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Asymptotic Solution for Systems of Semilinear Parabolic PDEs in Divergence Form with Measure Data

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Abstract

This paper considers the existence and uniqueness of an asymptotic solution of a monotone semilinear parabolic system in divergence form with measure data. The proof of the main result is probabilistic, which are those of stochastic analysis, Markov process and primarily Backward Stochastic Differential Equations (BSDEs). The probabilistic solution to the system is considered as some generalization of the notion of renormalized (or entropy) solution. It is shown for a Cauchy-Dirichlet problem of a monotone semilinear parabolic system in divergence form with measure data, there exists a unique probabilistic solution of the system under a mild integrability condition on the data.

Keywords: *Asymptotic Solution; BSDEs; Measure data; Parabolic; Semilinear*

2010 Mathematics Subject Classification: *35A01, 35A02, 35K41, 35K58, 35R60, 60H15, 60J25*

1. Introduction

The study of the existence of the solution of a semilinear parabolic equation is of great interest in the mathematical society due to several problems that appear in the physical sciences, chemical sciences, biology, engineering and applied mathematics which leads to mathematical models described by semilinear parabolic equations. Noticeable among these models are the Brusselator model describing some chemical reaction with two components; the Lokka-Volterra system, a competition model for two species; the Field-Noyes equation used to model the famous Belousou-Zhabotinsky reaction in chemical kinetics; the flame propagation model; model equations describing the morphogenesis of pattern. others are the Schnakenberg system; Fitz-Hugh-Nagumo equations; Hodgkin-Huxley equations.

Asymptotic analysis has been discussed in many types of problems, including the difference equations, singular differential equations, integral equations and special functions [1, 2, 3, 4, 5, 6, 7, 8]. It is noted that many of these differential equations and difference equations whose exact solutions are now known but can be approached via asymptotic analysis. Series solutions of the problems are mostly divergent, and therefore asymptotic analysis techniques are needed to evaluate the divergent representations of these problems.

Suppose $E \subset \mathbb{R}^d$, $d \geq 2$ is an open bounded domain, the Cauchy-Dirichlet problem for a monotone semilinear uniformly elliptic second-order parabolic system in divergence form with measure data is of the following form:

$$\begin{cases} \dfrac{\partial u^k}{\partial t} - L_t u^k = f^k(t,x,u) + \mu^k \qquad \text{in } E_T \quad \text{k=1,...,N} \\ u|_{\partial E}(t,.) = 0 \quad t \in [0,T) \\ u(0,.) = \varphi \quad \text{on } E \end{cases} \tag{1.1}$$

Where $E_T \equiv [0,T] \times E$ and $\mu^k, k = 1,\dots,N$ are bounded soft measures on $\mathbb{R}_+ \times E$ i.e bounded Borel measures absolutely continuous with respect to the parabolic capacity determined by the operator:

$$L_t = \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j} \left(a_{i,j}(t,x) \frac{\partial}{\partial x_i} \right) \tag{1.2}$$

Its coefficient $a : E_T \to \mathbb{R}^d \otimes \mathbb{R}^d$ is a measurable symmetric matrix-valued function such that for some $\gamma \geq 1$

$$\gamma^{-1}|\xi|^2 \leq \sum_{i,j=1}^{d} a_{i,j}(t,x)\xi_i\xi_j \leq \gamma|\xi|^2, \qquad \xi \in \mathbb{R}^d \tag{1.3}$$

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$f: E_T \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous monotone vector field. In the case of (1.1) being a scalar, results in an equation with a local operator of the form:

$$L = \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{i,j}\frac{\partial}{\partial x_i}\right) \tag{1.4}$$

The above system (1.1) is an evolution equation describing a reaction-diffusion problem that arises naturally in systems consisting of many interacting components and is widely used to describe pattern-formation phenomena in varieties of biological, chemical and physical systems. One of the major problems one faces when dealing with a system of (1.1) is to give a proper definition of a solution that guarantees the existence of only one solution. One possible attempt made, was to define a solution by a limit of approximation. The existence of a distributional solution so-called solution obtained by the limit of approximation (SOLA)was proved in [9]. [9] constructed an example of a discontinuous coefficient a_{ij} and u is the distributional solution with data $\mu = 0$ and $f = 0$ and u having the property that $u \in W_0^{1,q}(E)$ for every $q < d/d-1$. Since it was known that in general, one cannot expect that the solution belongs to the space $W_0^{1,q}(E)$ with $q > d/d-1$, the problem of the definition of a solution to (1.1) ensuring uniqueness arose.
However, [10] provided a solution to the problem by defining the solution in a duality sense in the case of a linear operator. Since his intervention, the theory of scalar equation with measure data and local operators (linear and nonlinear of Leray-Lion type) has attracted considerable attention for a result of an equation with measure data. According to [11], $u \in L^1(E_T)$ is a duality solution of the scalar form of (1.1) if

$$-\int_E \varphi w(0)dx + \int_{E_T} ugdxdt = \int_{E_T} wd\mu \tag{1.5}$$

for every $g \in L^\infty(E_T)$ and w is the solution to the backward problem

$$\begin{cases} -w_t - div(a^*(t,x)\nabla w) = g & \text{in } (0,T)\times E = E_T \\ w(T,x) = 0 \quad t \in E \\ w(t,x) = 0 \quad \text{on } (0,T)\times \partial E \end{cases} \tag{1.6}$$

where $a^*(t,x)$ is the transpose of $a(t,x)$. If $f \in L^1(E_T)$ then there exists a unique duality solution.
But for the nonlinear operator ($L = \Delta^\alpha$ with $\alpha \in (0,1]$)of the scalar form of (1.1), the existence of a distributional solution so-called solution obtained by the limit of approximation (SOLA) was proved in [12, 13, 14]. [12] examined the Dirichlet problem so that a function u has the property that $u \in L^q(0,T,W_0^{1,q}(E))$ for every $q < \dfrac{p(N+1)-N}{N+1}$ with $f \in \mathscr{M}(E_T), \varphi \in \mathscr{M}(E)$ and showed that $u \in C([0,T];H^{-s}(E))$ for s large enough is a distributional solution of Dirichlet scalar form of (1.1) with a being a non-linear monotone operator of the form $a(t,x,\nabla u)$. Also, [13] presented existence result of a distributional formulation of a scalar form of (1.1) with $f = 0$ for which φ and μ are bounded measures on E_T by approximating the equation with problems having regular data and using a compactness argument, so that a function $u \in L^1(0,T;W_0^{1,1}(E))$ is a weak solution if $a(t,x,u,\nabla u) \in L^1(E_T)^N$ and

$$-\int_{E_T} u\frac{\partial\phi}{\partial t}dxdt + \int_{E_T} a(t,x,u,\nabla u)\ldots\nabla\phi dxdt = \int_{E_T}\phi d\mu \tag{1.7}$$

for every $\phi \in C^\infty(\bar{E_T})$. In [14], uniqueness result of the existed result presented in [13] in the case where μ is a function in L^1 were shown. But this distributional formulation (SOLA) is not enough to ensure uniqueness due to the lack of regularity of the solution when the coefficient of the matrix is discontinuous as cited in [9], hence the problem of the definition of a solution to (1.1) ensuring uniqueness emerged. Nevertheless, the notion of the renormalized solution was first introduced in [15] for the study of the Boltzmann equation and was later adapted to the study of some nonlinear problems which guaranteed uniqueness. [16] proved existence and uniqueness result of the renormalized solution. A measurable u defined on E_T is a renormalized solution if $T_k(u) \in L^2(0,T;H_0^1(E))$, for any $k > 0$, $u \in L^\infty(0,T;L^1(E))$

$$\int_{\{(x,t):n\le|u(x,t)|\le n+1\}} |\nabla u|^2 dxdt \to 0 \qquad \text{as } n\to\infty \tag{1.8}$$

and if for any $s \in C^\infty(\mathbb{R})$ such that $S' \in C_0^\infty(\mathbb{R})$ (i.e S' has a compact support), then

$$\frac{\partial S(u)}{\partial t} - \text{div}[S'(u)A\nabla u] + s''(u)A\nabla u + dw[S'(u)\phi(u)] - S''(u)\phi(u)\nabla u = fS' \text{ in } D'(E_T) \tag{1.9}$$

and

$$S(u)(t=0) = S(u_0) \quad \text{ in } E \tag{1.10}$$

where $D'(E_T)$ is the derivative of an $L^\infty(E_T)$ function. Also, in [17], the existence and uniqueness of a renormalized solution for which the data f,g and φ respectively belong to $L^1(E_T),(L^{p^1}(E_T))^N$ and $L^1(E)$ was proved. Furthermore, the definition of a renormalized solution involving a more general nonlinear operator L_t of Leray-Lions type but with f not depending on u has been introduced in [18] for which $\varphi \in L^1(E)$ and $\mu \in M_0(E_T)$ for every μ which does not charge the set of zero capacity such that there exists a unique renormalized solution. [19] thereafter extended the notion of renormalized solution for general measure data μ and gave a definition of renormalized solution that does not depend on the decomposition of the regular part of μ under a certain assumption and proved the uniqueness of the solution. Likewise, in [20, 21, 22, 23] the framework of a renormalized solution is used.
Concurrently, the notion of entropy solutions has been proposed by [24] for the nonlinear elliptic problems. They introduced the concept of entropy solution, derived the basic apriori estimates on the measure of their level set and established the existence and uniqueness of

entropy solution for the Dirichlet elliptic problem. [25], thereafter, extended the notion of an entropy solution to a parabolic equation and then established the existence and uniqueness of the entropy solution. $u \in C([0,t];L^1(E))$ is an entropy solution for the scalar form of (1.1) with $\mu = 0$ such that for all $k > 0, T_k(u) \in L^p(E;W_0^{1,p}(E))$ and

$$\int_0^T \int_E T_k(u-\phi)(T) - \int_E T_k(\varphi - \phi(0)) + \int_0^T \langle \phi_E, T_k(u-\phi)\rangle + \int_0^T \int_E a(t,x)\nabla T_k(u-\phi) \leq \int_0^T \int_E f T_k(u-\phi) \tag{1.11}$$

$\forall L^p(E:W_0^{1,p}(E)) \cap L^\infty(E_T) \cap C([0,T];L^1(E))$ such that $\phi_t \in L^{P'}(E;W^{-1,p'})$ so that if $f \in L^1(E_T)$ and $\varphi \in L^1(E)$ then, there exists a unique entropy solution.
The notion of renormalized solution and entropy solution for the parabolic problem (1.1) in scalar form turns out to be equivalent as proved in [26]. The main tool of the uniqueness proof in the case of entropy and the renormalized solution was the fact that the truncates of the solutions belong to the energy space $W_0^{1,p}(E)$ as well as an estimate

$$\int_E |\nabla T_k(u)|^2 \leq k\|\mu\|_{M(E)}$$

on the decay of the energy of the solution on the sets where the solution is large which is true only if the datum μ belongs to $L^1(E)+W^{-1,p'}(E)$
To cover a larger class of operators, another possible attempt made was to define a solution via a non-linear Feynman-Kac formula. This stochastic approach to the scalar form of (1.1) has been developed in [27] which was defined as: a quasi-continuous function $u : E \to \mathbb{R}$ such that

$$u(x) = E_x\left(\int_0^\zeta f(X_t, u(X_t))dt + \int_0^\zeta dA_t^\mu\right) \tag{1.12}$$

with $\mathbb{X} = (X, P_x)$ being a time-space Markov process with lifetime ζ, E_x denotes the expectation with respect to P_x and A^μ is the additive functional of $\mathbb{X}$ associated with μ in the Revuz sense. It was proved in [27, 28, 29, 30, 31] that under mild integrability assumptions on the data, there exists a unique probabilistic solution of (1.1) when $N = 1$. The stochastic approach is simpler to investigate than the distributional formulation because the direct analysis of the distributional equation would generate many technical difficulties in using the fine topology while the stochastic approach avoids them. However, in [32], it was shown that the renormalized definition of the solution is equivalent to the probabilistic definition considered in [33] and shows that under mild integrability assumption on the data with f satisfying monotonicity condition, a quasi-continuous function u is a renormalized solution if and only if u can be represented by a suitable nonlinear Feynman-Kac formula.
When studying system (1.1) with f satisfying monotonicity condition

$$\langle f(t,x,y) - f(t,x,y'), y - y'\rangle \leq \alpha |y-y'|^2 \tag{1.13}$$

and the growth condition

$$f(.,.,0) \in L^1(E_T), \qquad \forall_{r>0, y\in\mathbb{R}^N} R^{0,T}\left(\sup_{|y|\leq r} |f(.,.,y)|\right) < \infty\, m_1\text{-almost surely (in short a.s)} \tag{1.14}$$

there's difficulty because showing that $f_u \equiv f(t,x,u)$ belong to $L^1(E_T)$ under growth condition is, in general, complicated and $u \in T_2^{0,1}$ is not certain which has to do with the weaker regularity of the solution of (1.1) and u, in general, does not admit the representation

$$u(s,x) = E'_{s,x}\int_0^{\zeta_\tau} dA_\theta \tag{1.15}$$

for some Addictive Functional (AF) of $\mathbb{X}'$, which implies that the integral on the right-hand side of (1.16) does not exist.

$$u(s,x) = E_{s,x}\left(\mathbf{1}_{\{\zeta^s > T\}}\varphi(\mathbf{X}_T) + \int_s^{T\wedge\zeta^s} f_u(\theta, \mathbf{X}_\theta)d\theta + \int_s^{T\wedge\zeta^s} dA_\theta^\mu\right) \tag{1.16}$$

The above comment shows that for systems, neither the distributional definition nor the probabilistic via nonlinear Feynman-Kac formula is applicable. For these reasons in [34], more general than in [27, 28], a probabilistic definition of a solution of the elliptic form of (1.1) is adopted. It uses the representation of u in terms of some backward stochastic differential equation (BSDE)

$$Y_t^{s,x} = \mathbf{1}_{\{\zeta > T_\tau\}}\varphi(\mathbf{X}_{T_\tau}) + \int_t^{\zeta_\tau} f(\mathbf{X}_\theta, Y_\theta^{s,x}, Z_\theta^{s,x})d\theta + \int_t^{\zeta_\tau} dA_\theta^\mu - \int_t^{\zeta_\tau} Z_\theta^{s,x} dB_\theta \tag{1.17}$$

In the case f is integrable, the representation reduces to (1.16).
The stochastic approach using BSDE (1.17) only requires quasi-integrability of $f(.,.,u)$ by making essential use of the Markov process $\mathbb{X}$ associated with L_t and therefore called stochastic Sobolev space (some space wider than $T_2^{0,1}$).

