Karaelmas Science and Engineering Journal
Journal home page: http://fbd.beun.edu.tr

Research Article

Lacunary A - Convergence

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Abstract

In this paper we introduce the concept of Lacunary A – Summability. We also give the relations between these summability and Lacunary A – statistical Summability. Following the concept of statistical A – limit superior and inferior, we give a definition of Lacunary A – limit superior and inferior which yields natural relationships among these ideas: x is Lacunary A-convergent if and only if $L_\theta(A) - \lim_{(n\to\infty)} supx = L_\theta(A) - \lim_{(n\to\infty)}$ infx. Lacunary A – core of x is also introduced and it is proved that a bounded sequence that A – summable to its Lacunary A – limit superior is Lacunary A-convergent.

Keywords: Lacunary convergence, Lacunary A – summability, Lacunary A – convergence, Lacunary A – core

1. Introduction

We now introduce some notation and basic definitions used in this paper. Let $A=(a_{nk})$ be a summable infinite matrix. For a given sequence $x:=\{x_k\}$, the A- transform of x, denoted by $Ax:=((Ax)_n)$, is given by $(Ax)_n = \sum_{k=1}^{\infty} a_{nk}x_k$ provided that the series converges for each $n\in\mathbb{N}$, the set of all natural numbers.

We say that A is regular if $\lim_{n\to\infty}(Ax)_n=L$ whenever $\lim_{n\to\infty} x_n=L$ (Freedman and Sember 1981). If $A=(a_{nk})$ be an infinite matrix, then Ax is the sequence whose nth term is given by $A_n(x)=\sum_{k=1}^{\infty} a_{nk}x_k$. Thus we say that x is A summable to L if $\lim_{n\to\infty} A_n(x)=L$. (Fridy and Miller 1991). The statistical convergence is depend on the density of subsets of $\mathbb{N}$. A subset K of $\mathbb{N}$ is said to have density $\delta(K) := \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \chi_K(k)$ where χ_K is the characteristic function of K (Fridy 1953). The A-density of K is defined by $\delta_A(K) := \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{\infty} a_{nk}\chi_K(k)$ provided the limit exists, where χ_K is the characteristic function of K. Then the sequence $x:=\{x_k\}$ is said to be A-statistically convergent to the number L if, for every $\varepsilon>0$, $\delta_A\{k\in\mathbb{N}: |x_k-L| \geq \varepsilon\}=0$ or equivalently $\lim_{n\to\infty}\sum_{k:|x_k-L|\geq\varepsilon} a_{nk}=0$. We denote this limit by st_A-$\lim_{n\to\infty} x=L$ (Duman et al. 2003). Let $\theta=\{k_r\}$ be a sequence of positive integers such that $k_0=0$, $0<k_{r-1}<k^r$ and $h_r:=k_r-k_{r-1}\to\infty$ $(r\to\infty)$. Then θ is called a *lacunary sequence*. The intervals determined by $[k_{r-1}, k_r]$ will be denoted by $I_r:=[k_{r-1}, k_r]$ and ratio $\frac{k_r}{k_{r-1}}$ will be denoted by η_r. Lacunary sequences have been studied in (Fridy and Orhan 1996, Aktuglu and Gezer 2009). A sequence $x:=\{x_k\}$ is called lacunary statistical convergent to L, if

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\{k\in I_r : |x_k-L|\geq\varepsilon\}\right|=0,$$

where $|K|$ denotes the cardinality of the set K. For a real number sequnce x, let N, M denote the sets:

$$N:=\{a\in\mathbb{R}: \delta\,\{k: x_k>a\}\neq 0\}$$

$$M:=\{b\in\mathbb{R}: \delta\,\{k: x_k<b\}\neq 0\}.$$

If x is a real number sequence, then the *statistical limit superior* of x and the *statistical limit inferior* of x is respectively given by

$$st-\lim_{n\to\infty} supx := \begin{cases} supN, & if\ N\neq\phi \\ +\infty, & if\ N=\phi \end{cases},$$

$$st-\lim_{n\to\infty} infx := \begin{cases} infM, & if\ M\neq\phi \\ +\infty, & if\ M=\phi \end{cases}$$

In (Mursaleen, et al. 2009) defined statistical A-summability as following. Let $x=\{x_k\}$ be a sequence of real numbers and $A=(a_{nk})$ be a nonnegative regular matrix. We say that x is statistical A-summability to L if for every $\varepsilon>0$,

$$\lim_{n\to\infty}\frac{1}{n}\left|\{i\leq n : |y_i-L|\geq\varepsilon\}\right|=0,$$

where $y_i=A_i(x)$.

