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The Monoid Rank and Monoid Presentation of Order-Preserving and Order-Decreasing Full Contraction Mappings

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Abstract
Let $n \in \mathbb{Z}^+$ and $X_n = \{1, 2, \ldots, n\}$ be a finite set. Let $\mathcal{O}DCT_n$ be the order-preserving and order-decreasing full contraction mappings on X_n. It is well known that $\mathcal{O}DCT_n$ is a monoid. In this paper, we have found the monoid rank and monoid presentation of $\mathcal{O}DCT_n$. In particular, we have proved that monoid rank of $\mathcal{O}DCT_n$ is $n-1$ for $n \in \mathbb{Z}^+$ and $< a_1, a_2, \ldots, a_{n-1} \mid a_i a_{n-1} = a_i \ \ (1 \leq i \leq n-1), a_i a_j = a_{j+1} a_i \ \ (1 \leq i \leq j \leq n-2) >$ is a monoid presentation of $\mathcal{O}DCT_n$ for $n \geq 3$.

Keywords: *Contraction mappings; Generating set; Monoid presentation.*

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1. Introduction

Let X be a non-empty set and let $\mathcal{T}_X$ be the full transformation semigroup on X. Every semigroup is isomorphic to a subsemigroup of full transformation semigroup [7]. So, the full transformation semigroup is ubiquitous in the semigroup theory. Let $n \in \mathbb{Z}^+$ and $X_n = \{1, 2, \ldots, n\}$ be a finite set. We use $\mathcal{T}_n$ instead of $\mathcal{T}_{X_n}$ for convenience.

Let M be a monoid and A be any subset of M. Then the submonoid of M by generated A (which is the smallest submonoid of M containing A) is denoted by $< A >$. If $< A >= M$ while the cardinality of A is a finite number, then M is called finitely generated monoid. With a similar idea, by replacing M by a semigroup S, one may define finitely generated semigroup as well.

The monoid rank of finitely generated monoid M is defined by

$$\text{rank}_M(M) = \min\{|A| :< A >= M\}.$$

Let $\mathcal{C}T_n$ be the full contraction transformations on X_n, it is defined by

$$\mathcal{C}T_n = \{\alpha \in \mathcal{T}_n \mid (\forall x, y \in X_n) \ \ |x\alpha - y\alpha| \leq x - y\}$$

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and $\mathcal{CT}_n$ is a submonoid of $\mathcal{T}_n$. Let $\mathcal{O}_n$ be the order-preserving full transformations on X_n and it is defined by

$$\mathcal{O}_n = \{\alpha \in \mathcal{T}_n \mid (\forall x, y \in X_n)\ x \leq y \implies x\alpha \leq y\alpha\}.$$

Let $\mathcal{S}_n$ be the symmetric group on X_n. Gomes and Howie have found the semigroup rank of $\mathcal{O}_n \setminus \mathcal{S}_n = \mathcal{O}_n \setminus \{1_S\}$ where 1_S is the identity mapping of $\mathcal{S}_n$ [5]. Let $\mathcal{C}_n$ be the order-preserving and order-decreasing transformations on X_n, it is called Catalan monoid on X_n and it is defined by

$$\mathcal{C}_n = \{\alpha \in \mathcal{O}_n \mid (\forall x \in X_n)\ x\alpha \leq x\}.$$

There are some papers about $\mathcal{C}_n$, in the literature such as [2, 6]. Adeshola and Umar defined a semigroup which is $\mathcal{O}_n \cap \mathcal{CT}_n$ and they used $\mathcal{OCT}_n$ instead of $\mathcal{O}_n \cap \mathcal{CT}_n$. The cardinalities of some equivalences on $\mathcal{OCT}_n$ has been investigated by Adeshola and Umar [1]. Let

$$\mathcal{D}_n = \{\alpha \in \mathcal{T}_n \mid (\forall x \in X_n)\ x\alpha \leq x\}$$