The manuscript is organized as follows: In section 2, a semilinear parabolic equation is introduced including notations, definitions, useful results and useful tools employed in the course of this work. Section 3 comprises some qualitative properties of the solution of the system in the stochastic Sobolev space using the theory of backward stochastic differential equations (BSDEs). And ends with section 4 of the conclusion

2. Preliminary Results and Mathematical background

2.1. Notations

The following notations used in this work are as follows:

$(\Omega,\mathscr{F},P)$	the probability space
$\mathscr{F}_t$	filtration
B	standard d-dimensional $\mathscr{F}_t$-Brownian motion
$\mathscr{A}$	the set of all $\mathscr{F}_t$ progressively measurable real-valued processes
$\mathscr{V}$	the subspace of $\mathscr{A}$ consisting of all increasing càdlàg processes Y such that $Y_0=0$
$\mathscr{V}_c$	the subspace of $\mathscr{A}$ consisting of all increasing continuous processes Y such that $Y_0=0$
$\mathscr{M}$	the space of all processes $Z\in\mathscr{A}$ such that $P\left(\int_0^T \lvert Z_t\rvert^2dt<\infty\right)^{p/2}=1$
$\mathscr{D}$	the space of all càdlàg processes in $\mathscr{A}$
$\mathscr{S}$	the space of all continuous processes in $\mathscr{A}$
$\mathscr{D}^p$, $p>0$	the space of all processes $Y\in\mathscr{D}$ such that $E\sup_{t\geq 0}\lvert Y_t\rvert^p<\infty$
$\mathscr{S}^p$, $p>0$	the space of all processes $Y\in\mathscr{S}$ such that $E\sup_{t\geq 0}\lvert Y_t\rvert^p<\infty$
$\mathscr{T}$	the set of all finite $\mathscr{F}_t$-stopping time
ΔY_t	càdlàg process Y such that $\Delta Y_t=Y_t-Y_{t-}$, $Y_{t-}=\lim_{s\nearrow t}Y_s$
$\lvert A\rvert$	trace AA^*, where A is an N×d dimensional real matrix
$\hat{x}$	$\hat{\text{sgn}}(x)=\mathbf{1}_{\{x\neq 0\}}\frac{x}{\lvert x\rvert}$, $x\in\mathbb{R}^N$
$\mathscr{D}^q\otimes\mathscr{M}^q$	the tensor product of $\mathscr{D}^q$ and $\mathscr{M}^q$
$H_0^1(H^{-1})$	Hilbert space (dual of Hilbert space)
$\mathbb{R}_+$	$[0,\infty)$
$\Omega=C(\mathbb{R}_+;\mathbb{R}^d)$	the space of continuous $\mathbb{R}^d$-valued function $\mathbb{R}_+$
X	the canonical process on Ω
$\mathscr{F}^0_{s,t}$	$\sigma(X_u,u\in(s,t))$
$\mathscr{F}_{s,\infty}$	the completion of $\mathscr{F}^0_{s,t}$ with respect to $\mathscr{P}$
$\mathscr{P}$	the family of $P_{s,\mu}$: μ is a measure on $\mathscr{B}(\mathbb{R}^d)$
$P_{s,\mu}(\cdot)$	$\int_{\mathbb{R}^d}P_{s,x}(\cdot)\mu dx$
$F_{s,t}$	completion of $\mathscr{F}^0_{s,t}$ in $\mathscr{F}_{s,\infty}$ with respect to $\mathscr{P}$
p	the fundamental solution for the operator L_t
$\mathbb{X}$	time-inhomogeneous Markov process
$\mathbb{X}'$	time-homogeneous Markov process with respect to the filtration $\mathscr{F}'_t$ associated with the operator $\frac{\partial}{\partial t}$
Ω'	$\mathbb{R}_+\times\Omega$
$P'_{s,x}(B)$	$P_{s,x}(\{\omega\in\Omega:(s,\omega)\in B\})$
$\mathbf{X}_t(s,w)$	$(s+t,X_{s+t}(\omega))\quad t\geq 0$
$\mathscr{F}'^0_t$	$\sigma(\mathbf{X}_u,u<t)$
$\mathscr{F}'^{\,0}_\infty$	$\sigma(\mathbf{X}_u,u<\infty)$
$\mathscr{F}'_\infty$	completion of $\mathscr{F}'^{\,0}_\infty$ with respect to $\mathscr{P}'$
$\mathscr{P}'$	the family P'_μ : μ is a probability measure on $\mathbb{R}_+\times\mathbb{R}^d$
F'_t	completion of $\mathscr{F}'^0_t$ in $\mathscr{F}'_\infty$ with respect to $\mathscr{P}'$
$p'(t,(s,x),\Gamma)$	transition density
μ	Radon measure on E
$\mathscr{M}_0(E)$	the set of all measures on E
$\mathscr{M}_0$	the set of all measures on $\mathbb{R}_+\times E$
$\mathscr{M}_{0,b}(E)$	the set of all bounded measures on E
$\mathscr{M}_{0,b}$	the set of all bounded measures on $\mathbb{R}_+\times E$
$\mathscr{W}$	the space of all $u\in L^2(\mathbb{R}_+;H_0^1(E))$ such that $\frac{\partial u}{\partial t}\in H^{-1}(E))$
$E_{s,x}(E'_{s,x})$	expectation with respect to $P_{s,x}(P'_{s,x})$
m_1	Lebesgue measure on $\mathbb{R}_+\times\mathbb{R}^d$
ζ^s	$\inf\{t\geq s,X_t\in\partial E\}$, the first exist of $(X,P_{s,x})$ from E
p'_E	transition density of the process $\mathbb{X}'$ killed on existing $\mathbb{R}_+\times E$
E_x	expectation with respect to P_x
A^μ	additive functional corresponding to a positive bounded soft measure μ
p_E	transition density of the process $\mathbb{X}$ killed on existing E
$G_E(x,y)$	the Green function for E
$\lvert E\rvert$	Lebesgue measure on E

$W^{1,p}(E),\ 1\leq p\leq\infty$	Sobolev spaces
$W^{0,1}(\mathbb{X}^{E_T})$	stochastic Sobolev space of functions depending on time
L^q	the dual space of L^p space
$T_k(u)$	truncator operator
∂E	the boundary of E
τ_k	$\inf\left\{t\geq 0:\int_0^t \lvert Z_s\rvert^2 ds\geq k\right\}$
$\nabla_{\mathbb{X}}u$	the stochastic gradient of u
$a\vee b$	$\max\{a,b\}$
$a\wedge b$	$\min\{a,b\}$
a^+	$\max\{0,a\}$
a^-	$\max\{0,-a\}$
$\mathbb{N}$	set of natural numbers
$\mathbb{N}^*$	$\mathbb{N}\backslash\{0\}$
$\mathbb{R}$	set of real numbers
$C_c^\infty(\mathbb{R}^d)$	space of functions $f:\mathbb{R}^d\to\mathbb{R}$ of class C^∞ with compact support.
$C_c(\mathbb{R}^d)$	space of continuous functions $f:\mathbb{R}^d\to\mathbb{R}$ with compact support.

2.2. Basic Tools

This section consists of some important lemma, and definitions, used mainly for the proof of qualitative properties of the solution in the next section.

2.2.1. Itô Tanaka Formula

The following multidimensional version of the Itô Tanaka formula will be frequently used, which serves as our basic tool in this work.

Lemma 2.1 ([35]). *Let $\{K_t\}_{t\in[0,T]}$ and $\{H_t\}_{t\in[0,T]}$ be two progressively measurable processes with values respectively in $\mathbb{R}^k$ and $\mathbb{R}^{k\times d}$ such that P-a.s $\int_0^t\left(\lvert K_s\rvert+\lvert H_s\rvert^2\right)ds<+\infty$. Let X be a progressively measurable process such that $X_t=X_0+\int_0^t K_s ds+\int_0^t H_s dB_s$ Then, for any $p\geq 1$, gives*

$$\begin{aligned}\lvert X_t\rvert^p &= \lvert X_0\rvert^p + p\int_0^t \lvert X_s\rvert^{p-1}\left\langle \hat{X}_s, K_s\right\rangle ds + p\int_0^t \lvert X_s\rvert^{p-1}\left\langle \hat{X}_s, H_s dB_s\right\rangle \\ &+\frac{p}{2}\int_0^t \lvert X_s\rvert^{p-2}\mathbf{1}_{X_s\neq 0}\left\{(2-p)\left(\lvert H_s\rvert^2-\left\langle \hat{X}_s, H_sH_s^*\hat{X}_s\right\rangle\right)+(p-1)\lvert H_s\rvert^2\right\}+\mathbf{1}_{p=1}A_t\end{aligned} \tag{2.1}$$

where $\{A_t\}_{t\in[0,T]}$ is a continuous increasing process with $A_0=0$ which increases only on the boundary of the random set $\{t\in[0,T],X_t=0\}$

Corollary 2.2 ([35]). *Under the assumption of Lemma (2.1) for every $0<t\leq T$ and $p\geq 1$ then*

$$\lvert X_t\rvert^p + c(p)\int_t^T \lvert X_s\rvert^{p-2}\mathbf{1}_{\{X_s\neq 0\}}\lvert H_s\rvert^2 = \lvert X_T\rvert^p + p\int_t^T \lvert X_s\rvert^{p-1}\left\langle \hat{X}_s, K_s\right\rangle ds - p\int_t^T \lvert X_s\rvert^{p-1}\left\langle \hat{X}_s, H_s dB_s\right\rangle \tag{2.2}$$

where $c(p)=p[(p-1)\wedge 1]/2$

2.3. Basic Definitions

The following definitions are important and serve as basic tools in this work.

Definition 2.3. A measurable function u is a quasi-continuous if the process $t\to u(\mathbf{X}_t)$ is continuous on $[0,\zeta_\tau],P'_{s,x}$-almost surely (a.s for short) for quasi-every (q.e for short) $(s,x)\in E_T$.

Definition 2.4. A measurable function u is a quasi-càdàg if the process $t\to u(\mathbf{X}_t)$ is càdàg on $[0,\zeta_\tau],P'_{s,x}$-a.s for q.e $(s,x)\in E_T$.

Definition 2.5. A Borel measurable function F on E is quasi-integrable if for q.e $(s,x)\in F$

$$P_{s,x}\left(\int_0^{\zeta\wedge T}\lvert f(X_\theta\rvert d\theta<\infty,\ T>0)\right)=1 \tag{2.3}$$

By qL^1, it denotes the set of all quasi-integrable functions on E.

Definition 2.6. We say that a process Y is of class (D) if $Y\in\mathscr{A}$ and the family $\{Y_\tau,\tau\in\mathscr{T}\}$ where $\mathscr{T}$ is the set of all finite F_t-stopping times, are uniformly integrable.

Definition 2.7. A measurable function u on E_T is of class (RD) if for q.e $(s,x)\in E_T$, the process $u(\mathbf{X})$ on $[0,\zeta_\tau]$ is of class (D) under the measure $P'_{s,x}$.

Definition 2.8. A Borel measurable function u on E_T is of class (RB) if for q.e

$$(s,x) \in E_T,\ P'_{s,x}\left(ess \sup_{\theta\in[0,\zeta_\tau]} |u(\mathbf{X}_\theta)| < \infty\right) = 1 \tag{2.4}$$

which implies that every quasi-cadlag function belongs to the class (RB).

Definition 2.9. A Borel measurable function u on E_T is of class (RM) if for q.e $(s,x) \in E_T$, $P'_{s,x}\left(\int_0^{\zeta_\tau} |u(\mathbf{X}_\theta)| < \infty\right) = 1.$

Definition 2.10. Stochastic Sobolev space of functions depending on time $(W^{0,1}(\mathbb{X}^{E_T}))$ denote the set of all $u \in RM$ for which there exists a sequence $\{u_n\} \subset C_c^\infty(E_T)$ such that for q.e $(s,x) \in E_T$,

$$\int_0^{\zeta_\tau} |(u_n - u)(\mathbf{X}_t)|^2 dt \to 0 \text{ in probability } P'_{s,x} \text{ as } n \to \infty \tag{2.5}$$

and

$$\int_0^{\zeta_\tau} |\nabla(u_n - u_m)(\mathbf{X}_t)|^2 dt \to 0 \text{ in probability } P'_{s,x} \text{ as } n,m \to \infty \tag{2.6}$$

Definition 2.11. Let $u_n, u \in W^{0,1}(\mathbb{X}^{E_T})$. $u_n \to u$ in $W^{0,1}(\mathbb{X}^{E_T})$ if $u_n \to u$ in RM and $\nabla_{\mathbb{X}} u_n \to \nabla_{\mathbb{X}} u$ in RM.

