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Review Article

A short history of the Cosserat spectrum problem

Yüksel Soykan*, Melih Göcen

Zonguldak Karaelmas University, Science and Art Faculty, Department of Mathematics, 67100, Zonguldak

Abstract

The aim of this paper is to give a short review about Cosserat spectrum.

Keywords: Cosserat spectrum, Cosserat eigenvalues, Dirichlet problem.

1. Introduction

The Cosserat (spectrum) eigenvalue problem is the Dirichlet problem for the Lamé equations of linear elasticity, where the Lamé parameter omega is considered as the eigenvalue parameter. The problem has a long history. The study of the Cosserat spectrum started with a series of nine papers [4]-[12] published between 1898 and 1901 by the French scientists E. and F. Cosserat. Their work was on the solutions of certain basic problems of static elasticity. Their main aim was to expand those solutions into eigenvectors.

We now define the terms "Cosserat spectrum" and "Cosserat eigenvalues". The homogeneous Navier equation

$$\Delta u+\omega \operatorname{graddiv} u=0$$

with homogeneous boundary condition

$$u|_{\partial\Omega}=0$$

admit non-trivial solutions when ω takes values in a set of points called Cosserat spectrum. The corresponding eigenvalues ω's of the Cosserat spectrum are called Cosserat eigenvalues. In other words, Cosserat eigenvalues are those values of a parameter ω, which admit non-trivial solutions to the Dirichlet problem:

$$\Delta u+\omega \operatorname{graddiv} u=0, \qquad u|_{\partial\Omega}=0 \qquad (1)$$

in a bounded domain $\Omega\subset R^3$ with a sufficiently smooth boundary $\partial\Omega$. Here $u=(u_1,u_2,u_3)$ is a vector function, and Δ is the Laplace operator.

If we set the constant ω as

$$\omega:=((\lambda+\mu)/\mu):=(1/(1-2\upsilon))$$

where λ, μ are the Lamé constants and υ is the Poisson ratio, then the equation $\Delta u+\omega \operatorname{graddiv} u=0$ may be viewed as the Lame equation.

As it is mentioned above, the problem (1) was first investigated by E. and F. Cosserat in [4]-[12] where they determined the Cosserat spectrum for some certain types of domain Ω such as: a ball

$$\Omega=\{x\in R^3:|x|<R\},$$

a spherical shell

$$\Omega=\{x\in R^3:R_1<|x|<R_2\}.$$

Cosserat spectrum theory was almost forgotten for a long time and was later extensively studied by S. Mihklin during the Sixties and Seventees in the 20th century. The work by Cosserat was continued by S. Mikhlin who determined the Cosserat spectrum for arbitrary domains. He published several papers between 1966 and 1973 in [43]-[48] and one in 1967 together with V.G.Maz'ya [42]. Further important results belong to A.N. Kozhevnikov [17]-[24].

* Corresponding author: yuksel_soykan@hotmail.com

There have been a lot of papers published on the Cosserat spectrum in the last two decades. All the papers [1]-[67] are related to the Cosserat spectrum.

For a detailed history of the Cosserat spectrum problem we refer to A Kozhevnikov's review article [23], see also [57]. There are some PhD theses on the Cosserat spectrum and for quite recent PhD thesis on it, see for example [29] and [54]. The period of research is not finished yet, and recently new results on the Cosserat spectrum were published, see for example Christian G. Simader [56]-[61], Stephan Weyers [66].

2. References

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