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LOGARITHMIC COEFFICIENTS OF STARLIKE FUNCTIONS CONNECTED WITH k-FIBONACCI NUMBERS

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Abstract. Let $\mathcal{A}$ denote the class of analytic functions f in the open unit disc $\mathbb{U}$ normalized by $f(0) = f'(0) - 1 = 0$, and let $\mathcal{S}$ be the class of all functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$. For a function $f \in \mathcal{S}$, the logarithmic coefficients δ_n $(n = 1, 2, 3, \ldots)$ are defined by

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \delta_n z^n \qquad (z \in \mathbb{U})$$

and it is known that $|\delta_1| \leq 1$ and $|\delta_2| \leq \frac{1}{2}\left(1 + 2e^{-2}\right) = 0,635 \cdots$. The problem of the best upper bounds for $|\delta_n|$ of univalent functions for $n \geq 3$ is still open. Let $\mathcal{SL}^k$ denote the class of functions $f \in \mathcal{A}$ such that

$$\frac{zf'(z)}{f(z)} \prec \frac{1 + \tau_k^2 z^2}{1 - k\tau_k z - \tau_k^2 z^2}, \quad \tau_k = \frac{k - \sqrt{k^2 + 4}}{2} \qquad (z \in \mathbb{U}).$$

In the present paper, we determine the sharp upper bound for $|\delta_1|, |\delta_2|$ and $|\delta_3|$ for functions f belong to the class $\mathcal{SL}^k$ which is a subclass of $\mathcal{S}$. Furthermore, a general formula is given for $|\delta_n|$ $(n \in \mathbb{N})$ as a conjecture.

1. Introduction

Let $\mathbb{C}$ be the set of complex numbers and $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of positive integers. Assume that $\mathcal{H}$ is the class of analytic functions in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, and let the class $\mathcal{P}$ be defined by

$$\mathcal{P} = \{p \in \mathcal{H} : p(0) = 1 \quad \text{and} \quad \Re(p(z)) > 0 \ (z \in \mathbb{U})\}.$$

For two functions $f, g \in \mathcal{H}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

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if there exists a Schwarz function

$$\omega \in \Omega := \{\omega \in \mathcal{H} : \omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \ (z \in \mathbb{U})\},$$

such that

$$f(z) = g(\omega(z)) \quad (z \in \mathbb{U}).$$

Indeed, it is known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U} \Rightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ consisting of functions f normalized by

$$f(0) = f'(0) - 1 = 0.$$

Each function $f \in \mathcal{A}$ can be expressed as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}). \tag{1}$$

We also denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $\mathcal{A}$ which are univalent in $\mathbb{U}$.

A function $f \in \mathcal{A}$ is said to be starlike of order $\alpha\,(0 \leq \alpha < 1)$, if it satisfies the inequality

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U}).$$

We denote the class which consists of all functions $f \in \mathcal{A}$ that are starlike of order α by $\mathcal{S}^*(\alpha)$. It is well-known that $\mathcal{S}^*(\alpha) \subset \mathcal{S}^*(0) = \mathcal{S}^* \subset \mathcal{S}$.

By means of the principle of subordination, Yılmaz Özgür and Sokól [13] defined the following class $\mathcal{SL}^k$ of functions $f \in \mathcal{S}$, connected with a shell-like region described by a function $\tilde{p}_k$ with coefficients depicted in terms of the k-Fibonacci numbers where k is a positive real number. The name attributed to the class $\mathcal{SL}^k$ is motivated by the shape of the curve

$$\Gamma = \left\{\tilde{p}_k\left(e^{i\varphi}\right) : \varphi \in [0, 2\pi) \setminus \{\pi\}\right\}.$$

The curve Γ has a shell-like shape and it is symmetric with respect to the real axis. For more details about the class $\mathcal{SL}^k$, please refer to [11, 13].

