



Circulant m -Diagonal Matrices Associated with Chebyshev Polynomials

Ahmet Öteleş

Department of Mathematics, Faculty of Education, Dicle University, Diyarbakir, Turkey

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Abstract

In this study, we deal with an m banded circulant matrix, generally called circulant m -diagonal matrix. This special family of circulant matrices arise in many applications such as prediction, time series analysis, spline approximation, difference solution of partial differential equations, and so on. We firstly obtain the statements of eigenvalues and eigenvectors of circulant m -diagonal matrix based on the Chebyshev polynomials of the first and second kind. Then we present an efficient formula for the integer powers of this matrix family depending on the polynomials mentioned above. Finally, some illustrative examples are given by using maple software, one of computer algebra systems (CAS).

1. Introduction

Multiplying a vector by a circulant matrix is equivalent to a well-known operation called a circular convolution. Convolution operations, and so circulant matrices, arise in number of applications: digital signal processing, image compression, physics/engineering simulations, number theory, coding theory, cryptography, etc. Numerical solutions of certain types of elliptic and parabolic partial differential equations with periodic boundary conditions often involve linear systems associated with circulant matrices [1]-[3].

A certain type of transformation of a set of numbers can be represented as the multiplication of a vector by a square matrix. Repetition of the operation is equivalent to multiplying the original vector by a power of the matrix. Solving some difference equations, differential and delay differential equations and boundary value problems, we need to compute the arbitrary integer powers of a square matrix [4, 5]. The powers of matrices are thus of considerable importance.

Computing the integer powers of circulant matrices depending on Chebyshev polynomials recently has been a very attractive problem [6]-[13]. For example, Rimas obtained a general expression for the entries of the r^{th} power ($r \in \mathbb{N}$) of the $n \times n$ real symmetric circulant $\text{circ}_n(0, 1, 0, \dots, 0, 1)$ (see [6] or [7] for the odd case and [8] or [9] for the even case). In [10], Gutiérrez obtained a general expression for the entries of the positive integer powers of complex symmetric circulant matrix given by

$$\begin{aligned} \text{circ}_n \left(b_0, b_1, \dots, b_{\frac{n-1}{2}}, b_{\frac{n-1}{2}}, \dots, b_1 \right) & \quad \text{if } n \text{ is odd,} \\ \text{circ}_n \left(b_0, b_1, \dots, b_{\frac{n}{2}-1}, b_{\frac{n}{2}}, b_{\frac{n}{2}-1}, \dots, b_1 \right) & \quad \text{if } n \text{ is even.} \end{aligned} \quad (1.1)$$

by generalizing the results derived by Rimas in [6]-[9].

In [11], Köken et al. obtained a general expression for the entries of the r^{th} power ($r \in \mathbb{N}$) of odd order circulant matrices of the type $\text{circ}_n(0, a, 0, \dots, 0, b)$. In [12], we presented a single expression for the integer powers of the circulant matrix $\text{circ}_n(a_0, a_1, 0, \dots, 0, a_{-1})$ of odd and even order by generalizing the results derived by Köken et al. in [11].



In the current study, we consider an $n \times n$ circulant m -diagonal matrix A_n , that clearly is as,

$$A_n = \text{circ}_n(a_0, a_1, \dots, a_m, 0, \dots, 0, a_{-m}, \dots, a_{-1}) \quad (1.2)$$

$$= \begin{bmatrix} a_0 & a_1 & \cdots & a_m & 0 & \cdots & 0 & a_{-m} & \cdots & a_{-1} \\ a_{-1} & a_0 & a_1 & \ddots & a_m & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & a_{-1} & a_0 & a_1 & \ddots & \ddots & \ddots & \ddots & \ddots & a_{-m} \\ a_{-m} & \ddots & a_{-1} & a_0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & a_{-m} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & a_m & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & a_m \\ a_m & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & a_1 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & a_{-m} & \cdots & a_{-1} & a_0 & a_1 \\ a_1 & \cdots & a_m & 0 & \cdots & 0 & a_{-m} & \cdots & a_{-1} & a_0 \end{bmatrix}$$

for all $3 \leq n \in \mathbb{N}$, where $1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$ and

$$a_{-i} = a_{n-i}, \quad i = 1, \dots, m. \quad (1.3)$$

We organize this paper as the following parts. In Section 2, we give some fundamental notations, definitions and important properties that we will need for the next sections. In Section 3, we introduce Lemma 3.1 and Theorem 3.3 that respectively give the statements of eigenvalues and eigenvectors of circulant m -diagonal matrix in (1.2) depending on the Chebyshev polynomials of the first and second kind. In Section 4, we obtain an efficient expression for the integer power of this matrix by means of the polynomials mentioned above. In Section 5, some illustrative examples are given. Finally, we will finish the paper with two Maple procedures.

