CONSTRUCTIVE MATHEMATICAL ANALYSIS
4 (2021), No. 1, pp. 81-90
http://dergipark.gov.tr/en/pub/cma
ISSN 2651 - 2939

Research Article

Heun equations and combinatorial identities

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ABSTRACT. Heun functions are important for many applications in mathematics, physics and in thus in interdisciplinary phenomena modelling. They satisfy second order differential equations and are usually represented by power series. Closed forms and simpler polynomial representations are useful. Therefore, we study and derive closed forms for several families of Heun functions related to classical entropies. By comparing two expressions of the same Heun function, we get several combinatorial identities generalizing some classical ones.

Keywords: Heun functions, entropies, combinatorial identities.

2020 Mathematics Subject Classification: 33E30, 94A17, 05A19.

Dedicated to Professor Francesco Altomare, on occasion of his 70th birthday, with esteem and friendship.

1. INTRODUCTION

Consider the general Heun equation (see, e.g., [15], [8], [9] and the references therein)

$$u''(x)+\left(\frac{\gamma}{x}+\frac{\delta}{x-1}+\frac{\epsilon}{x-a}\right)u'(x)+\frac{\alpha\beta x-q}{x(x-1)(x-a)}u(x)=0, \tag{1.1}$$

where $a\notin\{0,1\}$, $\gamma\notin\{0,-1,-2,\dots\}$ and $\alpha+\beta+1=\gamma+\delta+\epsilon$. Its solution $u(x)$ normalized by the condition $u(0)=1$ is called the *(local) Heun function* and is denoted by $Hl(a,q;\alpha,\beta;\gamma,\delta;x)$.

The confluent Heun equation is

$$u''(x)+\left(4p+\frac{\gamma}{x}+\frac{\delta}{x-1}\right)u'(x)+\frac{4p\alpha x-\sigma}{x(x-1)}u(x)=0, \tag{1.2}$$

where $p\neq 0$. The solution $u(x)$ normalized by $u(0)=1$ is called the *confluent Heun function* and is denoted by $HC(p,\gamma,\delta,\alpha,\sigma;x)$.

It was proved in [14] that

$$Hl\left(\frac{1}{2},-n;-2n,1;1,1;x\right)=\sum_{k=0}^{n}\left(\binom{n}{k}x^k(1-x)^{n-k}\right)^2, \tag{1.3}$$

$$Hl\left(\frac{1}{2},n;2n,1;1,1;-x\right)=\sum_{k=0}^{\infty}\left(\binom{n+k-1}{k}x^k(1+x)^{-n-k}\right)^2, \tag{1.4}$$

$$HC\left(n,1,0,\frac{1}{2},2n;x\right)=\sum_{k=0}^{\infty}\left(e^{-nx}\frac{(nx)^k}{k!}\right)^2. \tag{1.5}$$

Received: 14.10.2020; Accepted: 16.12.2020; Published Online: 01.03.2021
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DOI: 10.33205/cma.810478

More general results, providing closed forms of the functions $Hl\left(\frac{1}{2}, -2n\theta; -2n, 2\theta; \gamma, \gamma; x\right)$ and $Hl\left(\frac{1}{2}, 2n\theta; 2n, 2\theta; \gamma, \gamma; x\right)$, and explicit expressions for some confluent Heun functions can be found in [4].

In this paper, we give closed forms for several families of Heun functions and confluent Heun functions, extending (1.3), (1.4) and (1.5). Basic tools will be the results of [7] and [16] concerning the derivatives of Heun functions, respectively confluent Heun functions; see also [13] and [4].

By comparing two expressions of the same Heun function, we get several combinatorial identities; very particular forms of them can be traced in the classical book [5]. Recently, Ulrich Abel and Georg Arends gave in [1] purely combinatorial proofs of some similar identities presented in [2].

It is well known that the Heun functions and the Heun equations have important applications in Physics; see, e.g., [6]. Let us mention that the families of (confluent) Heun functions investigated in this paper are naturally related to some classical entropies: see [13], [14], [4], [3], [12].