Definition 1. *[13] Let k be any positive real number. The function $f \in \mathcal{S}$ belongs to the class $\mathcal{SL}^k$ if it satisfies the condition that*

$$\frac{zf'(z)}{f(z)} \prec \tilde{p}_k(z) \qquad (z \in \mathbb{U}), \tag{2}$$

where

$$\tilde{p}_k(z) = \frac{1 + \tau_k^2 z^2}{1 - k\tau_k z - \tau_k^2 z^2} = \frac{1 + \tau_k^2 z^2}{1 - (\tau_k^2 - 1) z - \tau_k^2 z^2} \tag{3}$$

with

$$\tau_k = \frac{k - \sqrt{k^2+4}}{2}. \tag{4}$$

For $k = 1$, the class $\mathcal{SL}^k$ reduces to the class $\mathcal{SL}$ which consists of functions $f \in \mathcal{A}$ defined by (1) satisfying

$$\frac{zf'(z)}{f(z)} \prec \tilde{p}(z)$$

where

$$\tilde{p}(z) := \tilde{p}_1(z) = \frac{1+\tau^2 z^2}{1-\tau z - \tau^2 z^2} \tag{5}$$

with

$$\tau := \tau_1 = \frac{1-\sqrt{5}}{2}. \tag{6}$$

This class was introduced by Sokól [10].

Definition 2. *[3] For any positive real number k, the k-Fibonacci sequence $\{F_{k,n}\}_{n\in\mathbb{N}_0}$ is defined recurrently by*

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1} \qquad (n \in \mathbb{N})$$

with initial conditions

$$F_{k,0} = 0, \qquad F_{k,1} = 1.$$

Furthermore n^{th} k-Fibonacci number is given by

$$F_{k,n} = \frac{(k-\tau_k)^n - \tau_k^n}{\sqrt{k^2+4}}, \tag{7}$$

where τ_k is given by (4).

For $k = 1$, we obtain the classic Fibonacci sequence $\{F_n\}_{n\in\mathbb{N}_0}$:

$$F_0 = 0, \qquad F_1 = 1, \qquad \text{and} \quad F_{n+1} = F_n + F_{n-1} \quad (n \in \mathbb{N}).$$

For more details about the k-Fibonacci sequences please refer to [7, 9, 12, 14].

Yılmaz Özgür and Sokól [13] showed that the coefficients of the function $\tilde{p}_k(z)$ defined by (3) are connected with k-Fibonacci numbers. This connection is pointed out in the following theorem.

Theorem 1. *[13] Let $\{F_{k,n}\}_{n\in\mathbb{N}_0}$ be the sequence of k-Fibonacci numbers defined in Definition 2. If*

$$\tilde{p}_k(z) = \frac{1+\tau_k^2 z^2}{1-k\tau_k z - \tau_k^2 z^2} := 1 + \sum_{n=1}^{\infty} \tilde{p}_{k,n} z^n, \tag{8}$$

then we have

$$\tilde{p}_{k,1} = k\tau_k, \quad \tilde{p}_{k,2} = \left(k^2+2\right)\tau_k^2, \quad \tilde{p}_{k,n} = \left(F_{k,n-1} + F_{k,n+1}\right)\tau_k^n \qquad (n \in \mathbb{N}). \tag{9}$$

It can be found the more results related to Fibonacci numbers in [7, 12, 14].

Remark 1. *[13] For each $k > 0$,*

$$\mathcal{SL}^k \subset \mathcal{S}^*(\alpha_k), \qquad \alpha_k = \frac{k}{2\sqrt{k^2+4}},$$

that is, $f \in \mathcal{SL}^k$ is a starlike function of order α_k, and so is univalent.

For a function $f \in \mathcal{S}$, the logarithmic coefficients δ_n $(n \in \mathbb{N})$ are defined by

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \delta_n z^n \qquad (z \in \mathbb{U}), \tag{10}$$

and play a central role in the theory of univalent functions. The idea of studying the logarithmic coefficients helped Kayumov [8] to solve Brennan's conjecture for conformal mappings. If $f \in \mathcal{S}$, then it is known that

$$|\delta_1| \leq 1$$

and

$$|\delta_2| \leq \frac{1}{2}\left(1 + 2e^{-2}\right) = 0,635\cdots$$

(see [2]). The problem of the best upper bounds for $|\delta_n|$ of univalent functions for $n \geq 3$ is still open.

The main purpose of this paper is to determine the upper bound for $|\delta_1|, |\delta_2|$ and $|\delta_3|$ for functions f belong to the univalent function class $\mathcal{SL}^k$. To prove our main results we need the following lemmas.