2. Preliminaries

In this part, we present some fundamental notations, definitions and necessary properties for the next parts. An $n \times n$ circulant matrix is defined in [14] as

$$C_n := \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \ddots & & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & c_2 \\ c_2 & & \ddots & \ddots & \ddots & c_1 \\ c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} & c_0 \end{bmatrix}$$

where $c_{ij} = c_{(j-i) \pmod n}$. It can be clearly seen from above that each row of C_n is a cyclic shift of the previous row. Since C_n has at most n distinct elements in each row, it is often represented by

$$C_n := \text{circ}_n(c_0, c_1, \dots, c_{n-1}).$$

Let $n \geq 1$ be a fixed integer and ω be the primitive n th root of unity; namely, $\omega = e^{\frac{2\pi}{n}} = \cos\left(\frac{2\pi}{n}\right) + \mathbf{i} \sin\left(\frac{2\pi}{n}\right)$, $\mathbf{i} = \sqrt{-1}$. The well-known eigenvalue decomposition of the matrix $C_n = \text{circ}_n(c_0, c_1, \dots, c_{n-1})$ is that

$$C_n = F_n^* D_n F_n \quad (2.1)$$

where $*$ denotes the conjugate transpose (i.e $F_n^* = \overline{F_n^T}$), F_n called $n \times n$ Fourier matrix that contains the eigenvectors of C_n such that

$$[F_n^*]_{u,v} = \frac{1}{\sqrt{n}} \omega^{(u-1)(v-1)}, \quad 1 \leq u, v \leq n$$

and $D_n = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ with

$$\lambda_k = \sum_{r=1}^n c_{r-1} \omega^{(k-1)(r-1)}, \quad 1 \leq k \leq n \quad (2.2)$$

are the eigenvalues of C_n [14].

It can be easily seen that the matrices F_n and F_n^* are symmetric:

$$F_n = F_n^T, F_n^* = (F_n^*)^T = \overline{F_n} \tag{2.3}$$

where we can deduce $[F_n]_{u,v} = [F_n]_{v,u}$. It is also one of fundamental property that the matrix F_n is unitary: $F_n F_n^* = F_n^* F_n = I$ [14].

In [15], we have the eigenvector $f^{(k)}$ of C_n corresponding to the eigenvalue λ_k in (2.2) as the following

$$f^{(k)} = \frac{1}{\sqrt{n}} \left(1, \omega^{k-1}, \omega^{2(k-1)}, \dots, \omega^{(n-1)(k-1)} \right)^T. \tag{2.4}$$

Since the product of two circulant matrices is again a circulant matrix, the r^{th} power ($r \in \mathbb{N}$) of C_n is also circulant and it is, from the well-known expression (2.1), obtained as

$$C_n^r = F_n^* D_n^r F_n = F_n^* \text{diag}(\lambda_1^r, \lambda_2^r, \dots, \lambda_n^r) F_n. \tag{2.5}$$

If C_n is nonsingular, then the expression (2.5) applies to negative integers.

Definition 2.1. The Chebyshev polynomial $T_n(x)$ of the first kind is a polynomial in x of degree n , defined by the relation

$$T_n(x) = \cos n\theta \quad \text{when } x = \cos \theta.$$

Definition 2.2. The Chebyshev polynomial $U_n(x)$ of the second kind is a polynomial of degree n in x defined by

$$U_n(x) = \sin(n+1)\theta / \sin \theta \quad \text{when } x = \cos \theta.$$

One can reach the following result about Chebyshev polynomials in [16]:

Let $T_k(x)$ and $U_k(x)$ ($k \in \mathbb{N} \cup \{0\}$) be the k^{th} degree Chebyshev polynomials of the first and second kind, respectively. Then

$$T_k(x) = \cos(k \arccos x) \quad \text{and} \quad U_k(x) = \frac{\sin((k+1) \arccos x)}{\sin(\arccos x)} \tag{2.6}$$

for $-1 \leq x \leq 1$. Moreover, one can find more applications related this polynomials in [17]-[19].

3. Eigenvalues and eigenvectors of circulant m -diagonal matrix

In this part, we give the expressions of eigenvalues and eigenvectors of $A_n = \text{circ}_n(a_0, a_1, \dots, a_m, 0, \dots, 0, a_{-m}, \dots, a_{-1})$ depending on Chebyshev polynomials of the first and second kind.

Lemma 3.1. Consider $3 \leq n \in \mathbb{N}$, $1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$ and $a_i \in \mathbb{R}$ ($i = 0, \pm 1, \dots, \pm m$). Let $A_n = \text{circ}_n(a_0, a_1, \dots, a_m, 0, \dots, 0, a_{-m}, \dots, a_{-1})$ be an $n \times n$ circulant matrix and $\alpha_k = \cos \frac{2\pi(k-1)}{n}$ for every $1 \leq k \leq n$. Then the eigenvalues of A_n are

$$\lambda_k = a_0 + \sum_{l=1}^m \left((a_l + a_{-l}) T_l(\alpha_k) + \mathbf{i}(a_l - a_{-l}) \text{sgn} \left(\frac{n}{2} + 1 - k \right) \sqrt{1 - \alpha_k^2} \lim_{j \rightarrow k} U_{l-1}(\alpha_j) \right) \tag{3.1}$$

where λ_k is the k^{th} eigenvalue of A_n and sgn denotes the signum function.