be the subsemigroup of $\mathcal{T}_n$ consisting of all order-decreasing transformations of X_n. Moreover, Adeshola and Umar defined a semigroup which is $\mathcal{OCT}_n \cap \mathcal{D}_n$ and they used $\mathcal{ODCT}_n$ instead of $\mathcal{OCT}_n \cap \mathcal{D}_n$ [1]. $\mathcal{ODCT}_n$ is called order-preserving and order-decreasing full contraction mappings. Also, $\mathcal{ODCT}_n = \mathcal{CT}_n \cap \mathcal{C}_n$ thus $\mathcal{ODCT}_n$ is a submonoid of $\mathcal{OCT}_n$ and submonoid of $\mathcal{C}_n$.

Let A be a set, then we denote by A^* the free monoid on A. Let $R \subseteq A^* \times A^*$ is a set of pairs of words. An element (r, s) of R is called a relation, and is usually written $r = s$ instead of (r, s). Monoid presentation is an ordered pair $< A \mid R >$ which is the quotient monoid $A^*/R^\#$ where $R^\#$ is the smallest congruence on A^* containing R. Let M be the monoid defined by $< A \mid R >$. Let $w_1, w_2 \in A^*$, if w_1 and w_2 are identical words on A^* then we write $w_1 \equiv w_2$, and we write $w_1 = w_2$ if they represent the same element of the monoid M, that is $(w_1, w_2) \in R^\#$. If $u_1 \equiv xry$ and $u_2 \equiv xsy$ where $x, y \in A^*$ and $(r, s) \in R$ or $(s, r) \in R$ then, we say u_2 is obtained from u_1 by an application of one relation from R. We say that $w_1 = w_2$ is a consequence of R, if w_1 and w_2 are identical words or if there exists a sequence $w_1 \equiv u_1 \to u_2 \to \ldots \to u_k \equiv w_2$ where each u_{i+1} is obtained from u_i $(1 \leq i \leq k-1)$ by an application of one relation from R. Let T be any monoid, let B be a generating set for T, and let $\phi : A \to B$ be an onto mapping. ϕ can be extended in a unique way $\bar{\phi} : A^* \to T$. The monoid T is said to satisfy relations R if for each $(u, v) \in R$ we have $u\bar{\phi} = v\bar{\phi}$. We refer the readers to two theses about semigroup and monoid presentations [3, 8].

2. Preliminaries

Let $\alpha \in \mathcal{T}_n$, then the kernel and image of α are defined by

$$\ker(\alpha) = \{(x, y) \in X_n \times X_n \mid x\alpha = y\alpha\}$$

$$\text{im}(\alpha) = \{x\alpha \mid x \in X_n\}.$$

Moreover, it is well known that if $\alpha, \beta \in \mathcal{T}_n$ then $\text{im}(\alpha\beta) \subseteq \text{im}(\beta)$ and $\ker(\alpha\beta) \supseteq \ker(\alpha)$.

Definition 2.1. Let A be a non-empty subset of X_n. If $x, y \in A$ and $x \leq z \leq y \implies z \in A$ for all $x, y \in A$, then A is called a convex subset of X_n.

If $\alpha \in \mathcal{T}_n$ is a contraction mapping then $\text{im}(\alpha)$ is a convex subset of X_n [4]. Thus if $\alpha \in \mathcal{ODCT}_n$ then $\text{im}(\alpha)$ is a convex subset of X_n. Moreover, from the definition of $\mathcal{ODCT}_n$ it is easy to see that if $\alpha \in \mathcal{ODCT}_n$ then $\text{im}(\alpha) = \{1, 2, \ldots, r\}$ for $1 \leq r \leq n$ and each equivalence kernel classes of α are convex subsets of X_n. Thus if $\alpha \in \mathcal{ODCT}_n$ then

$$\alpha = \begin{pmatrix} A_1 & A_2 & \ldots & A_r \\ 1 & 2 & \ldots & r \end{pmatrix}$$

for $1 \leq r \leq n$. Moreover, we have $x \geq i$ for $\forall x \in A_i$ and $\{A_1, A_2, \ldots A_r\}$ is a partition of X_n, if $a \in A_i$ and $b \in A_j$ for $1 \leq i < j \leq n$ then $a < b$.