Lemma 1. *[11] If $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ $(z \in \mathbb{U})$ and*

$$p(z) \prec \tilde{p}_k(z) = \frac{1 + \tau_k^2 z^2}{1 - k\tau_k z - \tau_k^2 z^2}, \qquad \tau_k = \frac{k - \sqrt{k^2+4}}{2},$$

then we have

$$|p_1| \leq k|\tau_k| \qquad \textit{and} \qquad |p_2| \leq \left(k^2+2\right)\tau_k^2.$$

The above estimates are sharp.

Lemma 2. *[5] If $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ $(z \in \mathbb{U})$ and*

$$p(z) \prec \tilde{p}_k(z) = \frac{1 + \tau_k^2 z^2}{1 - k\tau_k z - \tau_k^2 z^2}, \qquad \tau_k = \frac{k - \sqrt{k^2+4}}{2},$$

then we have

$$|p_3| \leq \left(k^3 + 3k\right)|\tau_k|^3.$$

The result is sharp.

Lemma 3. *[1] If $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ $(z \in \mathbb{U})$ and*

$$p(z) \prec \tilde{p}_k(z) = \frac{1 + \tau_k^2 z^2}{1 - k\tau_k z - \tau_k^2 z^2}, \qquad \tau_k = \frac{k - \sqrt{k^2+4}}{2},$$

then we have

$$\left|p_2-\gamma p_1^2\right|\leq k\left|\tau_k\right|\max\left\{1,\ \left|k^2+2-\gamma k^2\right|\frac{\left|\tau_k\right|}{k}\right\}$$

for all $\gamma\in\mathbb{C}$. *The above estimates are sharp.*

Lemma 4. *[2] Let* $p(z)=1+c_1z+c_2z^2+\cdots\in\mathcal{P}$. *Then*

$$|c_n|\leq 2 \qquad (n\in\mathbb{N}).$$

Lemma 5. *[4] Let* $p(z)=1+c_1z+c_2z^2+\cdots\in\mathcal{P}$. *Then*

$$2c_2=c_1^2+x\left(4-c_1^2\right)$$

for some x, $|x|\leq 1$, *and*

$$4c_3=c_1^3+2c_1\left(4-c_1^2\right)x-c_1\left(4-c_1^2\right)x^2+2\left(4-c_1^2\right)\left(1-|x|^2\right)z$$

for some z, $|z|\leq 1$.

Lemma 6. *[1] If the function* f *given by* (1) *is in the class* $\mathcal{SL}^k$, *then we have*

$$\left|a_3-\lambda a_2^2\right|\leq\begin{cases}\tau_k^2\left(k^2+1-\lambda k^2\right) & , \quad \lambda\leq\frac{2\left(k^2+1\right)\tau_k+k}{2k^2\tau_k}\\ \frac{k|\tau_k|}{2} & , \quad \frac{2\left(k^2+1\right)\tau_k+k}{2k^2\tau_k}\leq\lambda\leq\frac{2\left(k^2+1\right)\tau_k-k}{2k^2\tau_k}\\ \tau_k^2\left(\lambda k^2-k^2-1\right) & , \quad \lambda\geq\frac{2\left(k^2+1\right)\tau_k-k}{2k^2\tau_k}\end{cases}.$$

If $\frac{2\left(k^2+1\right)\tau_k+k}{2k^2\tau_k}\leq\lambda\leq\frac{k^2+1}{k^2}$, *then*

$$\left|a_3-\lambda a_2^2\right|+\left(\lambda-\frac{2\left(k^2+1\right)\tau_k+k}{2k^2\tau_k}\right)\left|a_2\right|^2\leq\frac{k\left|\tau_k\right|}{2}.$$

Furthermore, if $\frac{k^2+1}{k^2}\leq\lambda\leq\frac{2\left(k^2+1\right)\tau_k-k}{2k^2\tau_k}$, *then*

$$\left|a_3-\lambda a_2^2\right|+\left(\frac{2\left(k^2+1\right)\tau_k-k}{2k^2\tau_k}-\lambda\right)\left|a_2\right|^2\leq\frac{k\left|\tau_k\right|}{2}.$$

Each of these results is sharp.