Proof. Taking into account (2.2), (1.3) and $\omega^{(k-1)(n+2-r-1)} = \omega^{-(k-1)(r-1)}$ for all $2 \leq r \leq n$ (see [10]), we can write λ_k as

$$\begin{aligned} \lambda_k &= a_0 + a_1 \omega^{(k-1)1} + \dots + a_m \omega^{(k-1)m} + a_{n-m} \omega^{(k-1)(n-m)} + \dots + a_{n-1} \omega^{(k-1)(n-1)} \\ &= a_0 + a_1 \omega^{(k-1)} + \dots + a_m \omega^{(k-1)m} + a_{-m} \omega^{-(k-1)m} + \dots + a_{-1} \omega^{-(k-1)}. \end{aligned}$$

From the definition of ω , we get

$$\lambda_k = a_0 + \sum_{l=1}^m \left((a_l + a_{-l}) \cos \frac{2\pi(k-1)l}{n} + \mathbf{i}(a_l - a_{-l}) \sin \frac{2\pi(k-1)l}{n} \right). \tag{3.2}$$

Observe that from (2.6), we have

$$T_m \left(\cos \frac{2\pi(k-1)}{n} \right) = \cos \frac{2\pi(k-1)m}{n},$$

and

$$U_{m-1} \left(\cos \frac{2\pi(k-1)}{n} \right) = \frac{\sin \frac{2\pi(k-1)m}{n}}{\sin \frac{2\pi(k-1)}{n}}$$

where there exists indeterminate form $0/0$ for $k = 1$ and $k = \frac{n}{2} + 1$. Then we can construct the expression (3.2) as

$$\lambda_k = a_0 + \sum_{l=1}^m \left((a_l + a_{-l}) T_l \left(\cos \frac{2\pi(k-1)}{n} \right) + \mathbf{i}(a_l - a_{-l}) \sin \frac{2\pi(k-1)}{n} \lim_{j \rightarrow k} U_{l-1} \left(\cos \frac{2\pi(j-1)}{n} \right) \right).$$

Consequently, we reach the desired result by transforming $\cos \frac{2\pi(k-1)}{n} = \alpha_k$ and then

$$\sin \frac{2\pi(k-1)}{n} = \begin{cases} \sqrt{1-\alpha_k^2} & \text{if } \frac{n}{2} + 1 - k > 0, \\ 0 & \text{if } \frac{n}{2} + 1 - k = 0, \\ -\sqrt{1-\alpha_k^2} & \text{if } \frac{n}{2} + 1 - k < 0. \end{cases} \quad (3.3)$$

□

With the help of Lemma 3.1, we reach a nice result for the eigenvalues of $A_n = \text{circ}_n(a_0, a_1, \dots, a_m, 0, \dots, 0, a_{-m}, \dots, a_{-1})$. Since

$$\cos \frac{2\pi(n+2-k-1)}{n} = \cos \frac{2\pi(k-1)}{n}$$

and

$$\sin \frac{2\pi(n+2-k-1)}{n} = -\sin \frac{2\pi(k-1)}{n}.$$

We obtain that $\overline{\lambda_k} = \lambda_{n+2-k}$ ($2 \leq k \leq n$) from (3.2). Clearly, if we rewrite this eigenvalues in a diagonal matrix, then

$$\begin{aligned} D_n &= \text{diag} \left(\lambda_1, \lambda_2, \dots, \lambda_{\frac{n+1}{2}}, \overline{\lambda_{\frac{n+1}{2}}}, \dots, \overline{\lambda_2} \right) & \text{if } n \text{ is odd,} \\ D_n &= \text{diag} \left(\lambda_1, \lambda_2, \dots, \lambda_{\frac{n}{2}}, \lambda_{\frac{n}{2}+1}, \overline{\lambda_{\frac{n}{2}}}, \dots, \overline{\lambda_2} \right) & \text{if } n \text{ is even.} \end{aligned} \quad (3.4)$$

If we take $n = 8$ and $m = 2$ for the matrix A_n in (1.2), then, from Lemma 3.1, we get

$$\begin{aligned} \lambda_3 &= a_0 + (a_1 + a_{-1}) \cos \frac{\pi}{2} + \mathbf{i}(a_1 - a_{-1}) \sin \frac{\pi}{2} \\ &\quad + (a_2 + a_{-2}) \cos \pi + \mathbf{i}(a_2 - a_{-2}) \sin \pi \\ &= a_0 - a_2 - a_{-2} + \mathbf{i}(a_1 - a_{-1}) \end{aligned}$$

and

$$\begin{aligned} \lambda_7 &= a_0 + (a_1 + a_{-1}) \cos \frac{3\pi}{2} + \mathbf{i}(a_1 - a_{-1}) \sin \frac{3\pi}{2} \\ &\quad + (a_2 + a_{-2}) \cos 3\pi + \mathbf{i}(a_2 - a_{-2}) \sin 3\pi \\ &= a_0 - a_2 - a_{-2} - \mathbf{i}(a_1 - a_{-1}). \end{aligned}$$

As can be seen above, $\lambda_7 = \overline{\lambda_3}$.