Definition 2.12. The parabolic capacity of an open set $u \subset \mathbb{R}_+ \times E$ denoted as cap(u) define as cap$(u) = \inf\{\|u\|_{\mathscr{W}} : u \in W, u \geq 1_u \text{ in } \mathbb{R}_+ \times E\}$.

Definition 2.13. The parabolic capacity of an open set $u \subset E$ denoted as cap$_N(u)$ define as cap$_N(u) = \inf\{\|u\|_{H_0^1(E)} : u \in H_0^1(E), u \geq 1$ on u almost everywhere (a.e for short) in $E\}$.

Definition 2.14. Some properties are satisfied for quasi-every (q.e for short) $x \in E$ (respectively, $(s,x) \in \mathbb{R}_+ \times E$) if it is satisfied except for some Borel subset of E (respectively, $\mathbb{R}_+ \times E$) of cap$_N$ (respectively, cap) capacity zero.

Definition 2.15. Let μ be a Radon measure on E (respectively, $\mathbb{R}_+ \times E$), a measure μ is soft if μ charge no set of cap$_N$ (respectively, cap) capacity zero.

Definition 2.16. A pair $(Y^{s,x}, Z^{s,x})$ consisting of an $\mathbb{R}^N$ valued process $Y^{s,x}$ and $\mathbb{R}^d \times \mathbb{R}^N$-valued process $Z^{s,x}$ is a solution of BSDE$_{s,x}(\varphi, E, f + d\mu)$ if $Y^{s,x}, Z^{s,x}$ are $\{\mathscr{F}'_t\}$ progressively measurable, $Y^{s,x}$ is càdlàg,
$t \mapsto f(\mathbf{X}_t, Y_t^{s,x}, Z_t^{s,x}) \in L^1(0,\zeta_\tau), P'_{s,x}$-a.s, $P'_{s,x}\left(\int_t^{\zeta_\tau} |Z_\theta^{s,x}|^2 d\theta < \infty\right) = 1$
and

$$Y_t^{s,x} = \mathbf{1}_{\{\zeta > T_\tau\}}\varphi(\mathbf{X}_{T_\tau}) + \int_t^{\zeta_\tau} f(\mathbf{X}_\theta, Y_\theta^{s,x}, Z_\theta^{s,x})d\theta + \int_t^{\zeta_\tau} dA_\theta^\mu - \int_t^{\zeta_\tau} Z_\theta^{s,x} dB_\theta, \quad t \in [0,\zeta_\tau],\ P'_{s,x}\text{-a.s} \tag{2.7}$$

Definition 2.17. Consider the following systems:

$$\begin{cases} \dfrac{\partial u^k}{\partial t} - L_t u^k = f^k(x,u) + \mu^k & \text{in } E_T \quad \text{k=1,...,N} \\ u|_{\partial E}(t,.) = 0 \quad t \in [0,T) \\ u(0,.) = \varphi \quad \text{on } E \end{cases} \tag{2.8}$$

and

$$\begin{cases} \dfrac{\partial u^k}{\partial t} + L_t u^k = -f^k(x,u) - \mu^k & \text{in } E_T \quad \text{k=1,...,N} \\ u|_{\partial E}(t,.) = 0 \quad t \in [0,T) \\ u(T,.) = \varphi \quad \text{on } E \end{cases} \tag{2.9}$$

(1) A measurable function $u : E_T \to \mathbb{R}^N$ is a solution of (2.9) if

 (a) $(t,x) \longmapsto f(t,x,u(t,x)) \in qL^1(E_T), u \in W^{0,1}(\mathbb{X}^{E_T})$
 (b) u is of class (RD)
 (c) For q.e $(s,x) \in E_T$,

$$u(\mathbf{X}_t) = \mathbf{1}_{\{\zeta > T_\tau\}}\varphi(\mathbf{X}_{T_\tau}) + \int_t^{\zeta_\tau} f(\mathbf{X}_\theta, u(\mathbf{X}_\theta))d\theta + \int_t^{\zeta_\tau} dA_\theta^\mu - \int_t^{\zeta_\tau} \sigma\nabla_{\mathbb{X}} u(\mathbf{X}_\theta) dB_\theta \qquad t \in [0,\zeta_\tau],\ P'_{s,x}\text{-a.s} \tag{2.10}$$

(2) A measurable function $u : E_T \to \mathbb{R}^N$ is a solution of system (2.8) on $[0,T]$, if $\bar{u}(t,x) = u(T-t,x), (s,x) \in E_T$ is a solution of (2.9) with f_u replaced by $\bar{f}_u$, and μ replaced by $\bar{\mu}$ for which $\mu \in \mathscr{M}_{0,b}(E_T)$ such that $\int_{E_T} \bar{\eta} d\mu = \int_{E_T} \eta d\mu,\ \eta \in B_b(E_T)$ i.e.

$$\begin{cases} \dfrac{\partial \bar{u}^k}{\partial t} + L_{T-t}\bar{u}^k = -\bar{f}^k(t,x,\bar{u}) - \bar{\mu}^k \circ \iota_T^{-1} & \text{in } E_T \quad \text{k=1,...,N} \\ \bar{u}|_{\partial E}(t,.) = 0 \quad t \in [0,T) \\ \bar{u}(T,.) = \varphi \quad \text{on } E \end{cases} \tag{2.11}$$

where $\iota_T : E_T \to E_T,\ \iota_T(t,x) = (T-t,x)$

Remark 2.18. We have the following remarks:

(1) The integral equation (2.10) is the transformation of the partial differential equation (2.9).

(2) From Definition 2.16 and Definition 2.17, it follows that if u is a solution of (2.9) then for q.e $(s.x) \in E_T$, the solution of the Backward Differential Equation (BSDE) $(Y^{s,x}, Z^{s,x}) = (u(\mathbf{X}_t), \sigma\nabla_{\mathbb{X}}(\mathbf{X}_t)),\quad t \in [0,\zeta_\tau]$ is a solution of (2.9).

2.3.1. Basic Assumptions

The following hypotheses are considered in this work and are classified into parts as follows:

(A1) $E\left(|\xi|^p + \left(\int_0^\sigma |f(t,0)|dt\right)^p + |A|_\sigma^p\right) < +\infty$

(A2) There is $\mu \in \mathbb{R}$ such that $< y-y', f(t,y)-f(t,y') > \le \mu|y-y'|^2$ for every $t \ge 0,\ y,y' \in \mathbb{R}^N$ and a.e $(t,x) \in E_T$

(A3) For every $t \ge 0,\quad y \longmapsto f(t,y)$ is continuous

(A4) For every $r > 0,\quad E\int_0^\sigma \sup_{|y|\le r} |f(t,y)|dt < \infty$

(B1) $f(\cdot,\cdot,y)$ is measurable for every $y \in \mathbb{R}^N$ and $f(t,x,\cdot)$ is continuousfor a.e$(t,x) \in E_T$

(B2) There is $\alpha \in \mathbb{R}$ such that $< f(t,x,y)-f(t,x,y'), y-y' > \le \alpha|y-y'|^2$ for every $y,y' \in \mathbb{R}^N$ and a.e $(t,x) \in E_T$

(B3) $f(\cdot,\cdot,0) \in L^1(E_T), \mu \in M_{0,b}(E_T), \varphi \in L^1(E)$

(B4) $\forall r > 0, y \in \mathbb{R}^N,\quad R^{0,T}\left(\sup_{|y|\le r} |f(\cdot,\cdot y)|\right) < \infty, m_1$- a.e.

Remark 2.19. f satisfying (B2) is the monotonicity condition while f satisfying (B3) and (B4) is the growth condition of f.

2.3.2. Backward Stochastic Differential Equations

The theory of nonlinear backward stochastic differential equations (BSDEs for short) was developed by [36] from which there exist a unique and adapted square integrable solution to a BSDE of the type

$$y_t = \xi + \int_t^\sigma f(s,y_s,z_s) - \int_t^\sigma z_s dB_s, \qquad t \in [0,t] \tag{2.12}$$

provided the function f (called the generator) is Lipchitz in both variables y and z, ξ and $(f(t,0,0))_{0\le t\le T}$ are square integrable.
The theory of BSDEs is very important because of its connection with mathematical finance, stochastic control, partial differential equation, stochastic geometry etc.
Some useful results from [29] that will be helpful in the proof of the existence of a solution for BSDE. Before starting them, The following hypothesis is needed for the next three results.

(A) There is $\mu \in \mathbb{R}$ and a nonnegative progressively measurable process $\{f_t, t \ge 0\}$ such that $\langle \hat{y}, f(t,y)\rangle \le f_t + \mu\|y\|,\ \forall (t,y) \in \mathbb{R}_+ \times \mathbb{R}^N$

Result 2.20. Assume (A). Let (Y,Z) be a solution of BSDE$_{s,x}(\xi,\sigma,f+dA)$

$$Y_t = \xi + \int_t^\sigma f(s,Y_s)ds + \int_t^\sigma dA_s - \int_t^\sigma Z_s dB_s, \qquad 0 \le t \le \sigma\ P\text{-a.s} \tag{2.13}$$

If $Y \in \mathscr{D}^p$ and $\left(\int_0^\sigma f_t dt\right)^p + E|A|_\sigma^p < \infty$ for some $p > 0$, then $Z \in \mathscr{M}^p$ and there exist C_p depending only on p such that for every $a \ge \mu$

$$E\left(\int_0^\sigma e^{2at}|Z_t|^2 dt\right)^{p/2} \le C_p E\left(\sup_{0\le t\le\sigma} e^{apt}|Y_t|^p + \left(\int_0^\sigma e^{at} f_t dt\right)^p + \left(\int_0^\sigma e^{at} d|A|_t\right)^p\right) \tag{2.14}$$

Result 2.21. Assume (A). Let (Y,Z) be a solution of BSDE (2.13), If $Y \in \mathscr{D}^p$ and $\left(\int_0^\sigma f_t dt\right)^p + E|A|_\sigma^p < \infty$ for some $p > 1$, then $Z \in \mathscr{M}^p$ and there exist C_p depending only on p such that for every $a \ge \mu$

$$E\left(\sup_{0\le t\le\sigma} e^{apt}|Y_t|^p + \left(\int_0^\sigma e^{2at}|Z_t|^2 dt\right)^{p/2}\right) \le C_p E\left(e^{ap\sigma}|\xi|^p + \left(\int_0^\sigma e^{at} f_t dt\right)^p + \left(\int_0^\sigma e^{at} d|A|_t\right)^p\right) \tag{2.15}$$

Result 2.22. Assume that $\langle f(t,y),y\rangle \leq c|y|^2, y\in\mathbb{R}^N, t\geq 0$ for some $c\geq 0$ and $\|\xi\|_\infty + c + \||A|_\sigma\|_\infty \leq r < \infty$. If (Y,Z) is a solution of BSDE (2.13) such that Y is of class (D), then $\|Y\|_\infty \leq r$.

Result 2.23. Assume that (B1)-(B4) are satisfied with $\alpha \leq 0$. Let $n<m$ and $\delta Y_t = Y_t^m - Y_t^n$, $\delta Z_t = Z_t^m - Z_t^n$. Then for every $q\in(0,1)$,

$$\begin{aligned} E'_{s,x}\sup_{t\geq 0}|\delta Y_t|^q + \varepsilon E'_{s,x}\left(\int_0^\zeta |\delta Z_\theta|^2 d\theta\right)^{q/2} &\leq (1-q)^{-1}(1+2\varepsilon Cq)E'_{s,x}\Big(\ |\varphi(\mathbf{X}_{n-s})|^q \mathbf{1}_{\{\zeta>n-s\}} + |\varphi(\mathbf{X}_{m-s})|^q \mathbf{1}_{\{\zeta>m-s\}} \\ &+ \left(\int_n^\zeta d|A^\mu|_\theta\right)^q + \left(\int_n^\zeta |f(\mathbf{X}_\theta,0)|d\theta\right)^q\Big) \end{aligned} \tag{2.16}$$

for $\varepsilon = 0,1$ where C_q depend only on q

2.3.3. Stochastic Sobolev Space

Stochastic Sobolev space has been introduced in [34] which makes essential use of the Markov process $\mathbb{X}$ associated with operator L defined by

$$L = \sum_{i,j=1}^d \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial}{\partial x_i}\right) \tag{2.17}$$

in considering an elliptic system with measure data, there was a problem encountered as to whether the solution or its gradient belongs to $W_0^{1,q}(E)$ for $q\geq 1$ which is related to the lack of integrability stochastic Sobolev space was then introduced to overcome this difficulty.
This subsection covers some basic results attributed to stochastic Sobolev space, which will be referred to in the next section.
In [34], it is proved that every $u\in W^{0,1}\left(\mathbb{X}^{E_T}\right)$, there exists a unique a.e function $v\in\mathscr{B}(E_T)$ such that for every $\{u_n\}\subset C_c^\infty(E_T)$ satisfying (2.5) and (2.6),

$$\int_0^{\zeta_\tau} \|\nabla u_n(\mathbf{X}_t) - v(\mathbf{X}_t)\|^2 dt \to 0 \text{ in probability } P'_{s,x} \text{ as } n\to\infty \tag{2.18}$$

for q.e $(s,x)\in E_T$. Given $u\in W^{0,1}(\mathbb{X}^{E_T})$, $\nabla_{\mathbb{X}} u$ is the unique function v satisfying (2.5). From the construction of $\nabla_{\mathbb{X}} u$, it follows that $\nabla_{\mathbb{X}} u = \nabla u$ a.e if $u\in L^2(0,T;H_0^1(E))$, $u_n\to u$ in RM if (2.5) holds for q.e $(s,x)\in E_T$, $T_k(u)\to u$ in RM.