Throughout the paper, we shall use the notation

$$(x)_0 := 1, \quad (x)_k := x(x+1)\dots(x+k-1), \quad k \geq 1,$$

$$a_{nj} := 4^{-n}\binom{2j}{j}\binom{2n-2j}{n-j}, \tag{1.6}$$

$$r_{nj} := \binom{n}{j}^{-1} a_{nj}. \tag{1.7}$$

2. Heun functions

Let $\alpha\beta \neq 0$. As a consequence of the results of [7], we have (see [4, Prop. 1] and [4, (14)]):

$$Hl\left(\frac{1}{2}, \frac{1}{2}(\alpha+2)(\beta+2); \alpha+2, \beta+2; \gamma+1, \gamma+1; x\right) = \frac{\gamma}{\alpha\beta}(1-2x)^{-1}\frac{d}{dx}Hl\left(\frac{1}{2}, \frac{1}{2}\alpha\beta; \alpha, \beta; \gamma, \gamma; x\right).$$

From [4, (6)], [4, (22)] and (1.6), we obtain

$$Hl\left(\frac{1}{2}, -n; -2n, 1; 1, 1; x\right) = \sum_{j=0}^{n} a_{nj}(1-2x)^{2j}. \tag{2.8}$$

Theorem 2.1. *Let $0 \leq m \leq n$. Then*

$$\begin{aligned} &Hl\left(\frac{1}{2}, (2m+1)(m-n); 2(m-n), 2m+1; m+1, m+1; x\right) \\ &=4^m\binom{n}{m}^{-1}\binom{2m}{m}^{-1}\sum_{j=0}^{n-m}\binom{m+j}{m}a_{n,m+j}(1-2x)^{2j} \\ &=\sum_{j=0}^{n-m}4^j\binom{n-m}{j}\frac{(m+1/2)_j}{(m+1)_j}(x^2-x)^j. \end{aligned} \tag{2.9}$$

Proof. We shall prove the first equality by induction with respect to m. For $m = 0$, it follows from (2.8). Suppose that it is valid for a certain $m < n$. Then, (2) implies

$$\begin{aligned}
&Hl\left(\frac{1}{2}, \frac{1}{2}(2m+3)(m+1-n); 2(m+1-n), 2m+3; m+2, m+2; x\right) \\
=&\frac{(m+1)(1-2x)^{-1}}{2(m-n)(2m+1)}\frac{d}{dx}Hl\left(\frac{1}{2}, (2m+1)(m-n); 2(m-n), 2m+1; m+1, m+1; x\right) \\
=&\frac{(m+1)(1-2x)^{-1}}{2(m-n)(2m+1)}4^m\binom{n}{m}^{-1}\binom{2m}{m}^{-1}\sum_{i=1}^{n-m}\binom{m+i}{m}a_{n,m+i}(-4i)(1-2x)^{2i-1} \\
=&4^{m+1}\binom{n}{m+1}^{-1}\binom{2m+2}{m+1}^{-1}\sum_{j=0}^{n-m-1}\binom{m+1+j}{m+1}a_{n,m+1+j}(1-2x)^{2j},
\end{aligned}$$

and so the desired equality is true for $m+1$; this finishes the proof by induction.
In order to prove that the first member and the last member of (2.9) are equal, it suffices to use [4, Th. 1] with $\gamma = m+1$, $\theta = m + \frac{1}{2}$, and n replaced by $n - m$. $\square$

Corollary 2.1. *Let $0 \le i \le n-m$, $0 \le j \le n-m$. Then*

$$\sum_{j=i}^{n-m}(-1)^{j-i}\binom{n-m}{j}\frac{(m+1/2)_j}{(m+1)_j}\binom{j}{i} = 4^m\binom{n}{m}^{-1}\binom{2m}{m}^{-1}\binom{m+i}{m}a_{n,m+i} \tag{2.10}$$

and

$$\sum_{i=j}^{n-m}\binom{m+i}{m}\binom{i}{j}a_{n,m+i} = 4^{-m}\binom{n}{m}\binom{2m}{m}\frac{(m+1/2)_j}{(m+1)_j}\binom{n-m}{j}. \tag{2.11}$$