Corollary 3.2. Consider $3 \leq n \in \mathbb{N}$, $1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$ and $a_i \in \mathbb{R}$ ($i = 0, 1, \dots, m$). Let $B_n = \text{circ}_n(a_0, a_1, \dots, a_m, 0, \dots, 0, a_m, \dots, a_1)$ be an $n \times n$ symmetric circulant matrix and $\alpha_k = \cos \frac{2\pi(k-1)}{n}$ for every $1 \leq k \leq n$. Then the eigenvalues of B_n are

$$\mu_k = a_0 + 2 \sum_{l=1}^m a_l T_l(\alpha_k) \quad (3.5)$$

where μ_k is the k^{th} eigenvalue of B_n .

Proof. The proof can be straightforwardly obtained from Lemma 3.1. □

Since $\cos \frac{2\pi(n+2-k-1)}{n} = \cos \frac{2\pi(k-1)}{n}$, we can easily see that $\mu_k = \mu_{n+2-k}$ ($2 \leq k \leq n$) from Corollary 3.2. Clearly, if we rewrite this eigenvalues in a diagonal matrix again, then

$$\begin{aligned} D_n &= \text{diag} \left(\mu_1, \mu_2, \dots, \mu_{\frac{n+1}{2}}, \mu_{\frac{n+1}{2}}, \dots, \mu_2 \right) & \text{if } n \text{ is odd,} \\ D_n &= \text{diag} \left(\mu_1, \mu_2, \dots, \mu_{\frac{n}{2}}, \mu_{\frac{n}{2}+1}, \mu_{\frac{n}{2}}, \dots, \mu_2 \right) & \text{if } n \text{ is even.} \end{aligned}$$

If we take $n = 8$ and $m = 2$ for the matrix $B_n = \text{circ}_n(a_0, a_1, \dots, a_m, 0, \dots, 0, a_m, \dots, a_1)$, then, from Corollary 3.2, we get

$$\mu_3 = a_0 + 2 \left(a_1 \cos \frac{\pi}{2} + a_2 \cos \pi \right) = a_0 - 2a_2$$

and

$$\mu_7 = a_0 + 2 \left(a_1 \cos \frac{3\pi}{2} + a_2 \cos 3\pi \right) = a_0 - 2a_2.$$

As can be seen above, $\mu_7 = \mu_3$.

Now, from the expression (2.4), let us give the following result for the eigenvectors of $A_n = \text{circ}_n(a_0, a_1, \dots, a_m, 0, \dots, 0, a_{-m}, \dots, a_{-1})$ depending on Chebyshev polynomials of the first and second kind

Theorem 3.3. Let $3 \leq n \in \mathbb{N}$, $1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$ and $a_i \in \mathbb{R}$ ($i = 0, \pm 1, \dots, \pm m$) and $\alpha_k = \cos \frac{2\pi(k-1)}{n}$ for every $1 \leq k \leq n$. Then the eigenvector F_k of the matrix $A_n = \text{circ}_n(a_0, a_1, \dots, a_m, 0, \dots, 0, a_{-m}, \dots, a_{-1})$ corresponding to the eigenvalue λ_k given by (3.1) is

$$F_k = \begin{bmatrix} T_0(\alpha_k) + \text{isgn} \left(\frac{n}{2} + 1 - k \right) \sqrt{1 - \alpha_k^2} \lim_{j \rightarrow k} U_{-1}(\alpha_j) \\ T_1(\alpha_k) + \text{isgn} \left(\frac{n}{2} + 1 - k \right) \sqrt{1 - \alpha_k^2} \lim_{j \rightarrow k} U_0(\alpha_j) \\ \vdots \\ T_{n-1}(\alpha_k) + \text{isgn} \left(\frac{n}{2} + 1 - k \right) \sqrt{1 - \alpha_k^2} \lim_{j \rightarrow k} U_{n-2}(\alpha_j) \end{bmatrix}. \tag{3.6}$$

Proof. From (2.4), (2.6) and (3.3), the result can be easily obtained. □

We must note that each one of all circulant matrices also have the eigenvectors generated by F_k given by (3.6)

4. Integer powers of circulant m -diagonal matrix

In this part, by using the symmetric relationship between the eigenvalues in (3.4), we give the efficient expression to compute the integer power of the circulant m -diagonal matrix $A_n = \text{circ}_n(a_0, a_1, \dots, a_m, 0, \dots, 0, a_{-m}, \dots, a_{-1})$ based on Chebyshev polynomials of the first and second kind such that the method is faster than any of the classical methods which find the powers of A_n with an amount of computations.

Theorem 4.1. Consider $3 \leq n \in \mathbb{N}$, $1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$ and $a_i \in \mathbb{R}$ ($i = 0, \pm 1, \dots, \pm m$). Let $A_n = \text{circ}_n(a_0, a_1, \dots, a_m, 0, \dots, 0, a_{-m}, \dots, a_{-1})$ be an $n \times n$ nonsingular circulant m -diagonal matrix and $\alpha_k = \cos \frac{2\pi(k-1)}{n}$ for every $1 \leq k \leq n$. Then the (u, v) th entry of A_n^r is that

$$[A_n^r]_{u,v} = \frac{1}{n} (S_1 + S_2)$$

for all $r \in \mathbb{Z}$ and $1 \leq u, v \leq n$, where S_1 and S_2 are respectively such that