Result 2.24. If $u\in W^{0,1}(\mathbb{X}^{E_T}), r\in C^1(\mathbb{R})$ and there is $c>0$ such that $\|r'(t)\|\leq c$ for $t\in\mathbb{R}$, then $r(u)\in W^{0,1}(\mathbb{X}^{E_T})$ and $\nabla_{\mathbb{X}}(r(u)) = r'\nabla_{\mathbb{X}} u$

Result 2.25. Let $k\in\mathbb{R}$ and $u\in W^{0,1}(\mathbb{X}^{E_T})$. Then, $u\wedge k, u\vee k\in W^{0,1}(\mathbb{X}^{E_T})$ and

$$\begin{aligned} \nabla_{\mathbb{X}}(u\wedge k) &= \mathbf{1}_{(-\infty,k)}(u)\nabla_{\mathbb{X}} u = \mathbf{1}_{(-k,k]}(u)\nabla_{\mathbb{X}} u \text{ a.e} \\ \nabla_{\mathbb{X}}(u\vee k) &= \mathbf{1}_{(k,\infty)}(u)\nabla_{\mathbb{X}} u = \mathbf{1}_{[k,\infty)}(u)\nabla_{\mathbb{X}} u \text{ a.e} \end{aligned} \tag{2.19}$$

Result 2.26. If $u\in$RB and $T_k(u)\in W^{0,1}(\mathbb{X}^{E_T})$ for every $k\geq 0$, then $u\in W^{0,1}(\mathbb{X}^{E_T})$.

Result 2.27. If $u\in\mathscr{T}^{0,1}$ and $u\in$RB, then $u\in W^{0,1}(\mathbb{X}^{E_T})$

Result 2.28. If $u\in W^{0,1}(\mathbb{X}^{E_T})$ then, u is of class (RD).

2.3.4. Potential and Markov Processes

A family $A=\{A_{s,t}, 0\leq s\leq t\leq T\}$ of random variables is an additive function (AF) of $\mathbb{X}$ if $A_{s,\cdot}$ is a $(\{F_{s,t}\},P_{s,x})$-measurable càdlàg process and $P_{s,x}(A_{st} = A_{s,u}+A_{u,t}, s\leq u\leq t\leq T) = 1$ for q.e $(s,x)\in E_T$. For a given additive functional A of $\mathbb{X}$, its energy is given by

$$e(A) = \lim_{t\searrow 0}\frac{1}{2t}E_m A_t^2 \tag{2.20}$$

wherever the limit exists. If N is an AF such that for q.e $(s,x)\in E_T$ has $P_{s,x}$ almost all continuous trajectories, it is called continuous AF (CAF). If M is an AF such that for q.e $(s,x)\in E_T$, $E_{s,x}|M_{s,t}|^2<\infty$ and $E_{s,x}M_{s,t}=0$ for $t\in[s,T)$ it is called martingale AF (MAF). A CAF A of finite variation is square integrable if $\int_0^T E_{s,x}|A_{s,\cdot}|_T^2 ds<\infty$.
It is known from [37], that for a time-inhomogeneous Markov process $\mathbb{X}$ associated with the operator L_t defined in (1.2), the functional $A_t\equiv u(X_t)-u(X_0)$ admit the so-called Fukushima decomposition i.e for q.e $(s,x)\in\mathbb{R}_+\times\mathbb{R}^d$

$$u(X_t)-u(X_s) = N_{s,t}+M_{s,t} \qquad s\leq t,\ P_{s,x}\text{-a.s} \tag{2.21}$$

where N is a two-parameter continuous additive functional (MAF) of $\mathbb{X}$ of finite energy. Moreover

$$\left\langle M^i_{s,\cdot}, M^j_{s,\cdot}\right\rangle_t = \int_s^t a_{ij}(\theta,X_\theta)d\theta, \qquad s\leq t \tag{2.22}$$

which implies the process

$$B_{s,t} = \int_s^t \sigma^{-1}(\theta,X_\theta)dM_{s,\theta}, \qquad t\geq s \tag{2.23}$$

where $\sigma\cdot\sigma^T = a$ is a Brownian motion under $P_{s,x}$. It is also known from [38] that $B_{s,\cdot}$ is an $\{\mathscr{F}_{s,t}\}_{t\geq s}$-Brownian motion.
Also, it is known that for a time-homogeneous Markov process $\mathbb{X}'$ associated with the operator $\dfrac{\partial}{\partial t}+L_t, \mathbb{X}'$ admit the strict Fukushima decomposition

$$u(\mathbf{X}_t) = u(\mathbf{X}_0)+\mathbf{N}_t+\mathbf{M}_t \tag{2.24}$$

where $\mathbf{N}$ is a CAF of $\mathbb{X}'$ of zero energy and M is a MAF of $\mathbb{X}'$ of finite energy. In [32],

$$p(\mathbf{N}_t)(\omega') = N_{s,s+t}(\omega), \qquad p(\mathbf{M}_t)(\omega') = M_{s,s+t}(\omega),\ t\geq 0 \tag{2.25}$$

for $(\omega' = (s,\omega))$ where $p:\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}^d$ is the orthogonal projection.
Set $N_t = p(\mathbf{N}_t)$ and $M_t = p(\mathbf{M}_t)$, $t\geq 0$, then

$$N_t(\omega') = N_{s,s+t}(\omega), \qquad M_t(\omega') = M_{s,s+t}(\omega) \text{ for } \omega' = (s,\omega)\ t\geq 0 \tag{2.26}$$

and for $t\geq 0$, $\tau(t):\omega'\to\mathbb{R}_+$ by putting

$$\tau(t)(\omega') = s+t = \tau(0)(\omega)+t \quad \text{ for } \omega' = (s,w) \tag{2.27}$$

If ξ is a random variable on Ω then $\xi(\omega') = \xi(\omega)$ for $\omega' = s,\omega\in\Omega'$ with $\mathbf{X}_t = (\tau(t), X_{\tau(t)})$, $t\geq 0$ and

$$\zeta = \inf\{t\geq 0;\mathbf{X}_t\notin\mathbb{R}_+\times E\}, \quad \zeta_\tau = \zeta\wedge T_\tau, \quad T_\tau = T-\tau(0)$$

Moreover,

$$\left\langle M^i, M^j\right\rangle_t = \int_0^t a_{ij}(\mathbf{X}_\theta)d\theta, \qquad t\geq 0,\ P'_{s,x}\text{-a.s} \tag{2.28}$$

for every $(s,x)\in\mathbb{R}_+\times\mathbb{R}^d$, which implies the process

$$B_t = \int_0^t \sigma^{-1}(\mathbf{X}_\theta)dM_\theta, \quad t\geq 0 \tag{2.29}$$

where $\sigma\cdot\sigma^T = a$ is a Brownian motion under $P_{s,x}$ and also an $\{\mathscr{F}'_t\}_{t\geq 0}$-Brownian motion.
Let $\mathbb{X}' = (\{X_{s,t}, t\geq s\}, \{P_{s,x},(s,x)\in\mathbb{R}^+\times E\}, \{\mathscr{F}_t, t\geq 0\}, \zeta)$ be a diffusion process such that for every $t\geq 0$, then the transition operator

$$(P_{s,t}f)(x) = E_{s,x}f(\mathbf{X}_t) \tag{2.30}$$

A positive AF A of $\mathbb{X}'$ and a positive measure μ on $\mathbb{R}_+\times E$ are in the Revuz correspondence if

$$\langle\mu,f\rangle = \int_0^\infty\int_E fd\mu = \lim_{\alpha\to\infty}\alpha\int_0^\infty\int_E\left(E'_{s,x}\int_0^{\zeta_s} e^{-\alpha t}f(\mathbf{X}_t)dA_t\right)dm_1(s,x) \tag{2.31}$$

for every $f\in\mathscr{B}'(\mathbb{R}_+\times E)$. If $\langle\mu,1\rangle<\infty$, then A is called integrable under (2.31), the family of all integrable positive additive functionals of $\mathbb{X}'$ and the family of all bounded positive soft measures on $\mathbb{R}_+\times E$ are in one-to-one correspondence. The additive functional corresponding to a positive bounded soft measure μ, A^μ corresponds to μ if and only if for q.e $(s,x)\in\mathbb{R}_+\times E$

$$E'_{s,x}\int_0^{\zeta_s} f(\mathbf{X}_t)dA_t^\mu = \int_0^\infty\left(\int_0^\infty\int_E f(z)p'_E(t,(s,x),z)d\mu(z)\right)dt \tag{2.32}$$

for every $f\in\mathscr{B}_+(\mathbb{R}_+\times E)$.
The potential operator associated with $\mathbb{X}'$ is given as

$$R^{0,T}\mu(s,x) = \int_0^\infty\left(\int_0^T\int_E(t,(s,x),z)d\mu(z)\right)dt \tag{2.33}$$

so that (2.32) becomes

$$E'_{s,x}\int_0^{\zeta_s} f(\mathbf{X}_t)dA_t^\mu = R^{0,T}\mu(s,x)f(z) \tag{2.34}$$

hence

$$R^{0,T}\mu(s,x) = E'_{s,x}\int_0^{\zeta_s} dA_t^\mu \tag{2.35}$$

Result 2.29. Let $\mu\in\mathscr{M}_{0,b}(E_T)$. Then there exists a positive AF A^μ of $\mathbf{X}$ such that for every $(s,x)\in E_T$, if $R^{0,T}\mu(s,x)<\infty$ then

$$E'_{s,x}\int_0^{\zeta_s}\eta(\mathbf{X}_t)dA_t^\mu = \int_{E_T}\eta(\theta,y)p_E(s,x,t,y)d\mu(t,y)$$

for every bounded $\eta\in\mathscr{B}(E_T)$

Suppose the coefficient of the operator (1.2) does not depend on time i.e $a_{i,j}(t,x) = a_{i,j}(x)$ for $(t,x) \in E_T$, yields (2.17). The fundamental solution of A has the property that

$$p(s,x,t,y) = p(t-s,x,y) \quad \text{for any } t > s, x \neq y \tag{2.36}$$

Let $\mathbb{X} = (\{X_t, t \geq 0\}, \{P_x, x \in E\}, \{\mathscr{F}_t, t \geq 0\}, \zeta)$ be a diffusion process, the transition operator is given as

$$(P_t f)(x) = E_x f(X_t) \tag{2.37}$$

In the homogeneous case, a positive AF A of $\mathbb{X}(\mathbf{X} = \{(X, P_x) : x \in \mathbb{R}\})$ where $P_x = P'_{0,x}$ is a time-homogeneous Markov process with transition density $p(t,x,y) = p(0,x,t,y)$ and a positive measure μ on E are in the Revuz correspondence if

$$\langle \mu, f \rangle = \lim_{\alpha \to \infty} \alpha \int_0^\infty \int_E \left(E_x \int_0^{\zeta^0} e^{-\alpha t} f(X_t) dA_t \right) dm(x) \tag{2.38}$$

for every $f \in \mathscr{B}(E)$. If $\langle \mu, 1 \rangle < \infty$, then A is called integrable. The family of all integrable positive continuous additive functionals of $\mathbb{X}$ and the family of all bounded positive soft measures on E are in one-to-one correspondence via (2.38). A^μ corresponds to μ if and only if for q.e $x \in E$

$$E_x \int_0^{\zeta^0} f(X_t) dA_t^\mu = \int_0^\infty \int_E f(y) p_E(t,(s,x,y) d\mu(y) dt \tag{2.39}$$

$G_E(x,y) = \int_0^\infty p_E(t,x,y) dt$ is the green function on E. Then from (2.39),

$$E_x \int_0^{\zeta^0} f(X_t) dA_t^\mu \} = \int_E f(y) G_E(x,y) d\mu(y) dt \tag{2.40}$$

The potential operator associated with $\mathbb{X}$ is

$$R\mu(x) = \int_E G(x,y) d\mu(y) dt \tag{2.41}$$

hence,

$$R\mu(x) = E_x \int_0^{\zeta^0} dA_t^\mu \tag{2.42}$$

and by Result 2.29 $R\mu(x) < \infty$, for a fixed Borel positive measure on E.

