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Multilevel Evaluation of the General Dirichlet Series

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Abstract

In this Study, an accurate method for summing the general Dirichlet series is presented. Long range terms of this series are calculated by a multilevel approach. The Dirichlet series, in this technique, is decomposed into two parts, a local part and a smooth part. The local part vanishes beyond some cut off distance, " r_0 ", and it can be cheaply computed. The complexity of calculations depends on r_0 . The smooth part is calculated on a sequence of grids with increasing meshsize. Treating the smooth part using multilevels of grid points overcomes the high cost of calculating the long range terms. A high accuracy in approximating the smooth part is obtained with the same complexity of computing the local part. The method is tested on the Riemann Zeta function. Since there is no closed form for this function with odd integer orders, the method is applied for orders $s = 3, 5, 7$, and 9 . In comparison with the direct calculations, remarkable results are obtained for $s = 3$ and $s = 5$; the reason is the major effect of the long range terms. For $s = 7$, and $s = 9$, results obtained are better than those of direct calculations. The method is compared with efficient well known methods. The comparison shows the superiority of the multilevel method.

Keywords: Dirichlet series, Riemann Zeta Function, Multilevel Evaluation, Local Part, Smooth Part.
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1. Introduction

The general Dirichlet series is defined by the following infinite sum [5, 6]:

$$f(s) = \sum_{i=1}^{\infty} t_i e^{-s\lambda(i)}, \quad (1)$$

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where t_i , s are complex numbers and $\lambda(i)$ is a strictly increasing sequence of positive numbers that tends to *infinity*. The function (1) is considered to be one of the most important special functions in physics, mathematics, and engineering. In particular, Dirichlet series plays important role in differential and integral equations, for example [1]-[4]. Hence, developing fast and accurate techniques for approximating this function has been the focus of numerous researches during the last and the present centuries as in [7]-[17]. A very important and famous derivation of (1) is the Riemann Zeta function which is defines as follows:

If $\lambda(i) = \ln(i)$ and $t_i = (-1)^{i-1}$, then (1), becomes

$$f_1(s) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i^s}. \quad (2)$$

The Riemann Zeta function is defined by [8]

$$\zeta(s) = \frac{1}{(1 - 2^{1-s})} f_1(s). \quad (3)$$

For even integers, Euler [18] has found a closed form of (3),

$$\zeta(2k) = (-1)^{k-1} 2^{2k-1} \frac{B_{2k}}{(2k)!} \pi^{2k}, \quad (4)$$

where B_n is the n^{th} Bernoulli number [19]. Obviously, the more precise the approximated value of π is, the more accurate value of $\zeta(2k)$ can be obtained. For odd positive integers, no closed form is known, therefore, many analytical and numerical methods have been derived in order to accurately evaluate Riemann Zeta functions. An integral expression is known for these functions [20]:

$$\zeta(2k+1) = \frac{(-1)^{k+1} (2\pi)^{2k+1}}{2(2k+1)!} \int_0^1 B_{2k+1}(x) \cot(\pi x) dx, \quad (5)$$

where $B_{2k+1}(x)$ are Bernoulli polynomials [21, 22]. The closed form of the integral in (5) is still an open problem. The Fourier analysis and Katsuradas theorem are used to construct a rapidly converging series for $\zeta(2k+1)$ in [23] and [24] respectively. A remarkable expression for $\zeta(2k+1)$ has been derived in [25], and accelerated series for those functions is found in [26, 27]. Considerable irrationality results for values of Riemann Zeta functions at odd positive integers are obtained and presented in [27]-[33]. In fact, hundreds of researches have been carried out in both analytical and numerical approaches in order to shed light on the convergence of the Riemann Zeta functions and their applications. The advantage of any of these methods over one another is determined by the amount of work needed to reach a desired accuracy, and the speed of convergence.

The accuracy of direct calculations of (1) depends on the number of terms involved. Considering the long range terms certainly gives accurate results of the approximation, but this will increase the complexity of computations. Fast and accurate calculation of the long range part of infinite summations, in general, had been the subject of many researches from the beginning of the last century. The first effective summation methods were carried out by Madelung and Ewald in [34] and [35] respectively. Those methods were applied to sum the long range electrostatic interactions of a crystalline lattice. Since then, many improvements have been done on Ewald's method as in [36]-[38]. Alternative methods were introduced during the past decades in order to reduce the computational work in calculating the long range part of infinite summations, for example [39]-[42]. In this paper, a multilevel technique has been developed and presented in order to avoid the slow direct summation.