$$S_1 = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor + 1} \left(a_0 + \sum_{l=1}^m \left((a_l + a_{-l}) T_l(\alpha_k) + \mathbf{i}(a_l - a_{-l}) \text{sgn} \left(\frac{n}{2} + 1 - k \right) \sqrt{1 - \alpha_k^2} \lim_{j \rightarrow k} U_{l-1}(\alpha_j) \right) \right)^r \times \left(T_{|u-v|}(\alpha_k) + \text{isgn}(u-v) \text{sgn} \left(\frac{n}{2} + 1 - k \right) \sqrt{1 - \alpha_k^2} \lim_{j \rightarrow k} U_{|u-v|-1}(\alpha_j) \right)$$

and

$$S_2 = \sum_{k=2}^{\lfloor \frac{n+1}{2} \rfloor} \left(a_0 + \sum_{l=1}^m \left((a_l + a_{-l}) T_l(\alpha_k) - \mathbf{i}(a_l - a_{-l}) \text{sgn} \left(\frac{n}{2} + 1 - k \right) \sqrt{1 - \alpha_k^2} \lim_{j \rightarrow k} U_{l-1}(\alpha_j) \right) \right)^r \times \left(T_{|u-v|}(\alpha_k) - \text{isgn}(u-v) \text{sgn} \left(\frac{n}{2} + 1 - k \right) \sqrt{1 - \alpha_k^2} \lim_{j \rightarrow k} U_{|u-v|-1}(\alpha_j) \right).$$

Here $\lfloor x \rfloor$ and sgn denote the largest integer less than or equal to x and the signum function, respectively.

Proof. By using (2.5) and (2.3), we get

$$\begin{aligned} [A_n^r]_{u,v} &= [F_n^* D_n^r F_n]_{u,v} = \sum_{k=1}^n [F_n^*]_{u,k} [D_n^r F_n]_{k,v} = \sum_{k=1}^n [F_n^*]_{u,k} \lambda_k^r [F_n]_{k,v}, \\ &= \sum_{k=1}^n \lambda_k^r [F_n^*]_{u,k} [F_n]_{v,k} = \frac{1}{n} \sum_{k=1}^n \lambda_k^r \omega^{(u-1)(k-1)} \omega^{-(v-1)(k-1)} \end{aligned}$$

and then

$$[A_n^r]_{u,v} = \frac{1}{n} \sum_{k=1}^n \lambda_k^r \omega^{(k-1)(u-v)} \quad (4.1)$$

where λ_k is already obtained as in (3.1). Since $\overline{\lambda_k} = \lambda_{n+2-k}$, $2 \leq k \leq n$ and $\omega^{(n+2-k-1)(u-v)} = \omega^{-(k-1)(u-v)}$, the second half of the sum in (4.1) can be written as

$$\sum_{k=\frac{n+1}{2}+1}^n \lambda_k^r \omega^{(k-1)(u-v)} = \sum_{k=2}^{\frac{n+1}{2}} \lambda_{n+2-k}^r \omega^{(n+2-k-1)(u-v)} = \sum_{k=2}^{\frac{n+1}{2}} \overline{\lambda_k}^r \omega^{-(k-1)(u-v)}$$

for the case n is odd. The same observations can be applied in the case n even and the result is that

$$[A_n^r]_{u,v} = \frac{1}{n} \left[\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor + 1} \lambda_k^r \omega^{(k-1)(u-v)} + \sum_{k=2}^{\lfloor \frac{n+1}{2} \rfloor} \overline{\lambda_k}^r \omega^{-(k-1)(u-v)} \right] = \frac{1}{n} (S_1 + S_2).$$

Thus, from the expression above, we can write

$$S_1 = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor + 1} \lambda_k^r \left(\cos \frac{2\pi(k-1)(u-v)}{n} + \mathbf{i} \sin \frac{2\pi(k-1)(u-v)}{n} \right)$$

and

$$S_2 = \sum_{k=2}^{\lfloor \frac{n+1}{2} \rfloor} \overline{\lambda_k}^r \left(\cos \frac{2\pi(k-1)(u-v)}{n} - \mathbf{i} \sin \frac{2\pi(k-1)(u-v)}{n} \right).$$

Since, from (2.6),

$$T_{|u-v|} \left(\cos \frac{2\pi(k-1)}{n} \right) = \cos \frac{2\pi(k-1)(u-v)}{n}$$

and

$$U_{|u-v|-1} \left(\cos \frac{2\pi(k-1)}{n} \right) = \operatorname{sgn}(u-v) \frac{\sin \frac{2\pi(k-1)(u-v)}{n}}{\sin \frac{2\pi(k-1)}{n}}$$

with indeterminate form $0/0$ for $k=1$ and $k=\frac{n}{2}+1$, then

$$S_1 = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor + 1} \lambda_k^r \left(T_{|u-v|} \left(\cos \frac{2\pi(k-1)}{n} \right) + \mathbf{i} \operatorname{sgn}(u-v) \sin \frac{2\pi(k-1)}{n} \lim_{j \rightarrow k} U_{|u-v|-1} \left(\cos \frac{2\pi(j-1)}{n} \right) \right),$$

and

$$S_2 = \sum_{k=2}^{\lfloor \frac{n+1}{2} \rfloor} \overline{\lambda_k}^r \left(T_{|u-v|} \left(\cos \frac{2\pi(k-1)}{n} \right) - \mathbf{i} \operatorname{sgn}(u-v) \sin \frac{2\pi(k-1)}{n} \lim_{j \rightarrow k} U_{|u-v|-1} \left(\cos \frac{2\pi(j-1)}{n} \right) \right).$$

The theorem follows by substituting λ_k in (3.1) and $\cos \frac{2\pi(k-1)}{n} = \alpha_k$ into the above expressions. \square

From (2.5), we have the r^{th} power of any (symmetric) circulant matrix is also a (symmetric) circulant matrix.