Proof. It suffices to combine the last equality in (2.9) with

$$(x^2-x)^j = 4^{-j}\left((1-2x)^2-1\right)^j,$$

respectively with

$$(1-2x)^{2j} = \left(1+4(x^2-x)\right)^j.$$

$\square$

Example 2.1. *For $i = m = 0$, (2.10) reduces to*

$$\sum_{j=0}^{n}\left(-\frac{1}{4}\right)^j\binom{n}{j}\binom{2j}{j} = 4^{-n}\binom{2n}{n}, \tag{2.12}$$

which is (3.85) in [5].
For $j = m = 0$, (2.11) becomes

$$\sum_{i=0}^{n}\binom{2i}{i}\binom{2n-2i}{n-i} = 4^n, \tag{2.13}$$

which is (3.90) in [5].
From [4, (7)], [4, (23)] and (1.6), we know that

$$Hl\left(\frac{1}{2}, n+1; 2n+2, 1; 1, 1; x\right) = \sum_{j=0}^{n}a_{nj}(1-2x)^{2j-2n-1}. \tag{2.14}$$

Theorem 2.2. *For $m \geq 0$, we have*

$$\begin{aligned}
&Hl\left(\frac{1}{2}, (2m+1)(m+n+1); 2(m+n+1), 2m+1; m+1, m+1; x\right)\\
=&\binom{n+m}{n}^{-1}\sum_{j=0}^{n}\binom{2n+2m-2j}{2m}\binom{n+m-j}{m}^{-1}a_{nj}(1-2x)^{2j-2n-2m-1}\\
=&(1-2x)^{-2n-2m-1}\sum_{j=0}^{n}4^j\binom{n}{j}\frac{(1/2)_j}{(m+1)_j}(x^2-x)^j.
\end{aligned}$$

Proof. As in the proof of Theorem 2.1, the first equality can be proved by induction with respect to m, if we use (2.14) and (2). The equality of the first member and the last member follows from [4, Cor. 2] by choosing $\gamma = m+1$, $\theta = m+1/2$, and replacing n by $n+m+1$. □

Corollary 2.2. *Let $0 \leq i \leq n$, $0 \leq j \leq n$. Then*

$$\sum_{j=i}^{n}(-1)^{j-i}\binom{n}{j}\frac{(1/2)_j}{(m+1)_j}\binom{j}{i} = \binom{2n+2m-2i}{2m}\binom{n+m}{n}^{-1}\binom{n+m-i}{m}^{-1}a_{ni} \tag{2.15}$$

and

$$\sum_{i=j}^{n}\binom{2n+2m-2i}{2m}\binom{n+m-i}{m}^{-1}\binom{i}{j}a_{ni} = \binom{n+m}{n}\binom{n}{j}\frac{(1/2)_j}{(m+1)_j}. \tag{2.16}$$

The proof is similar to the proof of Corollary 2.1.
For $i = m = 0$, (2.15) reduces to (2.12), i.e., (3.85) in [5].
For $j = m = 0$, (2.16) reduces to (2.13), i.e., (3.90) in [5].

Let again $\alpha\beta \neq 0$. According to the results of [7] (see [4, Prop. 1] and [4, (15)]), we have

$$\begin{aligned}
&Hl\left(\frac{1}{2}, \frac{1}{2}(2\gamma-\alpha)(2\gamma-\beta); 2\gamma-\alpha, 2\gamma-\beta; \gamma+1, \gamma+1; x\right)\\
=&\frac{\gamma}{\alpha\beta}(1-2x)^{\alpha+\beta+1-2\gamma}\frac{d}{dx}Hl\left(\frac{1}{2}, \frac{1}{2}\alpha\beta; \alpha, \beta; \gamma, \gamma; x\right).
\end{aligned} \tag{2.17}$$

Using (2.8), (2.17) and the above methods of proof, we obtain the following identities:

$$\begin{aligned}
&Hl\left(\frac{1}{2}, (2k+1)(k-n); 2(k-n), 2k+1; 2k+1, 2k+1; x\right)\\
=&4^k\binom{n+k}{n}^{-1}\binom{n}{k}^{-1}\sum_{i=0}^{n-k}\binom{2n-2i}{2k}\binom{n}{i}r_{n,k+i}(1-2x)^{2i}\\
=&\sum_{j=0}^{n-k}4^j\binom{n-k}{j}\frac{(k+1/2)_j}{(2k+1)_j}(x^2-x)^j, \quad 0 \leq k \leq n.
\end{aligned} \tag{2.18}$$