The general multilevel approach in the context of general transformation, many body problems, and matrix multiplication has been initially proposed by Brandt in [43]. The approach is also used to develop the multilevel Monte-Carlo simulation method in statistical mechanics [44]. The method facilitates developing computational tools that describe, scale by scale, a material property at increasingly larger scales [45]. In the

frame of the method presented in this paper, the Riemann Zeta function is decomposed into two components, a local part and a smooth part. The local part vanishes beyond some cut-off radius r_{cut} and is computed directly and cheaply. The smooth part is accurately calculated on a sequence of grids with very low computer demand. In the next section of this paper, the developed scheme is presented in details, and the method is tested on Riemann Zeta functions at $n = 3, 5, 7$ and 9 . In section 3, the multilevel technique results are compared with those of two well known methods in [8] and [46]. Conclusions and future perspectives are presented in section 4.

2. Analysis of the Method

The analysis of the method in this paper starts from the following equivalent definition of the Dirichlet series (1):

$$f(s) = \frac{1}{2} \sum_{i=-\infty, i \neq 0}^{\infty} t_i G(i), \quad (6)$$

where $t_i = t_{-i}$ and

$$G(i) = e^{-s\lambda(|i|)}. \quad (7)$$

In the frame of the multilevel method, the kernel (7) is split into two parts:

$$G(i) = G_{loc}(i) + G_{sm}(i), \quad (8)$$

where

$$G_{loc}(i) = \begin{cases} e^{-s\lambda(|i|)} - P_m(i) & , |i| \leq r_0 \\ 0 & , |i| > r_0 \end{cases} \quad (9)$$

and

$$G_{sm}(i) = \begin{cases} P_m(i) & , |i| \leq r_0 \\ e^{-s\lambda(|i|)} & , |i| > r_0 \end{cases} \quad (10)$$

The cut off distance r_0 is a positive real number, and P_m is a polynomial of the form

$$P_m(r) = \sum_{j=0}^m c_j r^{2j}. \quad (11)$$

The part (9) is comprised of short-range influences, and it is called "local". It can be computed in a constant number of operations for a given r_0 . The part (10) is called "smooth" and it satisfies the assumption

$$\left. \frac{d^k}{dr^k} e^{-s\lambda(r)} \right|_{r=r_0} = \left. \frac{d^k}{dr^k} P_m(r) \right|_{r=r_0}, \quad 0 \leq k \leq m. \quad (12)$$

The coefficients c_j in (11) are determined by applying the smoothness criterion (12).

For simplicity, the values of the coefficients c_j in (11) can be universalized by changing the variable $|i|$ into $x_i = \frac{|i|}{r_0}$, so,

$$G_{loc}(i) = \begin{cases} e^{-s\lambda(r_0 x_i)} - \frac{1}{r_0^s} \tilde{P}_m(x_i) & , x_i \leq 1, i \neq 0 \\ 0 & , x_i > 1 \end{cases} \quad (13)$$

and

$$G_{sm}(i) = \begin{cases} \frac{1}{r_0^s} \tilde{P}_m(x_i) & , x_i \leq 1, i \neq 0 \\ e^{-s\lambda(r_0 x_i)} & , x_i > 1 \end{cases} \quad (14)$$

where

$$\tilde{P}_m(x) = \sum_{j=0}^m a_j x^{2j}, \quad (15)$$

and the coefficients a_j are determined by the assumption

$$\left. \frac{d^k}{dx^k} e^{-s\lambda(r_0x)} \right|_{x=1} = \left. \frac{d^k}{dx^k} \tilde{P}_m(x) \right|_{x=1}, \quad 0 \leq k \leq m. \quad (16)$$

Changing the variables from $|i|$ to $\frac{|i|}{r_0}$ is of great importance in calculations; it reduces the computational work because the values of a_j , in this case, can be found and saved in a pre-calculated table regardless the value of r_0 .