2. Lacunary A-Summability

In this section we define Lacunary A-sum-mability for a nonnegative regular matrix A and find its relationship with A- lacunary convergence.

Definition 2.1 Let $\theta=\{k_r\}$ be a lacunary sequence, $x=\{x_k\}$ be a sequence of real numbers and $A=(a_{nk})$ be a nonnegative regular matrix. We say that x is Lacunary A-summability to L if for every $\varepsilon>0$,

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\{k\in I_r : |y_k-L|\geq\varepsilon\}\right|=0,$$

where $y_k=A_k(x)$. In this case, we write $L_\theta-\lim_{n\to\infty} Ax=L$.

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Remark 2.1 If A is the identity matrix, then lacunary A-convergence coincides with the ordinary convergence. It is not hard to see that every convergent lacunary sequence is lacunary A-convergent.

Remark 2.2 Every A-summable sequence may not be lacunary A-convergent.

Remark 2.3 Note that every statistical convergent sequences is lacunary A-convergent.

The concepts of the statistical *A-limit superior* and *inferior* have been introduced in (Fridy and Orhan 1996).

Definition 2.2 Let $\theta = \{k_r\}$ be a lacunary sequence, $x = \{x_k\}$ be a sequence of real numbers and $y_i = A_i(x)$ be a nonnegative regular matrix. If x is a real number sequence, then the *lacunary A-limit superior* of x and the *lacunary A-limit inferior* of x are respectively given by

$$L_\theta(A)-\lim_{n\to\infty} sup x := \begin{cases} sup Y, & if \ Y \neq \phi \\ -\infty, & if \ Y = \phi \end{cases},$$

$$L_\theta(A)-\lim_{n\to\infty} inf x := \begin{cases} inf Z, & if \ Z \neq \phi \\ +\infty, & if \ Z = \phi \end{cases}$$

where $Y:=\{a\in\mathbb{R}: \delta\{i \in I_r: y_i > a\} \neq 0\}$ and

$Z:=\{b\in\mathbb{R}: \delta\{i \in I_r: y_i < b\} \neq 0\}$.

Now we give another lacunary analogue of a very basic property of convergent sequences (Mursaleen et al. 2009).

Definition 2.3 Let $y_i = A_i(x)$ and $\theta = \{k_r\}$ be a lacunary sequence, then the real numbers sequence x is said to be lacunary A-bounded if there is a number K such that

$\delta\{i \in I_r: |y_i| > K\} = 0.$

Theorem 2.1 If $\varphi = L_\theta(A)-\lim_{n\to\infty} sup x$ is finite, then for every positive number $\varepsilon>0$,

$\delta\{i \in I_r: |y_i| > \varphi-\varepsilon\} \neq 0$ and

$\delta\{i \in I_r: |y_i| > \varphi+\varepsilon\} = 0.$

Proof This is clear from the definition of *Lacunary A-limit inferior* and *Lacunary A-limit superior*.

Theorem 2.2 Let $\theta = \{k_r\}$ be lacunary sequence then for any real numbers sequence x, we have

$$L_\theta(A)-\lim_{n\to\infty} inf x \leq L_\theta(A)-\lim_{n\to\infty} sup x.$$