Result 2.30. Let $\mu \in \mathscr{M}_{0,b}(E_T)$ do not depend on time. Then, A^μ is continuous, for every $s \in [0,T]$, $R^{0,T}\mu(s,x) < \infty$ for q.e $x \in E$ and

$$A_t^\mu(0,\omega) = A_t^{\bar{\mu}}(\omega) \quad \text{for } P_x\text{-a.e } \omega \in \Omega. \tag{2.43}$$

Furthermore, from [39], there is c depending on $d, \wedge$ such that

$$p(t,x,y) \leq c t^{-d/2} \qquad \text{for } t > 0 \quad x,y \in \mathbb{R}^d \tag{2.44}$$

Therefore, there is c depending only on $d, \wedge$ such that

$$\sup_{x \in \mathbb{R}^d} E_x \zeta^0 \leq c|E|^{d/2} \tag{2.45}$$

where $|E|$ denotes the Lebesgue measure of E. Also, there exists a constant $a < 0$, $b > 0$ depending only on $d, \wedge, |E|$ such that for every $t > 0$

$$\sup_{x \in \mathbb{R}^d} P_x(\zeta^0 > t) \leq a e^{-bt}. \tag{2.46}$$

3. Qualitative Properties of the Solution

This section covers some qualitative properties of the solution of the system which include regularity of the solution, the existence of the solution, the uniqueness of the solution and large-time behaviour of the solution. For necessity, the following are shown: $(Y^{s,x}, Z^{s,x}) \in \mathscr{D}^q \otimes \mathscr{M}^q$, Y is of class (D); the regularity of $E_T \ni t \mapsto u(s,x) = E'_{s,x} Y_0^{s,x}$; the existence and uniqueness of the solution $(Y^{s,x} = u(\mathbf{X}_t), Z^{s,x} = \sigma \nabla_{\mathbb{X}} u(\mathbf{X}))$; the gradient of the solution in $L^q_{\text{loc}}(E_T)$ and lastly, the asymptotic behaviour of the solution as t becomes large.

3.1. Existence and Uniqueness of BSDE

We shall make use of the basic assumptions in subsection 2.3.1. To start with, consider the following proposition.

Proposition 3.1. *Assume (B1)-(B4). Let F be the set of those $(s.x) \in E_T$ for which the data $\zeta_\tau, f(\mathbf{X},.,.)\varphi(\mathbf{X}_{T_\tau})\mathbf{1}_{\{\zeta>T_\tau}, A^\mu$ satisfy assumptions (A1)-(A4) under $P'_{s,x}$. Then, cap$(E_T|F)=0$ and for every $(s,x)\in F$, there exists a unique solution $(Y^{s,x},Z^{s,x})$ of BSDE$_{s,x}(\varphi,E,f+d\mu)$ such that $(Y^{s,x},Z^{s,x})\in D^q\otimes M^q$ for $q\in(0,1)$ and $Y^{s,x}$ is of class(D).*

Proof. The condition

$$\varphi(\mathbf{X}_{T_\tau})\mathbf{1}_{\{\zeta>T_\tau\}}+E'_{s,x}\int_t^{\zeta_\tau} f(\mathbf{X}_\theta,\cdot,\cdot)d\theta+E'_{s,x}\int_t^{\zeta_\tau} dA^\mu_\theta \tag{3.1}$$

is finite i.e

$$\varphi(\mathbf{X}_{T_\tau})\mathbf{1}_{\{\zeta>T_\tau\}}+E'_{s,x}\int_t^{\zeta_\tau} f(\mathbf{X}_\theta,\cdot,\cdot)d\theta+E'_{s,x}\int_t^{\zeta_\tau} dA^\mu_\theta<\infty \tag{3.2}$$

is satisfied a.e $(s,x)\in E_T|F$, then it is satisfied for q.e $(s,x)\in F$.
Assume,

$$w\subset\{(s,x):\varphi(\mathbf{X}_{T_\tau})\mathbf{1}_{\{\zeta>T_\tau\}}+E'_{s,x}\int_t^{\zeta_\tau} f(\mathbf{X}_\theta,\cdot,\cdot)d\theta+E'_{s,x}\int_t^{\zeta_\tau} dA^\mu_\theta<\infty\} \tag{3.3}$$

and $\zeta_\tau=\inf\{t\geq 0:\tau(t),X_{\tau(t)}\in k\}\wedge T_\tau$ where k is a compact subset of $w=\infty$. Since $(\mathbf{X},P'_{s,x}))$ is a Feller process, ζ_τ is a $\{F'_t\}$-stopping time. By the strong Markov property with random shift

$$\begin{aligned}
P'_{s,x}(\zeta_\tau<T_\tau)&\leq P'_{s,x}\left(E'_{s,x}\left(|\varphi(\mathbf{X}_{T_\tau})|\mathbf{1}_{\{\zeta>T_\tau\}}+\int_t^{\zeta_\tau} f(\mathbf{X}_\theta,\cdot,\cdot)d\theta+\int_t^{\zeta_\tau} dA^\mu_\theta\right)=\infty,\zeta_\tau<T_\tau\right)\\
&=P'_{s,x}\left(E'_{s,x}\left(|\varphi(\mathbf{X}_{T_\tau})|\mathbf{1}_{\{\zeta>T_\tau\}}+\int_t^{\zeta_\tau} f(\mathbf{X}_\theta,\cdot,\cdot)d\theta+\int_t^{\zeta_\tau} dA^\mu_\theta|\mathscr{F}'_t\right)=\infty,\zeta_\tau<T_\tau\right)\\
&=P'_{s,x}\left(\varphi(\mathbf{X}_{T_\tau})\mathbf{1}_{\{\zeta>T_\tau\}}+E'_{s,x}\int_t^{\zeta_\tau} f(\mathbf{X}_\theta,\cdot,\cdot)d\theta|\mathscr{F}'_t+E'_{s,x}\int_t^{\zeta_\tau} dA^\mu_\theta|\mathscr{F}'_t=\infty,\zeta_\tau<T_\tau\right)\\
&=P_{s,x}\left(\varphi(X_{T_\tau})\mathbf{1}_{\{\zeta^s>T_\tau\}}+E_{s,x}\int_t^{\zeta^s} f(X_\theta,\cdot,\cdot)d\theta|\mathscr{F}_t+E_{s,x}\int_t^{\zeta^s} dA^\mu_\theta|\mathscr{F}_t=\infty,\zeta^s<T_\tau\right)
\end{aligned}$$

which by assumption equals zero for a.e $(s,x)\in E_T$. Thus, cap$(k)=0$ for any compact subset k of $(w=\infty)$. Since cap is the choquet capacity, it follows that cap$(\{w=0\})=0$ i.e cap$(E_T|F)=0$ which implies that in the case of Markov-type equations, (B1)-(B4) are analogous to (A1)-(A4).
For the second assertion:
Assume that
$r\equiv\|f(\cdot,0)\|_\infty+\|\xi_\tau\|_\infty+\||A|_\sigma\|_\infty<\infty$, where ξ and $\sup_t|f(t,0)|$ are bounded random variables. Let θ_r, be a smooth function such that $0\leq\theta_r\leq 1$, $\theta_r(y)=1$ for $|y|\leq r$ and $\theta_r(y)=0$ for $|y|\geq r+1$.

For $k\in\mathbb{N}$, let $T_n(y)=\dfrac{ny}{n\vee|y|}$ and

$$h_n(t,y)=\theta_r(y)(f(t,y)-f(t,0))\frac{n}{\psi_{r+1}(t)\vee n}+f(t,0),$$

$$\psi_r(t):=\sup_{|y|\leq r}|f(t,y)-f(t,0)|.$$

This function still satisfies condition (A2) but with a positive constant.
Choosing y and y' in $\mathbb{R}^d$, if $|y|>r+1$ and $|y'|>r+1$, the inequality is trivially satisfied and reduces to the case where $|y'|\leq r+1$. Thus,

$$\langle y-y',h_n(t,y)-h_n(t,y')\rangle=\theta_r(y)\frac{n}{\psi_{r+1}(t)\vee n}\langle y-y',f(t,y)-f(t,y')\rangle\frac{n}{\psi_{r+1}(t)\vee n}\left(\theta_r(y)\theta_r(y')\right)\langle y-y',f(t,y)-f(t,0)\rangle \tag{3.4}$$

The first term of the right-hand side of (3.4) is negative since the condition (A2) is in force for f with $\mu=0$. For the second term, one can use the fact that θ_r is $C(r)$ Lipschitz since $|y'|\leq r+1$

$$\left(\theta_r(y)-\theta_r(y')\right)\langle y-y',[f(t,y)-f(t,0)]\rangle\leq C(r)|y-y'|^2|f(t,y')-f(t,0)|\leq C(r)(\lambda_n+\psi_{r+1}(t))|y-y'|^2 \tag{3.5}$$

so that

$$\frac{n}{\psi_{r+1}(t)\vee n}\left(\theta_r(y)-\theta_r(y')\right)\langle y-y',[f(t,y)-f(t,0)]\rangle\leq C(r)(\lambda+1)n|y-y'|^2 \tag{3.6}$$

Hence, for each $n \in \mathbb{N}$, the BSDE associated with (ξ, h_n) has a unique solution (Y^n, Z^n) in the space $\mathscr{D}^2 \otimes \mathscr{M}^2$. Since $\langle y, h_n(t,y)\rangle \leq |y|\|f(t,0)\|_\infty + \lambda|y|$ and ξ are bounded. Result 2.22 shows that the process satisfies the inequality

$$\|Y^n\|_\infty \leq r \tag{3.7}$$

In addition, from Result 2.21

$$\|Z^n\|_{\mathscr{M}^2} \leq r' \tag{3.8}$$

where r' is another constant. As a byproduct (Y^n, Z^n) is a solution to BSDE associated to $(\xi^n, f_n + dA^n)$ where $\xi^n = T_n(\xi)$,

$$f_n(t,y) = f(t,y) - f(t,0) + T_n(f(t,0)) \qquad \text{and} \qquad A^n_t = \int_0^t \mathbf{1}_{\{|A|_s \leq n\}} dA_s$$

for this function (A2) is satisfied with $\mu = 0$.
For $i \in \mathbb{N}$, let $\bar{Y} = Y^{n+i} - Y^n, \bar{Z} = Z^{n+i} - Z^n, \bar{A} = A^{n+i} - A^n$, using the assumption of (A2) and Lipschitz continuity on f_{n+i} yields

$$e^{2\lambda^2 t}|\bar{Y}_t|^2 + \int_t^\sigma e^{2\lambda^2 s}|\bar{Z}_s|^2 ds \leq 2\int_t^\sigma e^{2\lambda^2 s}\left\langle \bar{Y}_s, f_{n+i}(s, Y^{n+i}_s) - f_n(s, Y^n_s)\right\rangle ds + 2\int_t^\sigma e^{2\lambda^2 s}\langle \bar{Y}_s, d\bar{A}_s\rangle - 2\int_t^\sigma e^{2\lambda^2 s}\langle \bar{Y}_s, \bar{Z}_s dB_s\rangle \tag{3.9}$$

But $\|\bar{Y}\|_\infty \leq 2r$ since $\|Y\|_\infty \leq r$, so that

$$e^{2\lambda^2 t}|\bar{Y}_t|^2 + \int_t^\sigma e^{2\lambda^2 s}|\bar{Z}_s|^2 ds \leq 4r\int_t^\sigma e^{2\lambda^2 s}|f_{n+i}(s, Y^{n+i}_s) - f_n(s, Y^n_s)|ds + 2\int_t^\sigma e^{2\lambda^2 s}\langle \bar{Y}_s, d\bar{A}_s\rangle - 2\int_t^\sigma e^{2\lambda^2 s}\langle \bar{Y}_s, \bar{Z}_s dB_s\rangle \tag{3.10}$$

Using Burkholder-Davis-Gundy inequality [41], hence for a constant c_p

$$c_p E\left[\int_t^\sigma \langle \bar{Y}_s, \bar{Z}_s dB_s\rangle\right] \leq d_p E\left[\int_t^\sigma |\bar{Y}_s|\|\bar{Z}_s\|^2 ds\right] \leq d_p E\left[\sup_{s\in[t,\sigma]} \bar{Y}_t \left(\int_t^\sigma \|\bar{Z}\|^2 ds\right)\right]$$

and thus

$$c_p E\left[\int_t^\sigma \langle \bar{Y}_s, \bar{Z}_s dB_s\rangle\right] \leq \frac{d_p}{2} E\left[\sup_{s\in[t,\sigma]} \bar{Y}_t\right] + \frac{d_p}{2} E\left[\int_t^\sigma \|\bar{Z}\|^2 ds\right] \tag{3.11}$$

where c_p and d_p are constants.
From conditioning of (3.10) and (3.11),

$$E\left[\sup_{s\in[t,\sigma]} \bar{Y}_t\right] + E\left[\int_t^\sigma \|\bar{Z}\|^2 ds\right] \leq 4rE\left[\int_t^\sigma |f_{n+i}(s, Y^{n+i}_s) - f_n(s, Y^n_s)|ds\right] + 4rE\left[\int_t^\sigma d\bar{A}_s\right] - 2\left(\frac{d_p}{2} E\left[\sup_{s\in[t,\sigma]} \bar{Y}_t\right] + \frac{d_p}{2} E\left[\int_t^\sigma \|\bar{Z}\|^2 ds\right]\right)$$

Further simplification yields

$$(1+d_p)\left(E\left[\sup_{s\in[t,\sigma]} \bar{Y}_t\right] + E\left[\int_t^\sigma \|\bar{Z}\|^2 ds\right]\right) \leq 4r\left(E\left[\int_t^\sigma |f_{n+i}(s, Y^{n+i}_s) - f_n(s, Y^n_s)|ds\right] + E\left[\int_t^\sigma d\bar{A}_s\right]\right)$$

so that,

$$E\left[\sup_{s\in[t,\sigma]} \bar{Y}_t\right] + E\left[\int_t^\sigma \|\bar{Z}\|^2 ds\right] \leq \frac{4}{(1+d_p)} r\left(E\left[\int_t^\sigma |f_{n+i}(s, Y^{n+i}_s) - f_n(s, Y^n_s)|ds\right] + E\left[\int_t^\sigma d\bar{A}_s\right]\right)$$

and then,

$$E\left[\sup_{s\in[t,\sigma]} \bar{Y}_t\right] + E\left[\int_t^\sigma \|\bar{Z}\|^2 ds\right] \leq Cr\left(E\left[\int_t^\sigma |f_{n+i}(s, Y^{n+i}_s) - f_n(s, Y^n_s)|ds\right] + E\left[\int_t^\sigma d\bar{A}_s\right]\right) \tag{3.12}$$

where $C = \dfrac{4}{(1+d_p)}$. The right-hand side of the inequality (3.12) tends to 0 as $n \to \infty$, hence, there is a Cauchy sequence in $\mathscr{D}^2 \otimes \mathscr{M}^2$ and the limit is a solution to the BSDE i.e $Y^n_t \to Y_t$ and $\int_t^\sigma Z^n_s dB_s \to \int_t^\sigma Z_s dB_s$.
Next is to show that $(Y,Z) \in \mathscr{D}^q \otimes \mathscr{M}^q$ assuming $\mu \leq 0$. From the previous step, it is shown that the BSDE has a unique solution in the space $\mathscr{D}^2 \otimes \mathscr{M}^2$.
For $m > n$, let $\delta Y = Y^m - Y^n, \delta Z = Z^m - Z^n, \delta\xi = \xi^m - \xi^n$ and $\tau_k = \inf\left\{t \geq 0; \int_0^t |\delta Z_s|^2 ds > k\right\}$.