As a consequence of (2.18), one gets

$$\begin{aligned}
&\sum_{j=i}^{n-k}(-1)^{j-i}\binom{n-k}{j}\frac{(k+1/2)_j}{(2k+1)_j}\binom{j}{i}\\
=&4^k\binom{n+k}{n}^{-1}\binom{n}{k}^{-1}\binom{2n-2i}{2k}\binom{n}{i}r_{n,k+i}, \quad 0 \leq i \leq n-k
\end{aligned} \tag{2.19}$$

and

$$\sum_{i=j}^{n-k}\binom{2n-2i}{2k}\binom{n}{i}\binom{i}{j}r_{n,k+i} = 4^{-k}\binom{n+k}{n}\binom{n}{k}\binom{n-k}{j}\frac{(k+1/2)_j}{(2k+1)_j}, \quad 0\le j\le n-k. \tag{2.20}$$

For $i=k=0$, (2.19) reduces to (2.12); for $j=k=0$, (2.20) becomes (2.13). Moreover,

$$\begin{aligned} &Hl\left(\frac{1}{2},(2k-1)(k+n);2(k+n),2k-1;2k,2k;x\right)\\ =&2^{2k-1}\binom{n+k-1}{k-1}^{-1}\binom{n-1}{k-1}^{-1}\sum_{i=0}^{n-k}\binom{2n-2i-2}{2k-2}\binom{n-1}{i}r_{n,k+i}(1-2x)^{1-2n+2i}\\ =&(1-2x)^{1-2n}\sum_{j=0}^{n-k}4^j\binom{n-k}{j}\frac{(k+1/2)_j}{(2k)_j}(x^2-x)^j, \quad 1\le k\le n. \end{aligned}\tag{2.21}$$

From (2.21), we derive

$$\sum_{j=i}^{n-k}(-1)^{j-i}\binom{n-k}{j}\frac{(k+1/2)_j}{(2k)_j}\binom{j}{i} = 2^{2k-1}\binom{n+k-1}{k-1}^{-1}\binom{n-1}{k-1}^{-1}\binom{2n-2i-2}{2k-2}\binom{n-1}{i}r_{n,k+i}, \quad 0\le i\le n-k, \tag{2.22}$$

$$\sum_{i=j}^{n-k}\binom{2n-2i-2}{2k-2}\binom{n-1}{i}\binom{i}{j}r_{n,k+i} = 2^{1-2k}\binom{n+k-1}{k-1}\binom{n-1}{k-1}\binom{n-k}{j}\frac{(k+1/2)_j}{(2k)_j}, \quad 0\le j\le n-k. \tag{2.23}$$

For $i=0$, $k=1$ and replacing n by $n+1$, from (2.22), we obtain

$$\sum_{j=0}^{n}\left(-\frac{1}{4}\right)^j\binom{n}{j}\binom{2j+1}{j} = \frac{1}{(n+1)4^n}\binom{2n}{n}. \tag{2.24}$$

With $j=0$, $k=1$ and replacing n by $n+1$, (2.23) yields

$$\sum_{i=0}^{n}(i+1)\binom{2i+2}{i+1}\binom{2n-2i}{n-i} = \frac{n+1}{2}4^{n+1}. \tag{2.25}$$

It is a pleasant calculation to prove (2.24) and (2.25) directly.

Using (2.14) and (2.17), we get

$$
\begin{aligned}
&Hl\left(\frac{1}{2},(2k+1)(k+n+1);2(k+n+1),2k+1;2k+1,2k+1;x\right)\\
&=4^k\binom{n+k}{n}^{-1}\binom{n}{k}^{-1}\sum_{j=0}^{n-k}\binom{2k+2j}{2j}\binom{n}{k+j}r_{nj}(1-2x)^{-2k-1-2j}\\
&=(1-2x)^{-2n-1}\sum_{j=0}^{n-k}4^j\binom{n-k}{j}\frac{(k+1/2)_j}{(2k+1)_j}(x^2-x)^j,\quad 0\le k\le n. \qquad (2.26)
\end{aligned}
$$