Consider $a_i \in \mathbb{R}$ ($i=0, \pm 1$) and let $A_4 = \mathbf{circ}_4(a_0, a_1, 0, a_{-1})$ be circulant tridiagonal matrix. Then, by using Theorem 4.1, we get $A_4^r = \mathbf{circ}_4(\tau_0, \tau_1, \tau_2, \tau_3)$ with

$$\begin{aligned} \tau_0 &= \frac{1}{4} [(a_0 + a_1 + a_{-1})^r + (a_0 - (a_1 + a_{-1}))^r + z^r + \overline{z}^r], \\ \tau_1 &= \frac{1}{4} [(a_0 + a_1 + a_{-1})^r - (a_0 - (a_1 + a_{-1}))^r - \mathbf{i}z^r + \mathbf{i}\overline{z}^r], \\ \tau_2 &= \frac{1}{4} [(a_0 + a_1 + a_{-1})^r + (a_0 - (a_1 + a_{-1}))^r - \mathbf{i}z^r - \mathbf{i}\overline{z}^r], \\ \tau_3 &= \frac{1}{4} [(a_0 + a_1 + a_{-1})^r - (a_0 - (a_1 + a_{-1}))^r + \mathbf{i}z^r - \mathbf{i}\overline{z}^r] \end{aligned}$$

where $z = a_0 + \mathbf{i}(a_1 - a_{-1})$.

If we take $m=1$ in Theorem 4.1, the expression given in [12, Theorem 2.1] can be easily seen.

Theorem 4.1 allows us to significantly reduce the computing process while finding the integer powers of the circulant m -diagonal matrix $A_n = \mathbf{circ}_n(a_0, a_1, \dots, a_m, 0, \dots, 0, a_{-m}, \dots, a_{-1})$

Theorem 4.2. Consider $3 \leq n \in \mathbb{N}$, $1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$ and $a_i \in \mathbb{R}$ ($i = 0, 1, \dots, m$). Let $B_n = \text{circ}_n(a_0, a_1, \dots, a_m, 0, \dots, 0, a_m, \dots, a_1)$ be an $n \times n$ nonsingular symmetric circulant m -diagonal matrix and $\alpha_k = \cos \frac{2\pi(k-1)}{n}$ for every $1 \leq k \leq n$. Then the (u, v) th entry of B_n^r is that

$$[B_n^r]_{u,v} = \frac{1}{n} \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor + 1} l_{n-2k+2} \left(a_0 + 2 \sum_{l=1}^m a_l T_l(\alpha_k) \right)^r T_{|u-v|}(\alpha_k)$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x and

$$l_s = \begin{cases} 1 & \text{if } s \in \{0, n\} \\ 2 & \text{in other cases.} \end{cases}$$

Proof. By using Theorem 4.1, we get

$$S_1 = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor + 1} \left(a_0 + 2 \sum_{l=1}^m a_l T_l(\alpha_k) \right)^r T_{|u-v|}(\alpha_k) + \text{isgn}(u-v) \text{sgn} \left(\frac{n}{2} + 1 - k \right) \sqrt{1 - \alpha_k^2} U_{|u-v|-1}(\alpha_k)$$

and

$$S_2 = \sum_{k=2}^{\lfloor \frac{n+1}{2} \rfloor} \left(a_0 + 2 \sum_{l=1}^m a_l T_l(\alpha_k) \right)^r T_{|u-v|}(\alpha_k) - \text{isgn}(u-v) \text{sgn} \left(\frac{n}{2} + 1 - k \right) \sqrt{1 - \alpha_k^2} U_{|u-v|-1}(\alpha_k).$$

Since

$$\begin{cases} \lfloor \frac{n+1}{2} \rfloor = \frac{n+1}{2} \text{ and } \lfloor \frac{n}{2} \rfloor + 1 = \frac{n+1}{2} & \text{if } n \text{ is odd,} \\ \lfloor \frac{n+1}{2} \rfloor = \frac{n}{2} \text{ and } \lfloor \frac{n}{2} \rfloor + 1 = \frac{n}{2} + 1 & \text{if } n \text{ is even.} \end{cases}$$