Lemma 7. *[6] If the function* f *given by* (1) *is in the class* $\mathcal{SL}^k$, *then*

$$\left|a_2a_4-a_3^2\right|\leq\tau_k^4.$$

The bound is sharp.

Lemma 8. *[6] If the function* f *given by* (1) *is in the class* $\mathcal{SL}^k$, *then*

$$\left|a_2a_3-a_4\right|\leq k\left|\tau_k\right|^3.$$

The bound is sharp.

2. The coefficients of $\log(f(z)/z)$

Theorem 2. *Let $f \in \mathcal{SL}^k$ be given by (1) and the coefficients of $\log(f(z)/z)$ be given by (10). Then*

$$|\delta_1| \le \frac{k}{2}|\tau_k|, \qquad |\delta_2| \le \frac{k^2+2}{4}\tau_k^2, \qquad |\delta_3| \le \frac{k^3+3k}{6}|\tau_k|^3, \tag{11}$$

where τ_k is defined by (4). Each of these results is sharp. The equalities are attained by the function $\tilde{p}_k$ given by (3).

Proof. Firstly, by differentiating (10) and equating coefficients, we have

$$\delta_1 = \frac{1}{2}a_2,$$

$$\delta_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right),$$

$$\delta_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right).$$

If $f \in \mathcal{SL}^k$, then by the principle of subordination, there exists a Schwarz function $\omega \in \Omega$ such that

$$\frac{zf'(z)}{f(z)} = \tilde{p}_k(\omega(z)) \quad (z \in \mathbb{U}), \tag{12}$$

where the function $\tilde{p}_k$ is given by (8). Therefore, the function

$$g(z) := \frac{1+\omega(z)}{1-\omega(z)} = 1 + c_1z + c_2z^2 + \cdots \quad (z \in \mathbb{U}) \tag{13}$$

is in the class $\mathcal{P}$. Now, defining the function $p(z)$ by

$$p(z) = \frac{zf'(z)}{f(z)} = 1 + p_1z + p_2z^2 + \cdots, \tag{14}$$

it follows from (12) and (13) that

$$p(z) = \tilde{p}_k\left(\frac{g(z)-1}{g(z)+1}\right). \tag{15}$$

Note that

$$\omega(z) = \frac{c_1}{2}z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)z^2 + \frac{1}{2}\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right)z^3 + \cdots$$

and so

$$\begin{aligned}\tilde{p}_k(\omega(z)) &= 1 + \frac{\tilde{p}_{k,1}c_1}{2}z + \left[\frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)\tilde{p}_{k,1} + \frac{1}{4}c_1^2\tilde{p}_{k,2}\right]z^2 \\ &\quad + \left[\frac{1}{2}\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right)\tilde{p}_{k,1} + \frac{1}{2}c_1\left(c_2 - \frac{c_1^2}{2}\right)\tilde{p}_{k,2} + \frac{c_1^3}{8}\tilde{p}_{k,3}\right]z^3 + \cdots.\end{aligned} \tag{16}$$

Thus, by using (13) in (15), and considering the values $\tilde{p}_{k,j}$ $(j = 1, 2, 3)$ given in (9), we obtain

$$p_1 = \frac{k\tau_k}{2} c_1, \tag{17}$$

$$p_2 = \frac{k\tau_k}{2} \left(c_2 - \frac{c_1^2}{2} \right) + \frac{\left(k^2+2\right)\tau_k^2}{4} c_1^2, \tag{18}$$

$$p_3 = \frac{k\tau_k}{2} \left(c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) + \frac{\left(k^2+2\right)\tau_k^2}{2} c_1 \left(c_2 - \frac{c_1^2}{2} \right) + \frac{\left(k^3+3k\right)\tau_k^3}{8} c_1^3. \tag{19}$$

On the other hand, a simple calculation shows that

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + \left(2a_3 - a_2^2\right) z^2 + \left(3a_4 - 3a_2 a_3 + a_2^3\right) z^3 + \cdots,$$

which, in view of (14), yields

$$a_2 = p_1, \qquad a_3 = \frac{p_1^2 + p_2}{2}, \qquad a_4 = \frac{p_1^3 + 3p_1 p_2 + 2p_3}{6}. \tag{20}$$