Now, using formulas (13) and (14), (6) can be expressed in the form

$$f(s) = \frac{1}{2}(U_{loc}(s) + U_{sm}(s)), \quad (17)$$

where

$$U_{loc}(s) = \sum_{|i| \leq [r_0], i \neq 0} t_i (e^{-s\lambda(|i|)} - \frac{1}{r_0^s} \tilde{P}_m(x_i)), \quad (18)$$

and

$$U_{sm}(s) = \frac{1}{r_0^s} \sum_{|i| \leq [r_0], i \neq 0} t_i \tilde{P}_m(x_i) + \sum_{|i| > [r_0]} t_i e^{-s\lambda(|i|)}. \quad (19)$$

The notation [...] refers to the greatest integer function. It is easy to see that calculating U_{loc} in (18) is computationally cheap for small values of r_0 . The main purpose of the method presented in this paper is accurately approximating the smooth part U_{sm} with computational work that is linear function of that of the local part.

The first step of the method is rewriting equation (19) by adding and subtracting the value

$$U_{self} = \frac{1}{2} \frac{\tilde{P}_m(x_0)}{r_0^s}. \quad (20)$$

This value denotes the "self contribution" of the computations, where $x_0 = 0$. This contribution is calculated once during the procedure. Using (20), equation (19) becomes

$$U_{sm}(s) = U_{sm}^{self}(s) - U_{self}, \quad (21)$$

where

$$U_{sm}^{self}(s) = \frac{1}{r_0^s} \sum_{|i| \leq [r_0]} t_i \tilde{P}_m(x_i) + E(r_0), \quad (22)$$

and

$$E(r_0) = \sum_{|i| > [r_0]} t_i e^{-s\lambda(|i|)}. \quad (23)$$

Omitting $E(r_0)$ from (22) gives the first approximation of (1),

$$f(s) \approx \frac{1}{2}(U_{loc}(s) + \tilde{U}_{sm}^{self}(s) - U_{self}), \quad (24)$$

where

$$\tilde{U}_{sm}^{self}(s) = \frac{1}{r_0^s} \sum_{|i| \leq [r_0]} t_i \tilde{P}_m(x_i). \tag{25}$$

More accurate approximation of $f(s)$ can be achieved by refining the value of $\tilde{U}_{sm}^{self}(s)$ as an approximation of $U_{sm}^{self}(s)$. This can be done by constructing a successive finite number L of coarse levels of equidistance integer points. At each level $l = 1, \dots, L$, the distance between any two neighboring grid points is $I_l = 2^{l-1}$, Table (1) shows the distances between grid points for $l = 1, 2$, and 3.

Table 1: Coarse levels grid points

l=3	-4	0	4
l=2	-2	0	2
l=1	-1	0	1

The construction of the coarse levels can be done by successive splitting of the kernel G in (7) into local and smooth parts at each level $l \geq 1$ as follows:

$$G(i) = \sum_{l=1} G_{loc}^{(l)}(i). \tag{26}$$

If $L = 1$, then $G_{loc}^{(1)} = G_{loc}(i)$ and $f(s)$ is approximated by (24). If $L > 1$, then we define, for $1 \leq l \leq L$,

$$G_{loc}^{(l)}(i) = \begin{cases} \frac{1}{r_0^s} \left(\frac{1}{2^{s(l-1)}} \tilde{P}_m\left(\frac{x_i}{2^{l-1}}\right) - \frac{1}{2^{sl}} \tilde{P}_m\left(\frac{x_i}{2^l}\right) \right) & , 0 \leq x_i \leq 2^{l-1} \\ e^{-s\lambda(r_0 x_i)} - \frac{1}{r_0^s} \frac{1}{2^{sl}} \tilde{P}_m\left(\frac{x_i}{2^l}\right) & , 2^{l-1} < x_i \leq 2^l \\ 0 & , x_i > 2^l \end{cases} \tag{27}$$

At each coarse Level l , if we let the cut-off distance $R(l) = 2^{l-1}r_0$, then the number of grid points inside $R(l)$ is the same in all levels, and the distance between any two neighboring grid points at level l is 2^{l-1} . Hence, from (27),

$$\tilde{U}_{sm}^{self}(n) = \sum_{l=1}^L \sum_{i=-[r_0]}^{[r_0]} t_i \frac{2^{l-1}}{\left(\frac{R(l)}{2}\right)^s} \psi(i), \tag{28}$$

where

$$\psi(i) = \begin{cases} \tilde{P}_m\left(\frac{|i|}{r_0}\right) - \frac{1}{2^s} \tilde{P}_m\left(\frac{|i|}{2r_0}\right) & , |i| \leq \left[\frac{1}{2}r_0\right] \\ e^{-s\lambda(|i|)} - \frac{1}{2^s} \tilde{P}_m\left(\frac{|i|}{2r_0}\right) & , \left[\frac{1}{2}r_0\right] < |i| \leq [r_0] \\ 0 & , |i| > [r_0] \end{cases} \tag{29}$$