3. The Monoid Rank of $ODCT_n$

In this section, we have found a minimal generating set of $\mathcal{O}DCT_n$ and we obtained the monoid rank of $\mathcal{O}DCT_n$. It is clear that $\mathcal{O}DCT_1 = \{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$ which is a clearly generated by empty set as a monoid and $\mathcal{O}DCT_2 = \{\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\}$ which is clearly generated by the element $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ as a monoid. Let $n \geq 3$ and $\mathcal{F}_r = \{\alpha \in \mathcal{O}DCT_n : |\mathrm{im}(\alpha)| = r\}$ for $1 \leq r \leq n$. Notice that $\mathcal{F}_n = \{\epsilon = \begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{pmatrix}\}$ where ϵ is the identity element of $\mathcal{O}DCT_n$.

Lemma 3.1. *Let $n \geq 3$. If $\alpha \in \mathcal{F}_r$ then $\alpha \in < \mathcal{F}_{r+1} >$ for $1 \leq r \leq n-2$.*

Proof. Let $n \geq 3$ and $\alpha \in \mathcal{F}_r$ for $1 \leq r \leq n-2$. Then we have

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_r \\ 1 & 2 & \dots & r \end{pmatrix}$$

where $1 \leq r \leq n-2$, so there exists i such that $|A_i| \geq 2$ for $1 \leq i \leq r$. Let x_i be the maximum element in A_i. Let β be a mapping such that

$$\beta = \begin{pmatrix} A_1 & \dots & A_{i-1} & A_i \setminus \{x_i\} & \{x_i\} & A_{i+1} & \dots & A_r \\ 1 & \dots & i-1 & i & i+1 & i+2 & \dots & r+1 \end{pmatrix}$$

for $i > 1$ and

$$\beta = \begin{pmatrix} A_1 \setminus \{x_1\} & \{x_1\} & A_2 & \dots & A_r \\ 1 & 2 & 3 & \dots & r+1 \end{pmatrix}$$

for $i = 1$. Then it is clear that $\beta \in \mathcal{F}_{r+1}$. Let γ be the mapping defined as

$$j\gamma = \left\{ \begin{array}{ll} j & \text{if } 1 \leq j \leq i \\ i & \text{if } j = i+1 \\ j-1 & \text{if } i+2 \leq j \leq r+1 \\ r+1 & \text{if } j > r+1 \end{array} \right\}$$

then it is clear that $\gamma \in \mathcal{F}_{r+1}$ and $\alpha = \beta\gamma$, so $\alpha \in < \mathcal{F}_{r+1} >$. □

Corollary 3.1. *$\mathcal{F}_r \subseteq < \mathcal{F}_{r+1} >$ for each $1 \leq r \leq n-2$.*

Corollary 3.2. *Since $\mathcal{F}_n$ is the set that has only the identity mapping of $\mathcal{O}DCT_n$ then we have $< \mathcal{F}_{n-1} > = \mathcal{O}DCT_n$ for $n \geq 3$.*

Corollary 3.3 ([1]). *$|\mathcal{F}_r| = \binom{n-1}{r-1}$ for $1 \leq r \leq n$.*

Corollary 3.4. *$\mathrm{rank}_M(\mathcal{O}DCT_n) \leq n-1$ for $n \in \mathbb{Z}^+$ since $|\mathcal{F}_{n-1}| = n-1$.*

Corollary 3.5 ([1]). *$|\mathcal{O}DCT_n| = 2^{n-1}$ for $n \geq 1$.*

Theorem 3.1. *$\mathrm{rank}_M(\mathcal{O}DCT_n) = n-1$ for $n \in \mathbb{Z}^+$.*