Substituting for a_2, a_3 and a_4 from (20), we obtain

$$\delta_1 = \frac{1}{2} p_1, \qquad \delta_2 = \frac{1}{4} p_2, \qquad \delta_3 = \frac{1}{6} p_3. \tag{21}$$

Using Lemma 1 and Lemma 2, we get the desired results. This completes the proof of theorem. □

Conjecture. Let $f \in \mathcal{SL}^k$ be given by (1) and the coefficients of $\log(f(z)/z)$ be given by (10). Then

$$|\delta_n| \leq \frac{F_{k,n-1} + F_{k,n+1}}{2n} |\tau_k|^n \qquad (n \in \mathbb{N}),$$

where $\{F_{k,n}\}_{n \in \mathbb{N}_0}$ is the Fibonacci sequence given by (7).

This conjecture has been verified for the values $n = 1, 2, 3$ by the Theorem 2.

Letting $k = 1$ in Theorem 2, we obtain the following consequence.

Corollary 1. *Let $f \in \mathcal{SL}$ be given by (1) and the coefficients of $\log(f(z)/z)$ be given by (10). Then*

$$|\delta_1| \leq \frac{1}{2} |\tau|, \qquad |\delta_2| \leq \frac{3}{4} \tau^2, \qquad |\delta_3| \leq \frac{2}{3} |\tau|^3,$$

where τ is defined by (6). Each of these results is sharp. The equalities are attained by the function $\tilde{p}$ given by (5).

Theorem 3. *Let $f \in \mathcal{SL}^k$ be given by* (1) *and the coefficients of* $\log(f(z)/z)$ *be given by* (10). *Then for any $\gamma \in \mathbb{C}$, we have*

$$\left|\delta_2 - \gamma\delta_1^2\right| \leq \frac{k\left|\tau_k\right|}{4} \max\left\{1, \left|k^2 + 2 - \gamma k^2\right| \frac{\left|\tau_k\right|}{k}\right\}.$$

Proof. By using (21), the desired result is obtained from the equality

$$\delta_2 - \gamma\delta_1^2 = \frac{1}{4}\left(p_2 - \gamma p_1^2\right) \qquad (\gamma \in \mathbb{C})$$

and Lemma 3. □

Letting $k = 1$ in Theorem 3, we obtain the following consequence.

Corollary 2. *Let $f \in \mathcal{SL}$ be given by* (1) *and the coefficients of* $\log(f(z)/z)$ *be given by* (10). *Then for any $\gamma \in \mathbb{C}$, we have*

$$\left|\delta_2 - \gamma\delta_1^2\right| \leq \frac{|\tau|}{4} \max\left\{1, \left|(3-\gamma)\tau\right|\right\}.$$

If we take $\gamma = 1$ in Theorem 3, then we obtain the following result.

Corollary 3. *Let $f \in \mathcal{SL}^k$ be given by* (1) *and the coefficients of* $\log(f(z)/z)$ *be given by* (10). *Then*

$$\left|\delta_2 - \delta_1^2\right| \leq \begin{cases} \frac{\tau_k^2}{2} & , \quad 0 < k \leq \frac{2}{\sqrt{3}} \\ \frac{k|\tau_k|}{4} & \quad k \geq \frac{2}{\sqrt{3}} \end{cases}.$$

Letting $k = 1$ in Corollary 3, we obtain the following consequence.

Corollary 4. *Let $f \in \mathcal{SL}$ be given by* (1) *and the coefficients of* $\log(f(z)/z)$ *be given by* (10). *Then*

$$\left|\delta_2 - \delta_1^2\right| \leq \frac{\tau^2}{2}.$$

3. The coefficients of the inverse function

Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk $\mathbb{U}$. In fact, the Koebe one-quarter theorem [2] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius 1/4. Thus every function $f \in \mathcal{A}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right).$$

In fact, for a function $f \in \mathcal{A}$ given by (1) the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots =: w + \sum_{n=2}^{\infty} A_n w^n. \tag{22}$$

Since $\mathcal{SL}^k \subset \mathcal{S}$, the functions f belonging to the class $\mathcal{SL}^k$ are invertible.