Equation (28) can be written in the form

$$\tilde{U}_{sm}^{self}(s) = \sum_{l=1}^L \frac{2^{l-1}}{\left(\frac{R(l)}{2}\right)^s} C, \tag{30}$$

where

$$C = \sum_{i=-[r_0]}^{[r_0]} t_i \psi(i). \tag{31}$$

The constant C is the same for all levels. So, letting L approaches *infinity* and summing the infinite geometric series in (30) gives

$$\tilde{U}_{sm}^{self}(s) = \frac{C}{r_0^s(1-2^{1-s})}. \quad (32)$$

Using (18), (20), and (32), the final approximation of $f(s)$ is

$$f(s) \approx \frac{1}{2} \left(\sum_{|i| \leq [r_0], i \neq 0} t_i (e^{-s\lambda(|i|)} - \frac{1}{r_0^s} \tilde{P}_m(\frac{|i|}{r_0}) - \frac{1}{2} \frac{\tilde{P}_m(0)}{r_0^s} + \frac{C}{r_0^s(1-2^{1-s})} \right). \quad (33)$$

Once the constant C in (31) is found, the complexity of approximating $f(s)$ in (33) is the same as in (18) with additional constant term. Finding C itself is done one time for a given r_0 , and its computational work is also the same as (18). Therefore, the accuracy and the whole complexity in approximating $f(s)$ in (33) is linearly proportional to the complexity of calculating the local part in (18). The number of grid points involved in calculation are the same in all levels because the cut-off distance is doubled in each coarse level in comparison with the finer one, and the distance between any two neighboring points are also doubled. The gain is that by increasing the level, more long range terms are involved in calculations.

3. Testing the Method

In this section, we will test the efficiency of the presented method on the Reimann Zeta function. Definition (3) can be written in the form,

$$\zeta(s) = \frac{1}{2} \left(\frac{1}{1-2^{1-s}} \right) \sum_{i=-\infty, i \neq 0}^{\infty} (-1)^{i-1} G(i), \quad (34)$$

where the kernel G is

$$G(i) = \frac{1}{|i|^s}, i \neq 0. \quad (35)$$

For a given cut-off distance $r_0 = r_{cut}$, from (33) and (34), the approximated Zeta function is

$$\zeta(s) \approx \frac{1}{2} \left(\frac{1}{1-2^{1-s}} \right) \left(\sum_{|i| \leq [r_{cut}], i \neq 0} (-1)^{i-1} \left(\frac{1}{|i|^s} - \frac{1}{r_{cut}^s} \tilde{P}_m\left(\frac{|i|}{r_{cut}}\right) \right) - \frac{1}{2} \frac{\tilde{P}_m(0)}{r_{cut}^s} + \frac{C}{r_{cut}^s(1-2^{1-s})} \right), \quad (36)$$

and the constant C is given by

$$C = \sum_{i=-[r_{cut}]}^{[r_{cut}]} (-1)^{i-1} \psi(i), \quad (37)$$

where

$$\psi(i) = \begin{cases} \tilde{P}_m\left(\frac{|i|}{r_{cut}}\right) - \frac{1}{2^s} \tilde{P}_m\left(\frac{|i|}{2r_{cut}}\right) & , |i| \leq [\frac{1}{2}r_{cut}] \\ \frac{1}{\left(\frac{|i|}{r_{cut}}\right)^s} - \frac{1}{2^s} \tilde{P}_m\left(\frac{|i|}{2r_{cut}}\right) & , [\frac{1}{2}r_{cut}] < |i| \leq [r_{cut}] \\ 0 & , |i| > [r_{cut}] \end{cases} \quad (38)$$

The method is tested on approximating the Riemann Zeta functions at $s = n$, where $n = 3, 5, 7$, and 9 . For $m = 1, \dots, 6$, the coefficients, a_j , of the polynomial $\tilde{P}_m(x)$ are shown in Tables (2), (3), (4), and (5).

The multilevel results for integer values of r_{cut} in the range $[2, 10]$ and $m = 2, 4, 6$ are shown in tables (6), (7), and (8).