By the Itô-Tanaka formula (Lemma 2.1), for $t \geq 0$,

$$\begin{aligned} |\delta Y_{t\wedge\tau_k}| &\leq |\delta Y_{\tau_k\wedge\sigma}| + \int_t^{\tau_k\wedge\sigma} \langle \delta\hat{Y}_s, f_m(s, Y^m_s) - f_n(s, Y^n_s)\rangle ds + \int_t^{\tau_k\wedge\sigma} \langle \delta\hat{Y_{s-}}, d(A^m_s - A^n_s)\rangle + \int_t^{\tau_k\wedge\sigma} \langle \delta\hat{Y}_s, \delta\hat{Z}_s dB_s\rangle \\ &\leq |\delta Y_{\tau_k\wedge\sigma}| + \int_t^{\tau_k\wedge\sigma} |f_m(s, Y^m_s) - f_n(s, Y^n_s)|ds + \int_t^{\tau_k\wedge\sigma} d|A^m - A^n|_s + \int_t^{\tau_k\wedge\sigma} \langle \delta\hat{Y}_s, \delta\hat{Z}_s dB_s\rangle \end{aligned}$$

and conditioning with respect to $\mathscr{F}_t$ yields

$$|\delta Y_{t\wedge\tau_k}| \leq E\left(Y_{\tau_k\wedge\sigma}| + \int_t^{\tau_k\wedge\sigma} |f_m(s,Y_s^m) - f_n(s,Y_s^n)|ds + \int_t^{\tau_k\wedge\sigma} d|A^m - A^n|_s|\mathscr{F}_t\right) \tag{3.13}$$

The process δY is continuous and belongs to a class (D). It follows that $P_{s,x}$-a.s, $\delta Y_{\tau_k} = Y_{\tau_k\wedge\sigma} \to Y_\sigma = 0$ as $k\to\infty$ and this convergence holds in L^1. As a byproduct, it is deduced that $E(\delta Y_{\tau_k}|\mathscr{F}_t)$ converges to 0 in the ucp, so that (3.13) becomes

$$|\delta Y_t| \leq E\left(\int_0^{\tau_k\wedge\sigma} |f_m(s,Y_s^m) - f_n(s,Y_s^n)|ds|\mathscr{F}_t\right) + E\left(\int_0^{\tau_k\wedge\sigma} d|A^m - A^n|_s|\mathscr{F}_t\right) \tag{3.14}$$

from which the following inequality is derived

$$|\delta Y_t| \leq E\left(|\xi|\mathbf{1}_{\{|\xi|>n\}} + \int_0^{\sigma} |f(s,0)|\mathbf{1}_{\{|f(s,0)|>n\}}ds + \int_0^{\sigma} \mathbf{1}_{\{|A|_s>n\}}d|A|_s|\mathscr{F}_t\right) \tag{3.15}$$

We have from Result 2.20 that

$$E\left[\sup_{s\in[t,\sigma]} |\delta Y_t|^q\right] \leq \frac{1}{1-q}E\left[|\xi|\mathbf{1}_{\{|\xi|>n\}} + \int_0^{\sigma} |f(s,0)|\mathbf{1}_{\{|f(s,0)|>n\}}ds + \int_0^{\sigma} \mathbf{1}_{\{|A|_s>n\}}d|A|_s\right]^q \tag{3.16}$$

Therefore, there exists Y of class(D) and belongs to $\mathscr{D}^q$ for each $q\in(0,1)$.
$Y^n \to Y$ in the norm $\|\cdot\|_1$ and in $\mathscr{D}^q$ for each $q\in(0,1)$, thus $(Y^n)_{\mathbb{N}}$ is a Cauchy sequence.
Now, $(\delta Y, \delta Z)$ solves the following BSDE

$$\delta Y_t = \xi^m - \xi^n + \int_t^{\sigma} F(s,\delta Y_s)ds + \int_0^{\sigma} dA_s - \int_0^{\sigma} dZ_s dB_s \tag{3.17}$$

where F stands for the random function

$$F(t,y) = f_m(t,Y_t^m) - f_n(t,Y_t^n) \tag{3.18}$$

since f_m is monotone, F satisfy the inequality

$$\langle y, F(t,y)\rangle \leq |y||f(t,0)|\mathbf{1}_{|f(t,0)|>n} \tag{3.19}$$

Thus, using Lebesgue Dominated convergence theorem, it can be deduced that for $q\in(0,1)$

$$E\left[\left(\int_0^{\sigma} |\delta Z_s|^2 ds\right)^{q/2}\right] \leq C_q E\left[\sup_t |\delta Y_t|^q + \left(\int_0^{\sigma} |f(s,0)|\mathbf{1}_{|f(s,0)|>n}ds\right)^q + \left(\int_0^{\sigma} \mathbf{1}_{\{|A|_{s>n}\}}d|A|_s\right)^q\right] \tag{3.20}$$

It follows that for each $q\in(0,1)$, $(Z^k)_k$ is a Cauchy sequence in $\mathscr{M}^q$ such that for every $q\in(0,1)$ and $t>0$

$$E\left[\int_0^{\sigma} (Z_s^n - Z_s)\,dB_s\right]^{q/2} \to 0 \qquad \text{as } n\to\infty \text{ in ucp} \tag{3.21}$$

and since the map $y\mapsto f(t,y)$ is continuous, by taking a limit in ucp that (Y,Z) solves the correct BSDE.
Let us consider (Y,Z) and (Y',Z') to be two solutions of BSDE$(\xi,\sigma,f+dA)$ such that Y,Y' are of class(D). Then $(\bar{Y},\bar{Z}) = (Y-Y', Z-Z')$ is a solution of the BSDE

$$\bar{Y}_t = \int_t^{\sigma} (f(s,Y_s) - f(s,Y_s'))ds - \int_t^{\sigma} \bar{Z}_s dB_s, \qquad t\geq 0 \tag{3.22}$$

Taking the conditional expectation of (3.10) and using (A2),

$$E\left[\sup_{s\in[t,\sigma]} \bar{Y}_s\right] + \left[\int_t^{\sigma} \|\bar{Z}_s\|^2 ds\right] \leq 0 \tag{3.23}$$

Thus, $(Y,Z) = (Y',Z')$.
Hence, it is shown that (Y,Z) is a unique solution to BSDE$(\xi,\sigma,f+dA)$ □

Before the prove of the existence of solutions to the system (2.9), there is a need to address the regularity of the function E_T such that $(s,x)\in E_T \mapsto u(s,x) = E'_{s,x}Y_0^{s,x}$.

3.2. Regularity of the Function E_T

Proposition 3.2. *Let F be a Borel subset of E_T such that $cap(E_T|F)=0$. Assume that for every $(s,x)\in F$, the real process $Y^{s,x}$ is continuous semimartingale under $P'_{s,x}$ such that $Y^{s,x}_{t\vee\zeta}=0$, $t\geq 0$ and there exist a Borel function v on E_T such that for every $(s,x)\in F$ and every $t\in[0,T_\tau]$*

$$v(\mathbf{X}_t)=Y^{s,x}_t,\quad P'_{s,x}\text{-a.s} \tag{3.24}$$

Thus, $u(s,x)=E'_{s,x}Y^{s,x}_0$ is a quasi-continuous version of v and for every $(s,x)\in F$

$$u(\mathbf{X}_t)=Y^{s,x}_t,\qquad t\in[0,\zeta_\tau],\quad P'_{s,x}\text{-a.s} \tag{3.25}$$

Proof. let $(s,x)\in F$. Since $Y^{s,x}$ is a continuous semimartingale there exists a finite variation continuous processes $R^{s,x}$ and $Z^{s,x}\in M$ such that

$$Y^{s,x}_t=1_{\{\zeta>T_\tau\}}v(X_{T_\tau})+\int_t^{\zeta_\tau}dR^{s,x}_\theta-\int_t^{\zeta_\tau}Z^{s,x}_\theta dB_\theta,\quad t\in[0,\zeta_\tau]\quad P'_{s,x}\text{-a.s} \tag{3.26}$$

Let $L^n=Y^{s,x}-\frac{1}{n}<Y^{s,x}=U$ i.e $L^n<U$.
By comparison principle, as seen in Theorem 1.3 of [40],

$$Y^n_t\leq Y^{n+1}_t,\qquad t\in[0,\zeta_\tau],\ P'_{s,x}\text{-a.s for every } n\geq 1,\ \zeta^n\leq\zeta^{n+1}\ P'_{s,x} \text{ and } f^n(t,Y^n_t,Z^n_t)\leq f^{n+1}(t,Y^{n+1}_t,Z^{n+1}_t)\,dt\otimes dP \text{ a.e.} \tag{3.27}$$

Therefore, $Y_t=\sup_{n\geq 1}Y^n_t$
which implies

$$Y^n_t\nearrow Y_t,\qquad t\in[0,\zeta_\tau],\ P'_{s,x}\text{-a.s} \tag{3.28}$$

Furthermore, if $Y^n_t=\sup_{l\leq 1}Y^{n,k}_t$ then,

$$Y^{n,k}_t\nearrow Y^n_t,\qquad t\in[0,\zeta_\tau],\ P'_{s,x}\text{-a.s} \tag{3.29}$$

Note that $f_k(t,y)=k(y-L^n_t)^-$ is the generator of the BSDE for $Y^{n,k}_t$
Let $f_{k,l}(t,y)=k(y-L^n_t)^--l(y-U_t)^+$ be the generator of the BSDE for $Y^{n,k,l}_t$. Since $Y^{s,x}-\dfrac{1}{n}<Y^{s,x}$ then $f_{k,l}(t,y)<f_k(t,y)$,
so that $Y^{n,k}_t=\inf\limits_{l\geq 1}Y^{n,k,l}$,
Hence,

$$Y^{n,k,l}_t\searrow Y^{n,k}_t,\qquad t\in[0,\zeta_\tau],\ P'_{s,x}\text{-a.s} \tag{3.30}$$

By (3.24)

$$f_{k,l}(.y)=k\left(y-v(\mathbf{X})+\frac{1}{n}\right)^--l(y-v(\mathbf{X}))^+,\ dt\otimes P'_{s,x}\text{-a.e on } [0,\zeta_\tau]\times\Omega \text{ for every } y\in\mathbb{R} \tag{3.31}$$

Let $h_{k,l}(t,x,y)=k\left(y-v(x)+\frac{1}{n}\right)^--l(y-v(x))^+$. So that,$Y^{n,k,l}_t=h_{k,l}(\mathbf{X_t})$, $t\in[0,\zeta_\tau]$, $P'_{s,x}$-a.s, where $h_{k,l}$ is a quasi-continuous version of the solution of PDE$(0,h_{k,l})$.
Let $h(s,x)=\limsup\limits_{k\to\infty}\liminf\limits_{l\to\infty}h_{k,l}(s,x)$, $(s,x)\in E_T$.