Theorem 4. *Let $f \in \mathcal{SL}^k$ be given by* (1), *and f^{-1} be the inverse function of f defined by* (22). *Then we have*

$$|A_2| \leq k\,|\tau_k|$$

and

$$|A_3| \leq \frac{k\,|\tau_k|}{2} \max\left\{1,\ 2\left|1 - k^2\right| \frac{|\tau_k|}{k}\right\}.$$

Each of these results is sharp.

Proof. Let the function $f \in \mathcal{A}$ given by (1) be in the class $\mathcal{SL}^k$, and f^{-1} be the inverse function of f defined by (22). Then using (20), we obtain

$$A_2 = -a_2 = -p_1 \tag{23}$$

and

$$A_3 = 2a_2^2 - a_3 = -\frac{1}{2}\left(p_2 - 3p_1^2\right).$$

The upper bound for $|A_2|$ is clear from Lemma 1. Furthermore by considering Lemma 3 we obtain the upper bound of $|A_3|$ as

$$|A_3| \leq \frac{k\,|\tau_k|}{2} \max\left\{1,\ 2\left|1 - k^2\right| \frac{|\tau_k|}{k}\right\}.$$

Finally, for the sharpness, we have by (8) that

$$\tilde{p}_k(z) = 1 + k\tau_k z + \left(k^2 + 2\right)\tau_k^2 z^2 + \cdots$$

and

$$\tilde{p}_k\left(z^2\right) = 1 + k\tau_k z^2 + \left(k^2 + 2\right)\tau_k^2 z^4 + \cdots.$$

From this equalities, we obtain

$$p_1 = k\tau_k \qquad \text{and} \qquad p_2 = \left(k^2 + 2\right)\tau_k^2$$

and

$$p_1 = 0 \qquad \text{and} \qquad p_2 = k\tau_k,$$

respectively. Thus, it is clear that the equality for $|A_2|$ is attained for the function $\tilde{p}_k(z)$; and the equality for the first value of $|A_3|$ is attained for the function $\tilde{p}_k(z^2)$, for the second value of $|A_3|$ is attained for the function $\tilde{p}_k(z)$. This evidently completes the proof of theorem. □

Remark 2. *It is worthy to note that the coefficient bound obtained for $|A_3|$ in Theorem 4 is the improvement of [11, Corollary 2.4].*

Theorem 5. *Let $f \in \mathcal{SL}$ be given by* (1), *and f^{-1} be the inverse function of f defined by* (22). *Then we have*

$$|A_2| \leq |\tau|, \qquad |A_3| \leq \frac{|\tau|}{2} \qquad \text{and} \qquad |A_4| \leq 2|\tau|^3.$$

Each of these results is sharp.

Proof. Let $f \in \mathcal{SL}$ be given by (1), and f^{-1} be the inverse function of f defined by (22). Then the upper bounds for $|A_2|$ and $|A_3|$ are obtained as a consequence of Theorem 4 when $k=1$. From (22), we have

$$-A_4 = 5a_2^3 - 5a_2a_3 + a_4.$$

By using (20) in the above equality, we obtain

$$-A_4 = \frac{8}{3}p_1^3 - 2p_1p_2 + \frac{1}{3}p_3.$$

By (17)-(19), this equality gives

$$A_4 = -\frac{\tau}{6}\left(c_3 - c_1c_2 + \frac{1-6\tau^2}{4}c_1^3\right).$$

By means of Lemma 5, we get

$$\begin{aligned} A_4 &= \frac{\tau}{6}\left[\frac{1}{4}c_1\left(4-c_1^2\right)x^2 - \frac{1}{2}\left(4-c_1^2\right)\left(1-|x|^2\right)z + \frac{3\tau^2}{2}c_1^3\right] \\ &= \frac{\tau}{24}\left[6\tau^2c_1^3 + \left(4-c_1^2\right)\left\{c_1x^2 - 2\left(1-|x|^2\right)z\right\}\right]. \end{aligned}$$

As per Lemma 4, it is clear that $|c_1| \leq 2$. Therefore letting $c_1 = c$, we may assume without loss of generality that $c \in [0,2]$. Hence, by using the triangle inequality, it is obtained that

$$|A_4| \leq \frac{|\tau|}{24}\left[6\tau^2c^3 + \left(4-c^2\right)\left\{c|x|^2 + 2\left(1-|x|^2\right)\right\}\right].$$