Taking into account that $r_{n,n-j}=r_{nj}$, from (2.26), we derive (2.19) and (2.20). Moreover,

$$
\begin{aligned}
&Hl\left(\frac{1}{2},(2k+1)(k-n);2(k-n),2k+1;2k+2,2k+2;x\right)\\
&=4^k\frac{2k+1}{n+1}\binom{n+k+1}{n}^{-1}\binom{n}{k}^{-1}\sum_{j=0}^{n-k}\binom{2k+2j+2}{2j}\binom{n+1}{k+j+1}r_{nj}(1-2x)^{2n-2k-2j}\\
&=\sum_{j=0}^{n-k}4^j\binom{n-k}{j}\frac{(k+1/2)_j}{(2k+2)_j}(x^2-x)^j,\quad 0\le k\le n. \qquad (2.27)
\end{aligned}
$$

From (2.27), we derive

$$
\begin{aligned}
&\sum_{j=i}^{n-k}(-1)^{j-i}\binom{n-k}{j}\frac{(k+1/2)_j}{(2k+2)_j}\binom{j}{i},\quad 0\le i\le n-k \qquad (2.28)\\
&=4^k\frac{2k+1}{n+1}\binom{n+k+1}{n}^{-1}\binom{n}{k}^{-1}\binom{2n-2i+2}{2k+2}\binom{n+1}{i}r_{n,k+i}
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{i=j}^{n-k}\binom{2n-2i+2}{2k+2}\binom{n+1}{i}\binom{i}{j}r_{n,k+i},\quad 0\le j\le n-k \qquad (2.29)\\
&=4^{-k}\frac{n+1}{2k+1}\binom{n+k+1}{n}\binom{n}{k}\binom{n-k}{j}\frac{(k+1/2)_j}{(2k+2)_j}.
\end{aligned}
$$

For $i=k=0$, (2.28) becomes

$$\sum_{j=0}^{n}\left(-\frac{1}{4}\right)^j\binom{n+1}{j+1}\binom{2j}{j}=\frac{2n+1}{4^n}\binom{2n}{n}. \qquad (2.30)$$

Let us recall the formula (7.6) in [5]:

$$\sum_{j=0}^{n}\left(-\frac{1}{4}\right)^j\binom{n}{j}\binom{2j}{j}\binom{j+h}{h}^{-1}=\frac{1}{4^n}\binom{2n+2h}{n+h}\binom{2h}{h}^{-1}. \qquad (2.31)$$

For $h=1$, (2.31) reduces to (2.30). For $j=k=0$, (2.29) becomes

$$\sum_{i=0}^{n}(2n-2i+1)\binom{2i}{i}\binom{2n-2i}{n-i}=(n+1)4^n, \qquad (2.32)$$

which can be proved also directly.

3. Confluent Heun functions

The hypergeometric function $v(t) = {}_1F_1(\alpha;\gamma;t)$ satisfies (see [10, p. 336], [11, 13.2.1]) $v(0) = 1$ and

$$tv''(t) + (\gamma - t)v'(t) - \alpha v(t) = 0. \tag{3.33}$$

Moreover (see [10, p. 338, 5.6], [11, 13.3.15]),

$${}_1F_1(\alpha+1;\gamma+1;t) = \frac{\gamma}{\alpha}\frac{d}{dt}{}_1F_1(\alpha;\gamma;t). \tag{3.34}$$

With the above notation, we have:

Theorem 3.3. *For $\alpha p \neq 0$, the confluent Heun function $HC(p,\gamma,0,\alpha,4p\alpha;x)$ satisfies*

$$HC(p,\gamma,0,\alpha,4p\alpha;x) = {}_1F_1(\alpha;\gamma;-4px), \tag{3.35}$$

$$HC(p,\gamma+1,0,\alpha+1,4p(\alpha+1);x) = -\frac{\gamma}{4p\alpha}\frac{d}{dx}HC(p,\gamma,0,\alpha,4p\alpha;x), \tag{3.36}$$

$$HC(p,\gamma+j,0,\alpha+j,4p(\alpha+j);x) = \frac{(-1)^j(\gamma)_j}{(4p)^j(\alpha)_j}\frac{d^j}{dx^j}HC(p,\gamma,0,\alpha,4p\alpha;x), \tag{3.37}$$

for all integers $j \geq 0$ with $(\alpha)_j \neq 0$.