Proof First consider the case in which $L_\theta(A)-\lim_{n\to\infty} sup x = -\infty$. This implies that $Y = \varnothing$, so for every $a \in \mathbb{R}$: $\delta\{i\in I_r: y_i>a\} = 0$ and $\delta\{i \in I_r: y_i \leq a\} = 1$, so for every $b\in\mathbb{R}$: $\delta\{i\in I_r: y_i<b\} \neq 0$. Hence, $L_\theta(A)-\lim_{n\to\infty} sup x = -\infty$. The case in which $L_\theta(A)-\lim_{n\to\infty} sup x = \infty$, can be proved similarly. Now, $\gamma := L_\theta(A)-\lim_{n\to\infty} sup x$ is finite and let $\varphi := L_\theta(A)-\lim_{n\to\infty} inf x$. Given ε > 0 we show that $\gamma + \varphi \in \mathbb{Z}$, so that $\varphi \leq \gamma + \varepsilon$. By Theorem 2.2, $\delta\{i\in I_r: y_i > \gamma + \varepsilon/2\} = 0$, because $\gamma = L_\theta(A)-\lim_{n\to\infty} sup x$. Similarly $\delta\{i\in I_r: y_i \leq \varphi + \varepsilon\} = 1$ Hence $\varepsilon + \gamma \in \mathbb{Z}$. By definition $\varphi := L_\theta(A)-\lim_{n\to\infty} inf x$, we conclude that $\varphi \leq \gamma + \varepsilon/2$; and since ε is arbitrary this gives us $\varphi \leq \gamma$.

From Theorem 2.1 and definition, it is clear that

$$\begin{aligned}\lim_{n\to\infty} inf x &\leq L_\theta(A)-\lim_{n\to\infty} inf x \\ &\leq L_\theta(A)-\lim_{n\to\infty} sup x \leq \lim_{n\to\infty} sup x\end{aligned}$$

for any sequnce x.

Theorem 2.3 A lacunary A-bounded sequence x is lacunary A-convergent if and only if $L_\theta(A)-\lim_{n\to\infty} inf x = L_\theta(A)-\lim_{n\to\infty} sup x$

Proof Let $\gamma := L_\theta(A)-\lim_{n\to\infty} sup x$ and $\varphi := L_\theta(A)-\lim_{n\to\infty} inf x$. First assume that $\gamma = \varphi$ and define $\varphi = L$. If ε>0 then

$\delta\{i\in I_r: y_i > L + \varepsilon/2\} = 0$ and $\delta\{i\in I_r: y_i < L - \varepsilon/2\} = 0$. Hence $L_\theta(A)-\lim_{n\to\infty} x = L$. Next assume $L_\theta(A)-\lim_{n\to\infty} x = L$ and ε>0. Then $\delta\{i\in I_r: |y_i - L| \geq \varepsilon\} = 0$, so

$\delta\{i\in I_r: y_i > L + \varepsilon\} = 0$

which implies that $L \leq \varphi$. On the other hand $\delta\{i\in I_r: y_i < L - \varepsilon/2\} = 0$, by using the Theorem 2.2, we have $\gamma = \varphi$.

(Osama and Edely 2009) proved β- statistical convergent relationship with β-summable. Similarly, we can give following theorem.

Theorem 2.4 If the number sequence x is bounded above and lacunary A-summability to the number $L_\theta(A)-\lim_{n\to\infty} sup x = L$, then x is lacunary A-convergent to L.

Proof Suppose that x is not lacunary A-convergent to L. Then by Theorem 2.3, $L_\theta(A)-\lim_{n\to\infty} inf x < L$, so there is a number $K < L$ such that $\delta\{k\in I_r: y_k < K\} \neq 0$. Let $K_1=\{k\in I_r: y_k < K\}$. Then for every ε>0, $\delta\{k\in I_r: y_k > L + \varepsilon\} = 0$. We can write $K_2=\{k\in I_r: K \leq y_k \leq L + \varepsilon\}$ and $K_3=\{k\in I_r: y_k > L + \varepsilon\}$, and let $P = sup y_k < \infty$. Since $\delta(K_1) \neq 0$, there are some n such that

$$\lim_n \sup \sum_{k\in K_1} a_{nk} \geq m > 0,$$

and for each $n, j\in\mathbb{N}$, $\sum_{k=1}^{\infty} |a_{nk}(j) x_k| < \infty$. Now