Proof. If $n = 1$ or $n = 2$ then result is clear, let $n \geq 3$. We have $\mathrm{rank}_M(\mathcal{O}DCT_n) \leq n-1$ from Corollary 3.4. Let

$$\mathcal{O}DCT_{(n,r)} = \{\alpha \in \mathcal{O}DCT_n : |\mathrm{im}(\alpha)| \leq r\}$$

for $1 \leq r \leq n-1$. It is clear that $\mathcal{O}DCT_{(n,r)}$ is an ideal of $\mathcal{O}DCT_n$. In particular, $\mathcal{O}DCT_{(n,n-2)}$ is an ideal of $\mathcal{O}DCT_n$. Moreover, there are $n-1$ different kernel classes in $\mathcal{F}_{n-1}$ and we have $\mathcal{F}_n = \{\epsilon\}$, so $\mathrm{rank}_M(\mathcal{O}DCT_n) \geq n-1$. Thus we have concluded that $\mathrm{rank}_M(\mathcal{O}DCT_n) = n-1$ for $n \in \mathbb{Z}^+$. □

4. The Monoid Presentation of $ODCT_n$

In this section, we have found a monoid presentation of $\mathcal{O}DCT_n$ for $n \geq 3$.

Proposition 4.1 ([8]). *Let A be a set and let M be any monoid. Then any mapping $\phi : A \to M$ can be extended in a unique way to a homomorphism $\overline{\phi} : A^* \to M$.*

Definition 4.1. Let M be any monoid, let B be a generating set of M, and let $\phi : A \to B$ be an onto mapping. By Proposition 4.1 the mapping ϕ can be extended in a unique way to an epimorphism $\overline{\phi} : A^* \to M$. Let $R \subseteq A^* \times A^*$ be a set of relations. The monoid M is said to satisfy relations R if for each $(u, v) \in R$ we have $u\overline{\phi} = v\overline{\phi}$.

Let M be a finite monoid, $A \subseteq M$ and $< A >= M$. Let $R \subseteq A^* \times A^*$ be a set of relations, and let $W \subseteq A^*$. It is well known that if

(i) the generators A of M satisfy all the relations from R

(ii) for each word $w \in A^*$ there exists a word $\overline{w} \in W$ such that $w = \overline{w}$ is a consequence of R

(iii) $|W| \leq |M|$

then $< A \mid R >$ is a monoid presentation of M.

Let $n \geq 3$ and α_i be the mapping defined as

$$\alpha_i = \begin{pmatrix} 1 & \dots & i-1 & i & i+1 & i+2 & \dots & n \\ 1 & \dots & i-1 & i & i & i+1 & \dots & n-1 \end{pmatrix}$$

for $2 \leq i \leq n-1$ and

$$\alpha_1 = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 1 & 1 & 2 & \dots & n-1 \end{pmatrix},$$

then it is clear that $\mathcal{F}_{n-1} = \{\alpha_i \mid 1 \leq i \leq n-1\}$.

Lemma 4.1. *Let $n \geq 3$ and α_i be defined as above then $\alpha_i\alpha_{n-1} = \alpha_i$ for $1 \leq i \leq n-1$. In particular, $(\alpha_{n-1})^2 = \alpha_{n-1}$.*

Proof. Let $n \geq 3$ and α_i be defined as above, then

$$\alpha_{n-1} = \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ 1 & 2 & \dots & n-1 & n-1 \end{pmatrix}.$$

$1(\alpha_i\alpha_{n-1}) = 1$ and $n(\alpha_i\alpha_{n-1}) = n-1$, we have $\mathrm{im}(\alpha_i\alpha_{n-1}) = \{1, 2, \dots, n-1\}$ from the definition of $\mathcal{O}DCT_n$. Moreover, $i(\alpha_i\alpha_{n-1}) = i$ and $(i+1)(\alpha_i\alpha_{n-1}) = i$, so $\alpha_i\alpha_{n-1} = \alpha_i$ for $1 \leq i \leq n-1$. ☐