Then

$$\begin{aligned} [B_n^r]_{u,v} &= \frac{1}{n} (S_1 + S_2) \\ &= \begin{cases} \frac{1}{n} \left[\left(a_0 + 2 \sum_{l=1}^m a_l \right)^r + 2 \sum_{k=2}^{\frac{n+1}{2}} \left(a_0 + 2 \sum_{l=1}^m a_l T_l(\alpha_k) \right)^r T_{|u-v|}(\alpha_k) \right] & \text{if } n \text{ is odd,} \\ \frac{1}{n} \left[\left(a_0 + 2 \sum_{l=1}^m a_l \right)^r + 2 \sum_{k=2}^{\frac{n}{2}} \left(a_0 + 2 \sum_{l=1}^m a_l T_l(\alpha_k) \right)^r T_{|u-v|}(\alpha_k) + \right. \\ \left. \left(a_0 + 2 \sum_{l=1}^m a_l T_l(\alpha_{\frac{n}{2}+1}) \right)^r T_{|u-v|}(\alpha_{\frac{n}{2}+1}) \right] & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

Therefore,

$$[B_n^r]_{u,v} = \frac{1}{n} \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor + 1} l_{n-2k+2} \left(a_0 + 2 \sum_{l=1}^m a_l T_l(\alpha_k) \right)^r T_{|u-v|}(\alpha_k)$$

which is desired. □

Consider $a_i \in \mathbb{R}$ ($i = 0, 1, 2$) and let $B_5 = \text{circ}_5(a_0, a_1, a_2, a_2, a_1)$ be a symmetric circulant pentadiagonal matrix. Then, from Theorem 4.2, we get $B_5^r = \text{circ}_4(\tau_0, \tau_1, \tau_2, \tau_2, \tau_1)$ with

$$\begin{aligned} \tau_0 &= \frac{1}{5} \left[(a_0 + 2a_1 + 2a_2)^r + 2 \left(a_0 + \frac{1}{\phi} a_1 - \phi a_2 \right)^r + 2 \left(a_0 - \phi a_1 + \frac{1}{\phi} a_2 \right)^r \right], \\ \tau_1 &= \frac{1}{5} \left[(a_0 + 2a_1 + 2a_2)^r + \frac{1}{\phi} \left(a_0 + \frac{1}{\phi} a_1 - \phi a_2 \right)^r - \phi \left(a_0 - \phi a_1 + \frac{1}{\phi} a_2 \right)^r \right], \\ \tau_2 &= \frac{1}{5} \left[(a_0 + 2a_1 + 2a_2)^r - \phi \left(a_0 + \frac{1}{\phi} a_1 - \phi a_2 \right)^r + \frac{1}{\phi} \left(a_0 - \phi a_1 + \frac{1}{\phi} a_2 \right)^r \right] \end{aligned}$$

where ϕ denotes the golden ratio.

Now, if we consider

$$\begin{cases} m = \lfloor \frac{n-1}{2} \rfloor = \frac{n-1}{2} & \text{for } n \text{ is odd,} \\ m = \lfloor \frac{n-1}{2} \rfloor + 1 = \frac{n}{2} & \text{for } n \text{ is even,} \end{cases}$$

in the symmetric circulant m -diagonal matrix $B_n = \text{circ}_n(a_0, a_1, \dots, a_m, 0, \dots, 0, a_m, \dots, a_1)$ and $a_{\frac{n}{2}} \neq 0$, then we get the symmetric circulant matrix in (1.1) discussed by Gutiérrez in [10]. And so, with the help of Theorem 4.2, we can straightforwardly reach the expression obtained by Gutiérrez in [10, Theorem 1] for positive integer powers of the matrix B_n in (1.1).

5. Illustrative examples

In this part, we give some illustrative examples. We will utilize Maple software in our calculations.

Example 5.1. Let $A_6 = \text{circ}_6(5, 4, 9, 0, 8, -2)$ be a circulant pentadiagonal matrix. we find the eigenvalues of A_6 by using (3.1) as

$$\begin{aligned}\lambda_1 &= 24, \\ \lambda_2 &= -2,5000 + 6,0621i, \\ \lambda_3 &= -4,5000 + 4,3301i, \\ \lambda_4 &= 20, \\ \lambda_5 = \overline{\lambda_3} &= -4,5000 - 4,3301i, \\ \lambda_6 = \overline{\lambda_2} &= -2,5000 - 6,0621i\end{aligned}$$

and from Theorem 4.1, the entries of A_6^3 as

$$A_6^3 = \text{circ}_6(3778, 1008, 3483, 938, 3651, 966).$$

Example 5.2. Let $A_9 = \text{circ}_9(-2, 3, -4, 9, 0, 0, 6, 5, -1)$ be a circulant heptadiagonal matrix. we find the eigenvalues of A_9 by using (3.1) as

$$\begin{aligned}\lambda_1 &= 16, \\ \lambda_2 &= -7,7942 - 3,6940i, \\ \lambda_3 &= -10,0923 - 1,73701i, \\ \lambda_4 &= 11,5000 + 11,2583i, \\ \lambda_5 &= -10,6133 + 9,7512i, \\ \lambda_6 = \overline{\lambda_5} &= -10,6133 - 9,7512i, \\ \lambda_7 = \overline{\lambda_4} &= 11,5000 - 11,2583i, \\ \lambda_8 = \overline{\lambda_3} &= -10,0923 + 1,73701i, \\ \lambda_9 = \overline{\lambda_2} &= -7,7942 + 3,6940i\end{aligned}$$

and from Theorem 4.1, the entries of A_9^4 as

$$A_9^4 = \text{circ}_9(-15410, 26041, 7866, -5401, 16331, 13209, -2024, 3458, 21466).$$