$$\begin{aligned}\sum_{k=1}^{\infty} a_{nk}(j) x_k &= \sum_{k\in K_1} a_{nk} x_k + \sum_{k\in K_2} a_{nk} x_k \\ &\quad + \sum_{k\in K_2} a_{nk} x_k \\ &\leq K \sum_{k\in K_1} a_{nk}(j) + (L+\varepsilon) \sum_{k=1}^{\infty} a_{nk}(j) \\ &\quad - (L+\varepsilon) \sum_{k\in K_1} a_{nk} + o(1) \\ &\leq L \sum_{k=1}^{\infty} a_{nk}(j) - m(L-K) + \varepsilon \left(\sum_{k=1}^{\infty} a_{nk}(j) - d \right) \\ &\quad + o(1)\end{aligned}$$

Since ε is arbitrary, it follows that

$$L_\theta(A)-\lim_{n\to\infty} \inf x \le L-m(L-K)<L$$

Hence x is not lacunary A-summable to L.

Theorem 2.5 Let $\theta=\{k_r\}$ be a lacunary sequence. Then statistical A-convergence implies lacunary A-convergence if and only if $\lim_{r\to\infty} \sup \eta_r < \infty$.

Proof First, assume that θ be a statistical A-convergent sequence and $\lim_{r\to\infty} \sup \eta_r < \infty$ then there exists a positive number M such that $\eta_r < M$ for all $r \ge 1$. Lettting

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\{k\in I_r:|y_k-L|\ge\varepsilon\}\right|=0,$$

and $\varepsilon>0$ we can then find an $r_0\in\mathbb{N}$ such that $\frac{1}{h_r}\left|\{k\in I_r:|y_k-L|\ge\varepsilon\}\right|=0$ for all $r>r_0$.

Now let $\sup_r \frac{1}{h_r}\left|\{k\in I_r:|y_k-L|\ge\varepsilon\}\right|$ and let n be any integer satisfying $k_{r-1}<n<k_r$ then

$$\frac{1}{n}\left|k\le n:|y_k-L|\ge\varepsilon\right| \le \frac{1}{n}\left|\{k\in I_r:|y_k-L|\ge\varepsilon\}\right|$$

$$\le \frac{1}{k_{r-1}}\left|\{k\in I_r:|y_k-L|\ge\varepsilon\}\right|$$

$$\le \frac{Kr_0}{k_{r-1}}+\frac{\varepsilon k_r-kr_0}{k_{r-1}}$$

$$\le \frac{Kr_0}{k_{r-1}}+\varepsilon\eta_r$$

$$\le \frac{Kr_0}{k_{r-1}}+\varepsilon M$$

and the sufficiency follows immediately. Conversely, assume that $\lim_{r\to\infty} \sup \eta_r < \infty$. Since $\theta=\{k_r\}$ is a lacunary sequence, we can choose a subsequence $\{k_{r(j)}\}$ of θ so that $k_{r(j)}>j$, and then, define

$$x_i=\begin{cases}1, & \text{if } k_{r(j)-1}<i\le 2k_{r(j)-1}, \quad \textit{for some } j=01,2,\ldots\\ 0, & \textit{otherwise}\end{cases}$$

and if $r\ne r(j)$, then $\frac{1}{h_r}\left|\{k\in I_r:|y_k-L|\ge\varepsilon\}\right|=0$. Thus

$$\frac{1}{h_r}\sum_{k\in I_r:|x_k-L|\ge\varepsilon} a_{nk}=\frac{k_{r(j)-1}}{k_{r(j)}-k_{r(j)-1}}<\frac{1}{j-1}$$

if $r\ne r(j)$, $\frac{1}{h_r}\,|\{k\le n:|y_k-L|\ge\varepsilon\}\,|=0$ for every

$$\frac{1}{k_{r(j)}}\sum_{k\in I_r:|x_k-L|\ge\varepsilon} a_{nk}\ge\frac{1}{k_{r(j)}}\left(k_{r(j)}-2k_{r(j)-1}\right)$$

$$=\left(1-\frac{2k_{r(j)-1}}{k_{r(j)}}\right)$$

$$>1-\frac{2}{j}$$

which converges to 1, and for $i=1,2,\ldots 2k_{r(j)-1}$,

$$\frac{1}{2k_{r(j)-1}}\frac{1}{h_r}\sum_{k\in I_r:|x_k-L|\ge\varepsilon} a_{nk}=\frac{k_{r(j)-1}}{2k_{r(j)-1}}=\frac{1}{2}$$

Then, it follows that x_k is not lacunary A-convergent.