Lemma 4.2. *Let $n \geq 3$ and α_i be defined as above then $\alpha_i\alpha_j = \alpha_{j+1}\alpha_i$ for $1 \leq i \leq j \leq n-2$.*

Proof. Let $n \geq 3$, α_i be defined as above and $1 \leq i \leq j \leq n-2$. It is clear that $1(\alpha_i\alpha_j) = 1$ and $n(\alpha_i\alpha_j) = n-1(\alpha_j) = n-2$ since $1 \leq i \leq j \leq n-2$. Thus $\mathrm{im}(\alpha_i\alpha_j) = \{1, 2, \dots, n-2\}$ from the definition of $\mathcal{O}DCT_n$. Moreover we have

$$i(\alpha_i\alpha_j) = i\alpha_j = i$$

$$(i+1)(\alpha_i\alpha_j) = i\alpha_j = i$$

$$(j+1)(\alpha_i\alpha_j) = j\alpha_j = j$$

$$(j+2)(\alpha_i\alpha_j) = (j+1)\alpha_j = j.$$

Also, $1(\alpha_{j+1}\alpha_i) = 1$ and $n(\alpha_{j+1}\alpha_i) = (n-1)\alpha_i = n-2$ thus $\mathrm{im}(\alpha_{j+1}\alpha_i) = \{1, 2, \dots, n-2\}$ from the definition of $\mathcal{O}DCT_n$. Moreover we have

$$i(\alpha_{j+1}\alpha_i) = i\alpha_i = i$$

$$(i+1)(\alpha_{j+1}\alpha_i) = (i+1)\alpha_i = i$$

$$(j+1)(\alpha_{j+1}\alpha_i) = (j+1)\alpha_i = j$$

$$(j+2)(\alpha_{j+1}\alpha_i) = (j+1)\alpha_i = j.$$

Therefore, $x(\alpha_i\alpha_j) = x(\alpha_{j+1}\alpha_i)$ for $\forall x \in X_n$. It follows that $\alpha_i\alpha_j = \alpha_{j+1}\alpha_i$ for $1 \leq i \leq j \leq n-2$. ☐

Definition 4.2. Let A be a finite set and $w = a_1 a_2 \ldots a_k$ for $a_i \in A$ and $1 \leq i \leq k$. Length of w is defined as k and we write $l(w) = k$ and if w is empty word then the length of w is defined as 0 (zero) and we write $l(w) = 0$.

Theorem 4.1. *Let $n \geq 3$. Let $A = \{a_1, a_2, \ldots, a_{n-1}\}$ and $R = \{a_i a_{n-1} = a_i \ \ (1 \leq i \leq n-1), a_i a_j = a_{j+1} a_i \ \ (1 \leq i \leq j \leq n-2)\}$. Then $< A \mid R >$ is a monoid presentation of $\mathcal{O}DCT_n$ for $n \geq 3$.*

Proof. Let $n \geq 3$. Let $A = \{a_1, a_2, \ldots, a_{n-1}\}$ and $R = \{a_i a_{n-1} = a_i \ \ (1 \leq i \leq n-1), a_i a_j = a_{j+1} a_i \ \ (1 \leq i \leq j \leq n-2)\}$. Let $f : A \to \mathcal{F}_{n-1}$ be the mapping such that $a_i f = \alpha_i$. There exists a unique epimorpish $\overline{f} : A^* \to \mathcal{O}DCT_n$ extending the f. Thus $\mathcal{O}DCT_n$ satisfies all the relations from R since Lemma 4.1 and Lemma 4.2. Let ε is the empty word and

$$W = \{a_{j_k} a_{j_{k-1}} \ldots a_{j_1} \mid n-1 \geq j_k > j_{k-1} > \ldots > j_1 \geq 1\} \cup \{\varepsilon\}.$$