Example 5.3. Let $B_7 = \text{circ}_7(1, -3, 2, 0, 0, 2, -3)$ be a symmetric circulant pentadiagonal matrix. we find the eigenvalues of B_7 by using (3.5) as

$$\begin{aligned}\lambda_1 &= -1, \\ \lambda_2 &= -3,6310, \\ \lambda_3 &= -1,2687, \\ \lambda_4 &= 8,8987, \\ \lambda_5 = \lambda_4 &= 8,8987, \\ \lambda_6 = \lambda_3 &= -1,2687, \\ \lambda_7 = \lambda_2 &= -3,6310\end{aligned}$$

and from Theorem 4.2, the entries of B_7^5 as

$$B_7^5 = \text{circ}_7(15771, -14485, 9987, -3388, -3388, 9987, -14485).$$

Examples 5.1, 5.2 and 5.3 can be also confirmed by means of Maple procedures given by Appendix A and B.

Appendix A. Following Maple procedure firstly generates a $n \times n$ circulant heptadiagonal matrix $A_n = \text{circ}_n(a_0, a_1, a_2, a_3, 0, \dots, 0, a_{-3}, a_{-2}, a_{-1})$ and then compute eigenvalues and the k^{th} power ($r \in \mathbb{Z}$) of it.

restart:

with(LinearAlgebra):

m:=3:

n:=n:

r:=r:

a[0]:=a[0]:

a[1]:=a[1]:

a[-1]:=a[-1]:

a[2]:=a[2]:

a[-2]:=a[-2]:


```

a[3]:=a[3]':
a[-3]:=a[-3]':
f:=(i,j)→ piecewise(i=j,a[0],j>i and j-i<m+1,a[j-i],i>j and i-j<m+1,a[j-i],n-j+i<m+1,a[j-i-n],n-i+j<m+1,a[j-i+n]):
A[n]:=Matrix(n,n,f);
alpha:=k → evalf(cos(2*Pi*(k-1)/n)):
lambda:=k → evalf((a[0]+sum((a[1]+a[-1])*ChebyshevT(l,alpha(k))+I*(a[1]-a[-1])*signum((n/2)+1-k)*sqrt(1-(alpha(k))^2)*limit
(ChebyshevU(1-1,alpha(j)),j=k),l=1..m))):
g:=(i,j)→ piecewise(i=j,lambda(i),0):
p:=(u,v)→evalf((1/n)*(sum((lambda(k)^r)*(ChebyshevT(abs(u-v),alpha(k))+I*signum(u-v)*signum((n/2)+1-k)*
sqrt(1-(alpha(k))^2)*limit(ChebyshevU(abs(u-v)-1,alpha(j)),j=k),k=1..floor(n/2)+1)+sum(conjugate(lambda(k))^r*
(ChebyshevT(abs(u-v),alpha(k))-I*signum(u-v)*signum((n/2)+1-k)*sqrt(1-(alpha(k))^2)*ChebyshevU(abs(u-v)-1,alpha(k))),
k=2..floor((n+1)/2))))):
eigenvalues_of_A[n]:=Matrix(n,n,g);
the_rth_power_of_A[n]:=Matrix(n,n,p);

```

Appendix B. Following Maple procedure firstly generates a $n \times n$ symmetric circulant heptadiagonal matrix $B_n = \text{circ}_n(a_0, a_1, a_2, a_3, 0, \dots, 0, a_3, a_2, a_1)$ and then compute eigenvalues and the k^{th} power ($r \in \mathbb{Z}$) of it.

```

restart:
with(LinearAlgebra):
m:=3':
n:=n':
a[0]:=a[0]':
a[1]:=a[1]':
a[2]:=a[2]':
a[3]:=a[3]':
r:=r':
f:=(i,j)→ piecewise(i=j,a[0],i>j and i-j<m+1,a[i-j],i<j and j-i<m+1,
a[j-i],i<j and j-i>n-m-1,a[n-(j-i)],i>j and i-j>n-m-1,a[n-(i-j)],0):
B[n]:=Matrix(n,n,f);
alpha:=k → evalf(cos(2*Pi*(k-1)/n)):
mu:=k → evalf(a[0]+2*sum(a[l]*ChebyshevT(l,alpha(k)),l=1..m)):
g:=(i,j)→ piecewise(i=j,mu(i),0):
l:=(s) → piecewise(s=0,1,s=n,1,2):
p:=(u,v)→ evalf((1/n)*((sum(((l(n-2*k+2)*(mu(k))^r)*(ChebyshevT(abs(u-v),alpha(k))),k=1..floor(n/2)+1))))):
eigenvalues_of_B[n]:=Matrix(n,n,g);
the_rt_power_of_B[n]:=Matrix(n,n,p);

```

6. Conclusion

There has been recently increasing research interest in circulant matrices in several areas, such as digital signal processing, image compression, physics/engineering simulations, number theory, coding theory, cryptography, and, naturally, linear algebra. This paper present eigenvalues, eigenvectors, powers of circulant m -diagonal matrix which is one type of circulant matrices by using some famous relations on chebyshev polynomials.

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