Then by (3.29) and (3.30),

$$Y^n_t=h(\mathbf{X_t}),\qquad t\in[0,\zeta_\tau],\ P'_{s,x}\text{-a.s for every } (s,x)\in F. \tag{3.32}$$

In particular, since $Y^n\in\mathscr{S}^2$, the function h is quasi-continuous. From what has already been proved it follows that for every $n\geq 1$, there exists a quasi-continuous function u_n such that

$$Y^n_t=u_n(\mathbf{X}_t)\qquad t\in[0,\zeta_\tau],\ P'_{s,x}\text{-a.s for every } (s,x)\in F. \tag{3.33}$$

Let

$$u(s,x)=\limsup_{n\to\infty}u_n(s,x),\ (s,x)\in E_T \tag{3.34}$$

Hence, $Y^n_t\nearrow Y_t$ which is the desired result (3.28).
Assuming that $Y^{s,x}$ is bounded. By the Itô-Tanaka formula (Lemma 2.1), $T_k(Y)$ is a semimartingale, so by the first part of the proof, the function u_k defined as $u_k(s,x)=E'_{s,x}T_k(Y^{s,x}_0)$, $(s,x)\in E_T$, is continuous and for $k\in\mathbb{N}$

$$T_k(Y^{s,x}_t)=u_k(\mathbf{X}_t),\qquad t\in[0,\zeta_\tau],\ P'_{s,x}\text{-a.s for every } (s,x)\in F. \tag{3.35}$$

But $u_k(s,x) = E_{s,x}T_k(Y_0^{s,x}) = T_k(E_{s,x}Y_0^{s,x})$ and since $Y_0^{s,x}$ is constant $P'_{s,x}$-a.s. Therefore, letting $k \to \infty$ in (3.35), and using the fact that

$$T_k(Y_t^{s,x}) \to Y_t^{s,x} \qquad \text{in RM}$$

result to

$$Y_t^{s,x} = u(\mathbf{X}_t) \tag{3.36}$$

which is the desired result □

Proposition 3.3. *Let F be a Borel subset E_T such that $cap(E_T|F) = 0$ and let $u : E_T \to \mathbb{R}$ be a Borel function. If $u(\mathbf{X})$ is a continuous semimartingale under $P'_{s,x}$ on $[0,\zeta_\tau]$ for $(s,x) \in F$, then $u \in W^{0,1}(\mathbb{X}^{E_T})$ and there exists Continuous Additive Functional (CAF) of finite variation such that*

$$u(\mathbf{X}_t) = u(s,x) + \int_0^t dA_\theta + \int_0^t \sigma\nabla_{\mathbb{X}} u(\mathbf{X}_\theta)dB_\theta \qquad t \in [0,\zeta_\tau],\ P'_{s,x}\text{-a.s} \tag{3.37}$$

for every $(s,x) \in F$

Proof. Assume u is bounded. By [34] and (3.33), it is known that

$$Y_t^n = u_n(\mathbf{X}_t),\ t \in [0,\zeta_\tau], \qquad \sigma\nabla_{\mathbb{X}} u_n(\mathbf{X}) = Z^n,\ dt \otimes P'_{s,x}\text{-a.e} \tag{3.38}$$

Since $Y = u(\mathbf{X})$ is a continuous semimartingale and the basic filtration is Brownian, there exists a finite variation continuous process $R^{s,x}$ and process $Z^{s,x} \in M$ such that

$$Y_t = Y_0 + \int_0^t dR_\theta^{s,x} + \int_0^t Z_\theta^{s,x} dB_\theta, \qquad t \in [0,\zeta_\tau] \tag{3.39}$$

Since the process Y is continuous from the proof of Proposition 3.2 and Dini's theorem, it follows that

$$P'_{s,x}\left(\sup_{t\in[0,\zeta_\tau]} |Y_t^n - Y_t|^2 > \varepsilon\right) \to 0 \tag{3.40}$$

Moreover, following Proposition (6.1) in [34]

$$dR^{n,+} \le \mathbf{1}_{\{Y_t^n = L_t^n\}} dR_t^+, \qquad dR^{n,-} \le \mathbf{1}_{\{Y_t^n = U_t^n\}} dR_t^- \tag{3.41}$$

Therefore, there exists a sequence of stopping times $\{\tau_k\}$ such that $\tau_k \le \tau_{k+1}$, $k \ge 1$, $\tau_k \to \zeta_\tau, P'_{s,x}$-a.s for q.e $(s,x) \in E_T$ and for every $k \ge 1$, the sequence $\{Y^{n,\tau_k}\}$. Therefore,

$$P'_{s,x}\left(\left\langle \int_0^t \left(Z_\theta^{n,s,x} - Z_\theta^{s,x}\right) dB_\theta \right\rangle_{\zeta_\tau}\right) > \varepsilon \to 0 \tag{3.42}$$

From (3.38)−(3.42), $u \in W^{0,1}(\mathbb{X}^{E_T})$ i.e $Y_t = u(\mathbf{X}_t)$ and $\sigma\nabla_{\mathbb{X}} u(\mathbb{X}^{E_T}) = Z^{s,x}, dt \otimes P'_{s,x}$-a.e
$u(s,x) = E'_{s,x}$ putting

$$A_t = u(\mathbb{X}_t) - u(s,x) - \int_0^t \sigma\nabla_{\mathbb{X}} u(\mathbb{X}_\theta)dB_\theta \tag{3.43}$$

as in (2.24). Then, (3.43) becomes

$$u(\mathbf{X}_t) = u(s,x) + A_t + \int_0^t \sigma\nabla_{\mathbb{X}} u(\mathbf{X}_\theta)dB_\theta$$

Since $u \in W^{0,1}(\mathbb{X}^{E_T})$, it implies that

$$Y_t = Y_0 + A_t + Z_\theta^{s,x} dB_\theta, \qquad t \in [0,\zeta_\tau] \tag{3.44}$$

Comparing (3.39) and (3.44), A is a Continuous Additive Function (CAF) of finite variation and $P'_{s,x}(R_t^{s,x} = L_t, t \in [0,\zeta_\tau]) = 1$ which proves the proposition in the case u is bounded. In the general case, (3.39) still holds.
By the Itô-Tanaka formula (Lemma 2.1), for every $k > 0$

$$T_k(u)(\mathbf{X}_t) = T_k(u(s,x)) + \int_0^t \mathbf{1}_{(-k,k]}(u(\mathbf{X}_\theta))dR_\theta^{s,x} + \int_0^t \mathbf{1}_{(-k,k]}(u(\mathbf{X}_\theta))Z_\theta^{s,x}dB_\theta + \frac{1}{2}\left(L_t^k - L_t^{-k}\right), \qquad t \in [0,\zeta_\tau] \tag{3.45}$$

where L^k (respectively, L^{-k}) is the local time of the process $u(\mathbf{X})$ at k (respectively,$-k$). Also, by (2.10)

$$T_k(u)(\mathbf{X}_t) = T_k(u(s,x)) + \int_0^t \mathbf{1}_{(-k,k]}(u(\mathbf{X}_\theta))dA_\theta^{s,x} + \int_0^t \mathbf{1}_{(-k,k]}\sigma\nabla_{\mathbb{X}} T_k(u(\mathbf{X}_\theta))dB_\theta \tag{3.46}$$

Comparing (3.45) and (3.46), it is observed that

$$Z^{s,x}\mathbf{1}_{(-k,k]}(u(\mathbf{X})) = \mathbf{1}_{(-k,k]}\sigma\nabla_{\mathbb{X}}(T_k(u))(\mathbf{X}),\ dt\otimes P'_{s,x} - \text{a.e} \tag{3.47}$$

Hence, by Result 2.25,

$$Z^{s,x}\mathbf{1}_{(-k,k]}(u(\mathbf{X})) = \mathbf{1}_{(-k,k]}(u(\mathbf{X}))\sigma\nabla_{\mathbb{X}}u(\mathbf{X}),\ dt\otimes P'_{s,x} - \text{a.e for every } k\geq 0, \tag{3.48}$$

further simplification yields

$$Z^{s,x} = \sigma\nabla_{\mathbb{X}}u(\mathbf{X}),\qquad dt\otimes P'_{s,x} - \text{a.e} \tag{3.49}$$

Since, $T_k(u)\in W^{0,1}(\mathbb{X}^{E_T})$ for every $k\geq 0$, u is quasi-continuous and $u\in$RB, there is a need to show that $u\in W^{0,1}\mathbb{X}^{E_T}$.
For q.e $(s,x)\in E_T$ and $\varepsilon>0$,

$$P'_{s,x}\left(\int_0^{\zeta_\tau}|T_k(u)-u|^2(\mathbf{X}_\theta)d\theta>\varepsilon\right)\leq P'_{s,x}\left(\int_0^{\zeta_\tau}|u|^2\mathbf{1}_{\{|u|>k\}}d\theta>\varepsilon\right)\to 0 \tag{3.50}$$

which shows by that $T_k(u)\to u$ in RM. By Result 2.25, for $k<l$,

$$\begin{aligned}
P'_{s,x}\left(\int_0^{\zeta_\tau}\nabla_{\mathbb{X}}|T_k(u)-T_l(u)|^2(\mathbf{X}_\theta)d\theta>\varepsilon\right) &= P'_{s,x}\left(\int_0^{\zeta_\tau}\nabla_{\mathbb{X}}|T_k(T_l(u))-T_l(u)|^2(\mathbf{X}_\theta)d\theta>\varepsilon\right)\\
&= P'_{s,x}\left(\int_0^{\zeta_\tau}\nabla_{\mathbb{X}}|T_l(u)\mathbf{1}_{\{|u|>k\}}-T_l(u)|^2(\mathbf{X}_\theta)d\theta>\varepsilon\right)\\
&= P'_{s,x}\left(\int_0^{\zeta_\tau}\nabla_{\mathbb{X}}|T_l(u)|^2\mathbf{1}_{\{|u|>k\}}(\mathbf{X}_\theta)d\theta>\varepsilon\right)\\
&\leq P'_{s,x}\left(\text{ess sup}_{\theta\in[0,\zeta_\tau]}|u(\mathbf{X}_\theta)|\geq k\right)
\end{aligned} \tag{3.51}$$

By the assumption that $u\in$RB, the right-hand side of the above inequality (3.51) tend to zero as $k\to\infty$ which shows that $\nabla_{\mathbb{X}}T_k(u)\to\nabla_{\mathbb{X}}T_l(u)$ in RM as $k,l\to\infty$. Consequently, $u\in W^{0,1}\mathbb{X}^{E_T}$ □

3.3. Existence and Uniqueness of Solution

This section considers the existence and the uniqueness of the system (2.9)

Theorem 3.4. *Assume (B1)-(B4). Then, there exists a unique solution* (2.9)*. Moreover, there exists a version of u (still denoted by u) such that*

$$u(\mathbf{X}_t) = \mathbf{1}_{\{\zeta>T_\tau\}}\varphi(\mathbf{X}_{T_\tau}) + \int_t^{\zeta_\tau} f(\mathbf{X}_\theta,u(\mathbf{X}_\theta))d\theta + \int_t^{\zeta_\tau} dA^\mu_\theta - \int_t^{\zeta_\tau}\sigma\nabla_{\mathbb{X}}u(\mathbf{X}_\theta)dB_\theta,\quad t\in[0,\zeta_\tau],\ P'_{s,x}\text{-a.s} \tag{3.52}$$

is satisfied for every $(s,x)\in F$

Proof ***(Existence Result)***. By Proposition 3.1, for every $(s,x\in F)$, there exists a solution $(Y^{s,x},Z^{s,x})$ of BSDE$_{s,x}(\varphi,E,f+d\mu)$ such that $(Y^{s,x},Z^{s,x})\in D^q\otimes M^q$ for $q\in(0,1)$ and $Y^{s,x}$ is of class (D).
Using Markov property,

$$\begin{aligned}
u(\mathbf{X}_t) &= E'_{\tau,X_{T_\tau}}Y_0^{s,x}\\
&= E'_{s,x}(Y_t^{s,x}\circ\theta_t|F_t)\\
&= E_{s,x}(Y_t^{s,x}|F_t)\\
&= Y_t^{s,x}
\end{aligned}$$

so, $u(\mathbf{X}_t)=Y_t^{s,x}, P'_{s,x}$-a.s for every $(s,x)\in F$ and every $t\in[0,t_\tau]$, where $u(s,x)=E'_{s,x}Y_0^{s,x}$
Consider a solution $(Y^{n,s,x},Z^{n,s,x})\in\mathscr{D}^q\otimes\mathscr{M}^q$, $q\in(0,1)$ of the BSDE$_{s,x}(\varphi,E,f+d\mu)$ such that $Y^{n,s,x}$ is of class (D) and $u_n(s,x)=E'_{s,x}Y_0^{n,s,x}$ and for every $(s,x)\in F$,

$$u_n(\mathbf{X}_t) = Y_t^{n,s,x},\ t\in[0,\zeta_\tau],\ P'_{s,x}\text{-a.s},\qquad \sigma\nabla u_n(\mathbf{X}) = Z^{n,s,x}\ dt\otimes P'_{s,x}\text{-a.s} \tag{3.53}$$

Applying Proposition 3.2 to each coordinate of the process $Y^{s,x}-Y^{n,s,x}$, there is a quasi-continuous function $v:E_T\to\mathbb{R}^N$ such that for every $(s,x)\in F$

$$Y_t^{s,x}-Y_t^{n,s,x} = v(\mathbf{X}_t)\qquad t\in[0,\zeta_\tau],P'_{s,x}\text{-a.s} \tag{3.54}$$

From (3.53) and (3.54),

$$Y_t^{s,x} = u_n(\mathbf{X}_t) + v(\mathbf{X}_t) \qquad t \in [0,\zeta_\tau], P'_{s,x}\text{-a.s} \tag{3.55}$$

But

$$v(s,x) = E'_{s,x}Y_0^{s,x} - E'_{s,x}Y_0^{n,s,x} = u(s,x) - u_n(s,x) \text{ for q.e } (s,x) \in E_T.$$

Therefore from (3.55),

$$Y_t^{s,x} = u_n(\mathbf{X}_t) + u(\mathbf{X}_t) - u_n(\mathbf{X}_t) \qquad t \in [0,\zeta_\tau], P'_{s,x}\text{-a.s}$$