Thus, for $\mu = |x| \leq 1$, we have

$$|A_4| \leq \frac{|\tau|}{24}\left[6\tau^2c^3 + \left(4-c^2\right)\left\{c\mu^2 + 2\left(1-\mu^2\right)\right\}\right] := F(c,\mu).$$

Now, we need to find the maximum value of $F(c,\mu)$ over the rectangle Π,

$$\Pi = \{(c,\mu) : 0 \leq c \leq 2,\ 0 \leq \mu \leq 1\}.$$

For this, first differentiating the function F with respect to c and μ, we get

$$\frac{\partial F(c,\mu)}{\partial c} = \frac{|\tau|}{24}\left[18\tau^2c^2 + \left(4-c^2\right)\left\{c\mu^2 + 2\left(1-\mu^2\right)\right\}\right]$$

and

$$\frac{\partial F(c,\mu)}{\partial \mu} = \frac{|\tau|}{12}\left(4-c^2\right)(c-2)\mu,$$

respectively. The condition $\frac{\partial F(c,\mu)}{\partial \mu} = 0$ gives $c = 2$ or $\mu = 0$, and such points (c, μ) are not interior point of Π. So the maximum cannot attain in the interior of Π. Now to see on the boundary, by elementary calculus one can verify the following:

$$\max_{0\le\mu\le1} F(0,\mu) = F(0,0) = \frac{|\tau|}{3}, \qquad \max_{0\le\mu\le1} F(2,\mu) = F(2,0) = 2|\tau|^3$$

$$\max_{0\le c\le2} F(c,0) = F(2,0) = 2|\tau|^3, \qquad \max_{0\le c\le2} F(c,1) = F(2,1) = 2|\tau|^3.$$

Comparing these results, we get

$$\max_{\Pi} F(c,\mu) = 2|\tau|^3$$

(see Figure 1). Also note that

$$\tilde{p}(z) = 1 + \tau z + 3\tau^2 z^2 + 4\tau^3 z^3 + \cdots$$

by (8) with $k = 1$. From this equality, we obtain

$$p_1 = \tau, \qquad p_2 = 3\tau^2 \qquad \textit{and} \qquad p_3 = 4\tau^3.$$

On the other hand, the sharpness of the upper bounds of $|A_2|$ and $|A_3|$ is known from Theorem 4 and it is seen that the equality for $|A_4|$ is attained for the function $\tilde{p}(z)$. This evidently completes the proof of theorem. □

Theorem 6. *Let $f \in \mathcal{SL}^k$ be given by (1), and f^{-1} be the inverse function of f defined by (22). Then for any $\gamma \in \mathbb{C}$, we have*

$$\left|A_3 - \gamma A_2^2\right| \le \frac{k|\tau_k|}{2} \max\left\{1,\ 2\left|1-(1-\gamma)k^2\right| \frac{|\tau_k|}{k}\right\}.$$

Proof. By using (20), the desired result is obtained from the equality

$$A_3 - \gamma A_2^2 = -\frac{1}{2}\left[p_2 - (3-2\gamma)p_1^2\right] \qquad (\gamma \in \mathbb{C})$$

and Lemma 3. □

Letting $k = 1$ in Theorem 6, we obtain following consequence.

Corollary 5. *Let $f \in \mathcal{SL}$ be given by (1), and f^{-1} be the inverse function of f defined by (22). Then for any $\gamma \in \mathbb{C}$, we have*

$$\left|A_3 - \gamma A_2^2\right| \le \frac{|\tau|}{2} \max\{1,\ 2|\gamma\tau|\}.$$

If we take $\gamma = 1$ in Theorem 6, then we obtain the following result.