Thus it is clear that $W \subseteq A^*$ and $|W| = 2^{n-1}$. Let $w \in A^*$ and $l(w) = m$. We will show that there exists $\overline{w} \in W$ such that $w = \overline{w}$ is a consequence of R. We use induction on m. If $m = 0$ or $m = 1$, then the result is clear. Let $m \geq 2$, then $w \equiv w_1 w_2$ where $l(w_1) = m-1$ and $l(w_2) = 1$. Thus $w_2 \in A$, moreover we have $w_1 = \overline{w_1}$ such that $\overline{w_1} \in W$ from the induction hypothesis. So $w = \overline{w_1} w_2$. If $\overline{w_1} \equiv \varepsilon$ then result is clear. Let $\overline{w_1} \not\equiv \varepsilon$. Then,

$$\overline{w_1} \equiv a_{t_p} a_{t_{p-1}} \ldots a_{t_1}$$

where $n-1 \geq t_p > t_{p-1} > \ldots > t_1 \geq 1$ and $w = a_{t_p} a_{t_{p-1}} \ldots a_{t_1} w_2$. If $w_2 \equiv a_{n-1}$ then

$$w = a_{t_p} a_{t_{p-1}} \ldots a_{t_1} a_{n-1}$$

$$w = a_{t_p} a_{t_{p-1}} \ldots a_{t_1}$$

so in this case $w = \overline{w_1}$ and $\overline{w_1} \in W$. Let $w_2 \equiv a_i$ where $1 \leq i \leq n-2$. Then

$$w = a_{t_p} a_{t_{p-1}} \ldots a_{t_1} a_i,$$

if $t_1 > i$ then we have $\overline{w} \equiv a_{t_p} a_{t_{p-1}} \ldots a_{t_1} a_i$ and $w = \overline{w}$, $\overline{w} \in W$. If $t_1 \leq i$ then

$$w = a_{t_p} a_{t_{p-1}} \ldots a_{t_2} a_{t_1} a_i$$

$$w = a_{t_p} a_{t_{p-1}} \ldots a_{t_2} a_{i+1} a_{t_1}.$$

If $p = 1$ then result is clear, let $p \geq 2$. If $t_2 > i+1$ then we have $\overline{w} \equiv a_{t_p} a_{t_{p-1}} \ldots a_{t_2} a_{i+1} a_{t_1}$ and $w = \overline{w}$, $\overline{w} \in W$. If $i + 1 = n - 1$ then

$$w = a_{t_p} a_{t_{p-1}} \ldots a_{t_2} a_{i+1} a_{t_1}$$

$$w = a_{t_p} a_{t_{p-1}} \ldots a_{t_2} a_{n-1} a_{t_1}$$

$$w = a_{t_p} a_{t_{p-1}} \ldots a_{t_2} a_{t_1},$$

so in this case $\overline{w} \equiv a_{t_p} a_{t_{p-1}} \ldots a_{t_2} a_{t_1}$ and $w = \overline{w}$, $\overline{w} \in W$. If $t_2 \leq i + 1 < n - 1$ then we have

$$w = a_{t_p} a_{t_{p-1}} \ldots a_{t_2} a_{i+1} a_{t_1}$$

$$w = a_{t_p} a_{t_{p-1}} \ldots a_{i+2} a_{t_2} a_{t_1}.$$

If we use the same algorithm, it is clear that finally we conclude that there exists a word $\overline{w} \in W$ such that $w = \overline{w}$ is a consequence of R. Moreover, $|W| = |\mathcal{O}DCT_n| = 2^{n-1}$, it follows that $< A \mid R >$ is a monoid presentation of $\mathcal{O}DCT_n$ for $n \geq 3$. □

5. Conclusion

In this paper we have found monoid rank of $\mathcal{O}DCT_n$ for $n \in \mathbb{Z}^+$. Moreover since $\mathcal{O}DCT_1$ is a trivial monoid and $\mathcal{O}DCT_2$ is a monogenic monoid, we give a monoid presentation of $\mathcal{O}DCT_n$ for $n \geq 3$. Recently, the rank of $\mathcal{O}CT_n$ and the rank of $\mathcal{O}RCT_n$ have been found, finding presentation problem can be considered on those semigroups as a future work.

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Author's contributions

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