Definition 2.4 If x is a lacunary A-bounded sequence, then the lacunary A-core of x is the closed interval

$$\left[L_\theta(A)-\lim_{n\to\infty}\inf x,\ L_\theta(A)-\lim_{n\to\infty}\sup x\right].$$

In this case x is not lacunary A- bounded, $L_\theta(A)$ - *core*$\{x\}$ is defined accordingly as either $[L_\theta(A)-\lim_{n\to\infty} \inf x, \infty)$ or $(-\infty, L_\theta(A)-\lim_{n\to\infty} \sup x]$ and $(-\infty,\infty)$.

It is clear from (Theorem 2.3) that for any sequence x; $L_\theta(A)-$ *core* $\{x\}\subseteq$ *core* $\{x\}$ where *core* $\{x\}$ is usual core.

Lemma 2.1 Let A satisfy $\sup_n\sum_{k\in I_r} a_{nk}<\infty$, then $\lim_{n\to\infty} \sup Ax \le \lim_{n\to\infty} \sup x$ for every $x\in l_\infty$ if and only if A is regular, $\lim_{n\to\infty}\sum_{k\in I_r} a_{nk}=0$ such that $\delta_A\{I_r\}=0$ and $\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{nk}=1$.

Proof (⇒) Let A satisfies $\lim_{n\to\infty} \sup Ax \le L_\theta(A)-\lim_{n\to\infty} \sup x$ and $x\in l_\infty$, then $L_\theta(A)-\lim\sup x\le \lim\sup x$ and since $Ax\in l_\infty$.

$\sup_n\sum_{k\in I_r}|a_{nk}|<\infty$. By $\lim\sup Ax \le L_\theta(A)-\lim\sup x$ we have

$$-L_\theta(A)-\lim\sup(-x)\le -\lim\sup(-Ax)$$

$$\le \lim\sup Ax\le L_\theta(A)-\lim\sup x \text{ and}$$

$$L_\theta(A)-\lim\inf x\le \lim\inf Ax$$

$$\le \lim\sup Ax\le L_\theta(A)-\lim\sup x.$$

If $x\in l_\infty$ and x is lacunary A-convergent, we have $L_\theta(A)-\lim\inf x=L_\theta(A)-\lim\sup x$. So $\lim Ax\le L_\theta(A)-\lim x$. Hence A is regular and $\lim_{n\to\infty}\sum_{k\in I_r}|a_{nk}|=0$, such that $\delta_A\{I_r\}=0$. Also since $L_\theta(A)-\lim\sup x\le \lim\sup x$ and by hipotesis $\lim\sup Ax \le \lim\sup x$ and so

$$\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{nk}=1$$

(⇐) Let A be regular such that $\delta_A\{I_r\}=0$, $\lim_{n\to\infty}\sum_{k\in I_r}|a_{nk}|=0$. If $x\in l_\infty$ then $Ax\in l_\infty$ and $L_\theta(A)-\lim\sup x$ is finite. Given $\varepsilon>0$ and $Y:=\{k: x_k>L_\theta(A)-\lim\sup x+\varepsilon\}$. Thus $\delta_A\{Y\}=0$ and if $k\notin Y$ then $x_k<L_\theta(A)-\lim\sup x+\varepsilon$.

For a fixed positive integer m we write

$$(Ax)_n=\sum_{k<m} a_{nk}x_k+\sum_{k\ge m} a_{nk}x_k$$

$$\le \|x\|_\infty\sum_{k<m}|a_{nk}|+\|x\|_\infty\sum_{k\ge m}\left(|a_{nk}|-a_{nk}\right)$$

$$\le \|x\|_\infty+\sum_{k<m}|a_{nk}|L_\theta(A)$$

$$-\left(\lim\sup x+\varepsilon\right)\sum_{k\in Y,k\ge m}|a_{nk}|$$

$$+\|x\|_\infty\sum_{k\ge m}\left(|a_{nk}|-a_{nk}\right)$$

By using the regularity of A, we have $\lim\sup(Ax)_n\le L_\theta(A)-\lim\sup x+\varepsilon$.