Corollary 6. *Let $f \in \mathcal{SL}^k$ be given by (1), and f^{-1} be the inverse function of f defined by (22). Then*

$$\left|A_3 - A_2^2\right| \le \begin{cases} \tau_k^2 & , \quad 0 < k \le \frac{2}{\sqrt{3}} \\ \\ \frac{k|\tau_k|}{2} & , \quad k \ge \frac{2}{\sqrt{3}} \end{cases}.$$

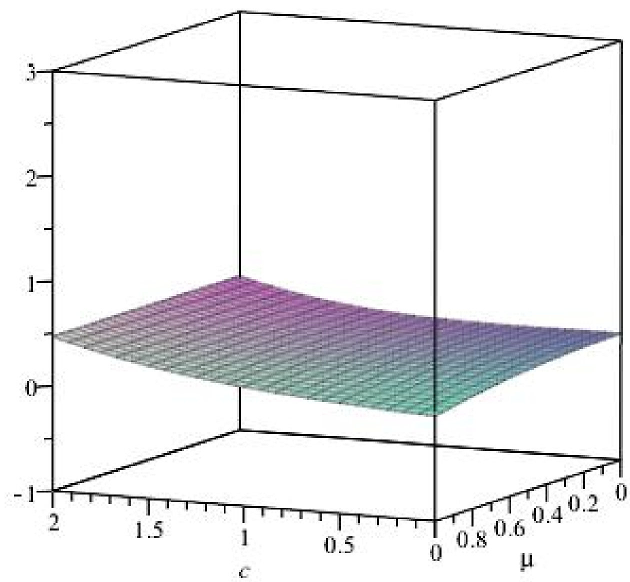

FIGURE 1. Mapping of $F(c,\mu)$ over Π

Letting $k=1$ in Corollary 6, we obtain the following consequence.

Corollary 7. *Let $f \in \mathcal{SL}$ be given by (1), and f^{-1} be the inverse function of f defined by (22). Then*

$$\left|A_3 - A_2^2\right| \leq \tau^2.$$

Theorem 7. *Let $f \in \mathcal{SL}^k$ be given by (1), and f^{-1} be the inverse function of f defined by (22). Then*

$$\left|A_2A_4 - A_3^2\right| \leq \begin{cases} \left(1+k^2\right)\tau_k^4 & , \quad 0 < k \leq \frac{2}{\sqrt{3}} \\ \tau_k^4 + \frac{k^3|\tau_k|^3}{2} & , \quad k \geq \frac{2}{\sqrt{3}} \end{cases}$$

and

$$\left|A_2A_3 - A_4\right| \leq \begin{cases} 4k\left|\tau_k\right|^3 & , \quad 0 < k \leq \frac{2}{\sqrt{3}} \\ k\left|\tau_k\right|^3 + \frac{3k^2\tau_k^2}{2} & , \quad k \geq \frac{2}{\sqrt{3}} \end{cases}.$$

Proof. Let $f \in \mathcal{SL}^k$ be of the form (1) and its inverse f^{-1} be given by (22). Then we obtain

$$\left|A_2A_4 - A_3^2\right| = \left|a_2^2\left(a_2^2 - a_3\right) + \left(a_2a_4 - a_3^2\right)\right|$$

and

$$\left|A_2A_3 - A_4\right| = \left|3a_2\left(a_2^2 - a_3\right) - \left(a_2a_3 - a_4\right)\right|.$$

Hence, applying triangle inequality, we have

$$\left|A_2A_4 - A_3^2\right| \leq \left|a_2\right|^2\left|a_3 - a_2^2\right| + \left|a_2a_4 - a_3^2\right|$$

and

$$\left|A_2A_3 - A_4\right| \leq 3\left|a_2\right|\left|a_3 - a_2^2\right| + \left|a_2a_3 - a_4\right|,$$

respectively. On the other hand, from Lemma 6, we obtain

$$\left|a_3 - a_2^2\right| \leq \begin{cases} \tau_k^2 & , \quad 0 < k \leq \frac{2}{\sqrt{3}} \\ \frac{k\left|\tau_k\right|}{2} & , \quad k \geq \frac{2}{\sqrt{3}} \end{cases}. \tag{24}$$

Furhermore, we get

$$\left|a_2\right| \leq k\left|\tau_k\right| \tag{25}$$

by using (23) together with Lemma 1. Now, by considering Lemma 7 and Lemma 8, we get the desired estimates. □

Letting $k = 1$ in Theorem 7, we obtain the following consequence.

Corollary 8. *Let $f \in \mathcal{SL}$ be given by* (1), *and f^{-1} be the inverse function of f defined by* (22). *Then*

$$\left|A_2A_4 - A_3^2\right| \leq 2\tau^4$$

and

$$\left|A_2A_3 - A_4\right| \leq 4\left|\tau\right|^3.$$

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