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