Proof. According to (1.2), the function $u(x) = HC(p,\gamma,0,\alpha,4p\alpha;x)$ satisfies $u(0) = 1$ and

$$xu''(x) + (4px+\gamma)u'(x) + 4p\alpha u(x) = 0. \tag{3.38}$$

From (3.33) and (3.38), it is easy to deduce that $u(x) = v(-4px)$, and this entails (3.35). Now, (3.36) is a consequence of (3.35) and (3.34); (3.37) can be proved by induction with respect to j. Let us remark that (3.36) coincides with (30) in [4]. □

Corollary 3.3. *Let $K_n(x) := HC\left(n,1,0,\frac{1}{2},2n;x\right)$ be the function given by (1.5). Then*

$$K_n(x) = {}_1F_1\left(\frac{1}{2};1;-4nx\right) \tag{3.39}$$

and

$$K_n(x) = \frac{1}{\pi}\int_{-1}^{1} e^{-2nx(1+t)}\frac{dt}{\sqrt{1-t^2}}. \tag{3.40}$$

Proof. (3.39) follows from (3.35) with $\alpha = 1/2$, $\gamma = 1$ and $p = n$. By using (3.39) and [10, p. 338, 5.9], [11, 13.4(i)], we get (3.40). Let us remark that (3.40) coincides with (69) in [14]. □

Using (3.37) with $p = n$, $\gamma = 1$, $\alpha = 1/2$, we get

$$HC\left(n,j+1,0,j+\frac{1}{2},2n(2j+1);x\right) = \frac{(-1)^j}{n^j}\binom{2j}{j}^{-1}K_n^{(j)}(x), \quad j \geq 0. \tag{3.41}$$

From (3.41) and [4, (34)], we obtain

$$K_n^{(j)}(0) = (-n)^j\binom{2j}{j}, \tag{3.42}$$

which is (35) in [4].
On the other hand, (3.40) implies (with $t = \sin\varphi$)

$$\begin{aligned} K_n^{(j)}(0) =& \frac{(-2n)^j}{\pi}\sum_{k=0}^{j}\binom{j}{k}\int_{-\pi/2}^{\pi/2}\sin^k\varphi d\varphi \\ =& (-2n)^j\sum_{i=0}^{[j/2]}\binom{j}{2i}\binom{2i}{i}4^{-i}. \end{aligned}$$

Combined with (3.42), this produces

$$\sum_{i=0}^{[j/2]}\binom{j}{2i}\binom{2i}{i}4^{-i} = 2^{-j}\binom{2j}{j},$$

which is (3.99) in [5].
Finally, we give closed forms for some families of confluent Heun functions.

Theorem 3.4. *(i) For $0 \le j \le n$, we have*

$$HC\left(p, j+\frac{1}{2}, 0, j-n, 4p(j-n); x\right) = \frac{(2j)!}{j!}\sum_{k=0}^{n-j}\binom{n-j}{k}\frac{(n-k)!}{(2n-2k)!}(16px)^{n-j-k}. \tag{3.43}$$

(ii) More generally, for $0 \le j \le n$ and $\lambda > -1$,

$$\begin{aligned} & HC\left(p, j+1+\lambda, 0, j-n, 4p(j-n); x\right) \\ =& \frac{(\lambda+1)_j\Gamma(\lambda+1)}{\Gamma(n+\lambda+1)}\sum_{k=0}^{n-j}(\lambda+n+1-k)_k\binom{n-j}{k}(4px)^{n-j-k}. \end{aligned} \tag{3.44}$$

Proof. By using the relation between the function ${}_1F_1$ and the Hermite polynomials (see [10, p. 340, 5.16], [10, p. 235, (4.51)], [11, 13.6.16]), we have

$${}_1F_1\left(-n; \frac{1}{2}; x\right) = n!\sum_{k=0}^{n}\frac{1}{k!(2n-2k)!}(-4x)^{n-k}. \tag{3.45}$$

From (3.35) and (3.45), it follows that

$$HC\left(p, \frac{1}{2}, 0, -n, -4pn; x\right) = n!\sum_{k=0}^{n}\frac{(16px)^{n-k}}{k!(2n-2k)!}. \tag{3.46}$$

Now, (3.43) is a consequence of (3.46) and (3.37).
In order to prove (3.44), we need the relation between ${}_1F_1$ and the Laguerre polynomials (see [10, p. 340, 5.14], [11, 13.6.19]):

$${}_1F_1(-n; \lambda+1; x) = \frac{n!\Gamma(\lambda+1)}{\Gamma(n+\lambda+1)}L_n^\lambda(x), \quad \lambda > -1, \tag{3.47}$$

where (see [10, p. 245, (4.61)], [11, 18.5.12])

$$L_n^\lambda(x) = \sum_{k=0}^{n}(-1)^k\frac{(\lambda+k+1)_{n-k}}{k!(n-k)!}x^k. \tag{3.48}$$