Since ε is arbitrary we complete the proof.

3. Rates of Lacunary A-Convergent

Like (Duman et al. 2003, Fridy 1978) defined rates of statistical *A*-convergence. Here we define rates of lacunary *A*-convergence.

Definition 3.1 Let $A = (a_{nk})$ be a nonnegative regular summability matrix, $\theta = \{k_r\}$ be a lacunary sequence and $hr :=$ $k_r - k_{r-1} \to \infty$ as $r \to \infty$. We say that the sequence $x = \{xk\}$ is lacunary convergent to the number with the rate of $o(h_r)$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{h_r} \sum_{k\in I_r:|x_k - L|\geq\varepsilon} a_{nk} = 0\cdot$$

In this case, it is denoted by $L_\theta(A) - o(h_r) = x_k - L$ $(k\to\infty)$.

Definition 3.2 Let $\theta = \{k_r\}$ be a lacunary sequence and $A = (a_{nk})$ be a nonnegative regular summability matrix and let h_r be sequence $x = \{x_k\}$ is lacunary convergent *A*-bounded with the rate of $O(h_r)$ if for every $\varepsilon > 0$,

$$\sup_n \frac{1}{\mathrm{h_r}} \sum_{k\in I_r:|(x_k - L)|\geq\varepsilon} a_{nk} < \infty\,.$$

In this case, it is denoted by $L_\theta(A) - o(h_n) = x_k - L$.

Theorem 3.3 Let $x = \{x_k\}$, $y = \{y_k\}$ be two sequences and $\{\tilde{k}n\},\{k_n\}$ lacunary sequences.

Assume that $A = (a_{nk})$ is a nonnegative regular summability matrix, $h_r := k_r - k_{r-1} \to\infty$ and $t_r := \tilde{k}_r - \tilde{k}_{r-1} \to\infty$. If for some real number L, $\tilde{L}$ we have $L_\theta(A) - o(h_r) = x_k - L$ and (as $k\to\infty$), $L_\theta(A) - o(h_r) = x_k - L$ then for $p_r = max\{h_r, t_r\}$.

i) $(x_k - L) \pm (y_k - \tilde{L}) = L_\theta(A) - o(p_k)$.

ii) $(x_k - L)(y_k - \tilde{L}) = L_\theta(A) - o(p_k)$.

Proof

i) $$\frac{1}{p_r} = \sum_{\substack{k\in I_r \\ |(x_k - L)\pm(yk-\tilde{L})|\geq\varepsilon}} a_{nk} \leq \frac{1}{\mathrm{h_r}} \sum_{\substack{k\in I_r \\ |(x_k - L)|\geq\varepsilon/2}} a_{nk} + \frac{1}{\mathrm{t_r}} \sum_{\substack{k\in I_r \\ |(y_k - \tilde{L})|\geq\varepsilon/2}} a_{nk}$$

so $(x_k - L) \pm (y_k - \tilde{L}) = L_\theta(A) - o(p_k)$.

ii) $$\frac{1}{p_r} \sum_{\substack{k\in I_r \\ |(x_k - L)(y_k-\tilde{L})|\geq\varepsilon}} a_{nk} \leq \frac{1}{h_r} \sum_{\substack{k\in I_r \\ |(x_k - L)|\geq\sqrt{\varepsilon}/2}} a_{nk}$$

$$\frac{1}{t_r} \sum_{\substack{k\in I_r \\ |(y_k - \tilde{L})|\geq\sqrt{\varepsilon}/2}} a_{nk}$$

4. Results

We study the concepts of lacunary A-convergent and lacunary A-core and proved several important properties of lacunary sequence.

5. Acknowledgements

The author would like to thank the referees and Yusuf Kaya for their careful reading of this paper.

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