Hence,

$$Y_t^{s,x} = u(\mathbf{X}_t) \qquad t \in [0,\zeta_\tau], P'_{s,x}\text{-a.s} \tag{3.56}$$

It follows that u is quasi-cádlág and belongs to RB.
Let $\bar{u}(s,x) = \lim_{n\to\infty} u_n(s,x)$, $(s,x) \in E_T$ and

$$\bar{u}(\mathbf{X}_t) = \bar{Y}_t^{s,x},\ t \in [0,\zeta_\tau],\ P'_{s,x}\text{-a.s for q.e } (s,x) \in E_T. \tag{3.57}$$

We have that $\bar{u}$ is quasi-cádlág, such that $\bar{u}$ belongs to $\mathscr{T}_2^{0,1}$ i.e $T_k(\bar{u}) \to \bar{u}$ in RM. Since $\bar{u}$ is quasi-cádlág, it belongs to RB.
Following Result 2.27, $\bar{u} \in W^{0,1}(\mathbb{X}^{E_T})$ and $u = \bar{u}$ q.e. Therefore, applying Proposition 3.3 to each coordinate of the function v, it follows that $u \in W^{0,1}(\mathbb{X}^{E_T})$ and $u(\mathbf{X}) = Z^{s,x}\, dt \otimes P'_{s,x}$-a.s for every $(s,x) \in F$. Also from Result 2.28, u is of class (RD). Thus, u is a solution to (2.9). □

Uniqueness Result. Assume that $\alpha \leq 0$. Let $(Y^{s,x},Z^{s,x}),(\bar{Y}^{s,x},\bar{Z}^{s,x})$ be the solutions of BSDE$_{s,x}(\varphi,E,f+d\mu)$ such that $Y^{s,x}$ and $\bar{Y}^{s,x}$ are of class (D). Then $(\vec{Y}^{s,x},\vec{Z}^{s,x}) = (Y^{s,x}-\bar{Y}^{s,x},Z^{s,x}-\bar{Z}^{s,x})$ is a solution of BSDE

$$\vec{Y}_t^{s,x} = \int_t^{\zeta_\tau} (f(\mathbf{X}_\theta,Y_\theta^{s,x}) - f(\mathbf{X}_\theta,\bar{Y}_\theta^{s,x}))d\theta - \int_t^{\zeta_\tau} \vec{Z}_\theta^{s,x} hB_\theta, \qquad t \geq 0 \tag{3.58}$$

Assume, $\sigma_k = \inf\{t \geq 0; \int_t^{\zeta_\tau} |\vec{Z}_\theta^{s,x}|^2 \geq k\}$, by the Itô-Meyer formula,

$$|\vec{Y}_t^{s,x}| \leq |\vec{Y}^{s,x}_{\sigma_k\wedge\zeta_\tau}| + \int_t^{\sigma_k\wedge\zeta_\tau} \left\langle f(\mathbf{X}_\theta,Y_\theta^{s,x}) - f(\mathbf{X}_\theta,\bar{Y}_\theta^{s,x}), \hat{\text{sgn}}(\vec{Y}_t^{s,x})\right\rangle - \int_t^{\sigma_k\wedge\zeta_\tau} \left\langle \hat{\text{sgn}}(\vec{Y}_\theta^{s,x}), \vec{Z}_\theta^{s,x} dB_\theta\right\rangle \tag{3.59}$$

By property B2 (under basic assumptions in subsection 2.3.1) and $\alpha \leq 0$

$$\int_t^{\sigma_k\wedge\zeta_\tau} \left\langle f(\mathbf{X}_\theta,Y_\theta^{s,x}) - f(\mathbf{X}_\theta,\bar{Y}_\theta^{s,x}), \hat{\text{sgn}}(\vec{Y}_t^{s,x})\right\rangle \to 0 \tag{3.60}$$

Hence (3.59) becomes

$$|\vec{Y}_t^{s,x}| \leq -\int_t^{\sigma_k\wedge\zeta_\tau} \left\langle \hat{\text{sgn}}(\vec{Y}_\theta^{s,x}), \vec{Z}^{s,x} dB_\theta\right\rangle, \quad t \geq 0. \tag{3.61}$$

Taking the conditional expectation with respect to F_t on both sides of (3.61)

$$|\vec{Y}_t^{s,x}| \leq E\left(-\int_t^{\sigma_k\wedge\zeta_\tau} \left\langle \hat{\text{sgn}}(\vec{Y}_\theta^{s,x}), \vec{Z}^{s,x} dB_\theta\right\rangle |F_t\right) \tag{3.62}$$

and then letting $k \to \infty$ and using the fact that $\vec{Y}^{s,x}$ is of class (D), therefore,

$$|\vec{Y}^{s,x}| = 0,\ t \geq 0 \quad \text{which implies } Y_t^{s,x} = \bar{Y}_t^{s,x} \tag{3.63}$$

□

Next is to show that u has weak derivatives in $L^q_{\text{loc}}(E_T)$

Proposition 3.5. *Let u be a solution of the system (2.9). Then $\nabla_{\mathbb{X}} u \in L^q_{loc}(E_T)$ for every $q \in (0,1)$.*

Proof. Since u is of class (RD),$u(\mathbf{X})$ is of class (D) on $[0,\zeta_\tau]$ under $P'_{s,x}$ for q.e $(s,x) \in E_T$. Therefore, $(u(\mathbf{X}),\sigma\nabla_{\mathbb{X}}u(\mathbf{X}))$ is a unique solution of BSDE$_{s,x}(\varphi,E,f+d\mu)$ and $u \in \mathscr{D}^q$ and $\nabla_{\mathbb{X}}u \in \mathscr{M}^q$ for $q \in (0,1)$. Applying the Itô-Tanaka (Lemma 2.1) to (3.52) and then apply (B2) and the fact that u is of class (RD) yields

$$|u(\mathbf{X}_t)| \leq E'_{s,x}\left(|\varphi(\mathbf{X}_{T_\tau})|\mathbf{1}_{\{\zeta_\tau > T_\tau\}} + \int_0^{\zeta_\tau} |f(\mathbf{X}_\theta,0)|d\theta + \int_0^{\zeta_\tau} d|A^\mu|_\theta\ |\mathscr{F}'_T\right) \tag{3.64}$$

By Burholder-Davis-Gundy inequality [41], it is obtained that for every $q \in (0,1)$,

$$E'_{s,x} \sup_{0\le t\le \zeta_\tau} |u(\mathbf{X}_t)|^q \le \frac{1}{1-q} E'_{s,x}\left(1+|\varphi(\mathbf{X}_{T_\tau})|\mathbf{1}_{\{\zeta_\tau > T_\tau\}} + \int_0^{\zeta_\tau} |f(\mathbf{X}_\theta,0)|d\theta + \int_0^{\zeta_\tau} d|A^\mu|_\theta\right)$$

$$E'_{s,x} \sup_{0\le t\le \zeta_\tau} |u(\mathbf{X}_t)|^q \le (1-q)^{-1} E'_{s,x}\left(1+|\varphi(\mathbf{X}_{T_\tau})|\mathbf{1}_{\{\zeta_\tau > T_\tau\}} + \int_0^{\zeta_\tau} |f(\mathbf{X}_\theta,0)|d\theta + \int_0^{\zeta_\tau} d|A^\mu|_\theta\right) \tag{3.65}$$

By the above estimate and Result 2.20

$$E'_{s,x}\left(\int_t^{\zeta_\tau} |\sigma\nabla_{\mathbb{X}} u(\mathbf{X}_\theta)|^2 d\theta\right)^{q/2} \le c(q) E'_{s,x}\left(1+|\varphi(\mathbf{X}_{T_\tau})|\mathbf{1}_{\{\zeta_\tau > T_\tau\}} + \int_0^{\zeta_\tau} |f(\mathbf{X}_\theta,0)|d\theta + \int_0^{\zeta_\tau} d|A^\mu|_\theta\right) \tag{3.66}$$

for every $q \in (0,1)$. Since $q \in (0,1)$, it is obtained that

$$\begin{aligned} E'_{s,x}\left(\int_0^{\zeta_\tau} |\sigma\nabla_{\mathbb{X}} u(X_\theta)|^2 d\theta\right)^{q/2} &\ge \wedge^{-1} E'_{s,x}\left(\int_0^{\zeta_\tau} |\nabla_{\mathbb{X}} u(X_\theta)|^q d\theta \cdot \zeta_\tau{}^{q/2-1}\right) \\ &\ge \wedge^{-1} T^{q/2-1} E'_{s,x}\left(\int_0^{\zeta_\tau} |\nabla_{\mathbb{X}} u(\mathbf{X}_\theta)|^q d\theta\right) \\ &= \wedge^{-1} T^{q/2-1} \iint_{E_T} |\nabla_{\mathbb{X}} u(\theta,y)|^q p_E(s,x,\theta,y) d\theta dy < \infty \end{aligned}$$

for q.e $(s,x) \in E_T$, where p_E is the transition density of the process $\mathbb{X}$ killed on existing E. $p_E(s,x,.,.)$ is continuous and strictly positive on $(s,T]\times E$ [39], hence, $\nabla_{\mathbb{X}} u \in L^q_{\text{loc}}(E_T)$. □

Remark 3.6. From [31], it follows that if u is a probabilistic solution of (2.9) such that $f(.,u) \in L^1(E_T)$, then, $u \in \mathscr{T}_2^{0,1}, u \in L^q(0,T;W_0^{1,q}(E))$ for $q \in \left[1, \frac{d+2}{d+1}\right)$ and u is a renormalized (entropy) solution of (2.9)

4. Conclusion

The existence of the solution of a semilinear parabolic system with measure data is considered in this paper. The methods of proof are those of stochastic analysis, the Markov process and mainly Backward Stochastic Differential Equations (BSDEs). The proof of the main result on existence and uniqueness of

$$u(\mathbf{X}_t) = \mathbf{1}_{\{\zeta > T_\tau\}}\varphi(\mathbf{X}_{T_\tau}) + \int_t^{\zeta_\tau} f(\mathbf{X}_\theta, u(\mathbf{X}_\theta))d\theta + \int_t^{\zeta_\tau} dA^\mu_\theta - \int_t^{\zeta_\tau} \sigma\nabla_{\mathbb{X}} u(\mathbf{X}_\theta)dB_\theta \tag{4.1}$$

is probabilistic. The approach made was to show that $\text{cap}(E_T\backslash F) = 0$ and for every $(s,x) \in F$, there exist $\{\mathscr{F}'_t\}$ progressively measurable processes consisting of an $\mathbb{R}^N$-valued process $Y^{s,x}$ and an $\mathbb{R}^d \times \mathbb{R}^N$-valued process $Z^{s,x}$ which are the solution of BSDE

$$Y_t^{s,x} = \mathbf{1}_{\{\zeta > T_\tau\}}\varphi(\mathbf{X}_{T_\tau}) + \int_t^{\zeta_\tau} f(\mathbf{X}_\theta, Y_\theta^{s,x}, Z_\theta^{s,x})d\theta + \int_t^{\zeta_\tau} dA^\mu_\theta - \int_t^{\zeta_\tau} Z_\theta^{s,x} dB_\theta, \quad t \in [0,\zeta_\tau], P'_{s,x}\text{-a.s} \tag{4.2}$$

such that $(Y^{s,x}, Z^{s,x} \in \mathscr{D}^q \otimes \mathscr{M}^q)$ and $Y^{s,x}$ is of class (D). Then $u(s,x) = E'_{s,x} Y_0^{s,x}$ is set for $(s,x) \in E_T$ and show that u is quasi-continuous. Using the Markov property, $u(\mathbf{X}_t) = Y_t^{s,x}, P'_{s,x}$-a.s for every $(s,x) \in F$ and $t \in [0,T_\tau]$, where F is a Borel subset of E_T such that $\text{cap}(E_T\backslash F) = 0$. Lastly, it was shown that u is quasi-càdlàg, belongs to $W^{0,1}(\mathbb{X}^{E_T})$ and the representation

$$Y_t^{s,x} = u(\mathbf{X}_t), t \in [0,\zeta_\tau], P'_{s,x}\text{-a.s} \quad , Z_t^{s,x} = \sigma\nabla_{\mathbb{X}} u(\mathbf{X}) \quad dt \otimes P'_{s,x}\text{-a.s} \tag{4.3}$$

holds quasi-every (q.e for short) with u having weak derivatives in $L^q_{\text{loc}}(E_T)$. The probabilistic solution u to the system is generally weak and could be considered as some generalization of the notion of the renormalized solution because if $f_u \subset L^1(E_T)$ then $u \in \mathscr{T}_2^{0,1}, u \in L^q(0,T;W^{1,q}(E))$ for $q \in \left[1, \frac{d+2}{d+1}\right)$ and u is a renormalized solution to the system as presented by [31]. The results were proved for systems with f satisfying conditions for which the usual monotonicity methods do not apply, it only requires f to satisfy a mild integrability condition and allow f to depend on x.

In the case of uniqueness, choosing any two solutions of Backward Stochastic Differential Equation (BSDE) which are of class (D), and taking the conditional expectation with respect to F_t, letting $k \to \infty$, it was shown that the solution is unique.

Hence, for a Cauchy-Dirichlet problem of a monotone semilinear parabolic system in divergence form with measure data, there exists a unique probabilistic solution of the system under a mild integrability condition on the data.

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