From (3.35), (3.47) and (3.48), we get

$$HC\left(p, \lambda+1, 0, -n, -4pn; x\right) = \frac{n!\Gamma(\lambda+1)}{\Gamma(n+\lambda+1)}L_n^\lambda(-4px). \tag{3.49}$$

Combined with (3.37), (3.49) produces(3.44), and this concludes the proof. $\square$

4. Other combinatorial identities

Let us return to (2.10). Since

$$\frac{(m+1/2)_j}{(m+1)_j}=4^{-j}\binom{2m+2j}{m+j}\binom{2m}{m}^{-1}, \tag{4.50}$$

it becomes

$$\begin{aligned}&\sum_{j=i}^{n-m}(-1)^{j-i}4^{-j}\binom{n-m}{j}\binom{2m+2j}{m+j}\binom{j}{i}\\ =&4^{m-n}\binom{n}{m}^{-1}\binom{m+i}{m}\binom{2m+2i}{m+i}\binom{2n-2m-2i}{n-m-i}.\end{aligned}$$

Set $i+m=r$, $j=r-m+k$, and replace n by $n+r$; we get

$$\begin{aligned}&\sum_{k=0}^{n}\left(-\frac{1}{4}\right)^k\binom{n+r-m}{n-k}\binom{2r+2k}{r+k}\binom{r-m+k}{k}\\ =&4^{-n}\binom{n+r}{m}^{-1}\binom{r}{m}\binom{2r}{r}\binom{2n}{n},\quad n\geq 0, r\geq m\geq 0.\end{aligned} \tag{4.51}$$

Here are some particular cases of (4.51).

$$r=m=n:\quad \sum_{k=0}^{n}\left(-\frac{1}{4}\right)^k\binom{n}{k}\binom{2n+2k}{n+k}=4^{-n}\binom{2n}{n}.$$

$$r=m:\quad \sum_{k=0}^{n}\left(-\frac{1}{4}\right)^k\binom{n}{k}\binom{2r+2k}{r+k}=4^{-n}\binom{n+r}{r}^{-1}\binom{2r}{r}\binom{2n}{n}.$$

$$m=0:\quad \sum_{k=0}^{n}\left(-\frac{1}{4}\right)^k\binom{n+r}{n-k}\binom{2r+2k}{r+k}\binom{r+k}{k}=4^{-n}\binom{2r}{r}\binom{2n}{n}.$$

$$r=n:\quad \sum_{k=0}^{n}\left(-\frac{1}{4}\right)^k\binom{2n-m}{n-k}\binom{2n+2k}{n+k}\binom{n-m+k}{k}=4^{-n}\binom{2n}{m}^{-1}\binom{n}{m}\binom{2n}{n}^2.$$

$$m=n:\quad \sum_{k=0}^{n}\left(-\frac{1}{4}\right)^k\binom{r}{n-k}\binom{2r+2k}{r+k}\binom{r-n+k}{k}=4^{-n}\binom{n+r}{r}^{-1}\binom{r}{n}\binom{2r}{r}\binom{2n}{n}.$$

Now, let us return to (2.11); use (4.50), set $j+m=r$, $i=r-m+k$, and replace n by $n+r$. We get

$$\begin{aligned}&\sum_{k=0}^{n}\binom{r+k}{m}\binom{r+k-m}{k}\binom{2r+2k}{r+k}\binom{2n-2k}{n-k}\\ =&4^{n}\binom{n+r}{m}\binom{2r}{r}\binom{n+r-m}{n},\quad n\geq 0, r\geq m\geq 0.\end{aligned} \tag{4.52}$$

For $r=m=n$, (4.52) reduces to

$$\sum_{k=0}^{n}\binom{n+k}{n}\binom{2n+2k}{n+k}\binom{2n-2k}{n-k}=4^n\binom{2n}{n}^2.$$

Clearly, there are many other particular cases of (4.52).
Several other particular combinatorial identities can be obtained starting with other general formulas from the preceding sections, but we omit the details.

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