Table 2: Polynomial Coefficients for $n = 3$

m	a_0	a_1	a_2	a_3	a_4	a_5	a_6
1	2.5	-1.5	0	0	0	0	0
2	4.375	-5.25	1.875	0	0	0	0
3	6.5625	-11.8125	8.4375	-2.1875	0	0	0
4	9.0234	-21.6563	23.2031	-12.0313	2.4609	0	0
5	11.7305	-35.1914	50.2734	-39.1016	15.9961	-2.7070	0
6	14.6631	-52.7871	94.2627	-97.7539	59.9854	-20.3027	2.9326

Table 3: Polynomial Coefficients for $n = 5$

m	a_0	a_1	a_2	a_3	a_4	a_5	a_6
1	3.5	-2.5	0	0	0	0	0
2	7.8750	-11.25	4.375	0	0	0	0
3	14.4375	-30.9375	24.0625	-6.5625	0	0	0
4	23.4609	-67.0313	78.2031	-42.6563	9.0234	0	0
5	35.1914	-125.6836	195.5078	-159.9609	67.6758	-11.7305	0
6	49.8545	-213.6621	415.4541	-453.2227	287.6221	-99.7090	14.6631

Direct calculations of (34) requires ignoring the terms beyond some cut-off distance, say, the r_{cut} used in calculating (36). In this case, the comparison between the approximation and the direct calculations makes sense. For this purpose, the following "direct" zeta function is defined:

$$\zeta_{r_{cut}}(n) = \frac{1}{2} \left(\frac{1}{1 - 2^{1-n}} \right) \sum_{i=-[r_{cut}], i \neq 0}^{[r_{cut}]} \frac{(-1)^{i-1}}{|i|^n} \tag{39}$$

Table (9) shows $\zeta(n)$ using the direct calculations (39) up to 12 digits and shows the r_{cut} needed to reach this accuracy. For the purpose of examining the accuracy of the multilevel method, the relative error is calculated considering the values of Zeta functions in table (9) as "exact" values.

For $n = 3$, Figures (1), (2), (3) show the direct evaluation, and the approximation using $m = 2, 4$, and 6 respectively. Figure (4) shows the percentage relative error in calculating $\zeta(3)$ using $m = 2$. Figure (5) shows the percentage relative error in calculating $\zeta(3)$ using $m = 2, 4$, and 6. It is easy to notice that the approximated solution is more accurate than the direct calculations, and the accuracy increases by increasing m for small values of r_{cut} . Figure (6) shows the approximation and the direct calculations for $\zeta(5)$ using $m = 6$. Figures (7), and (8) show the error in calculating $\zeta(7)$ and $\zeta(9)$ using $m = 2$ and $m = 4$ respectively. In all cases, for small values of r_{cut} , the error is very small in comparison with the direct calculations.

Table (10) shows the r_{cut} needed in multilevel technique in order to reach the same results in table (9). One can easily see the efficiency of the multilevel method.

Table 4: Polynomial Coefficients for $n = 7$

m	a_0	a_1	a_2	a_3	a_4	a_5	a_6
1	4.5	-3.5	0	0	0	0	0
2	12.375	-19.25	7.875	0	0	0	0
3	26.8125	-62.5625	51.1875	-14.4375	0	0	0
4	50.2734	-156.4063	191.9531	-108.2813	23.4609	0	0
5	85.4648	-332.3633	543.8672	-460.1953	199.418	-35.1914	0
6	135.3	-631.5	1291.7	-1457.3	947.2	-334.3	49.9

Table 5: Polynomial Coefficients for $n = 9$

m	a_0	a_1	a_2	a_3	a_4	a_5	a_6
1	5.5	-4.5	0	0	0	0	0
2	17.875	-29.25	12.375	0	0	0	0
3	44.6875	-109.6875	92.8125	-26.8125	0	0	0
4	94.9609	-310.7813	394.4531	-227.9063	50.2734	0	0
5	180.4	-738.1	1249.1	-1082.6	477.6	-85.5	0
6	315.7	-1550	3278.9	-3788.9	2507.4	-897.4	135.3

Table 6: Multilevel calculations using $m = 2$

r_{cut}	$\zeta(3)$	$\zeta(5)$	$\zeta(7)$	$\zeta(9)$
2	1.167968750	1.0200520833333333	1.0010540674603174	0.99921875000000004
3	1.2054183813443073	1.0377127470405934	1.0085074623254779	1.0020362284696094
4	1.2014212902681327	1.0368409500854658	1.0083390697846069	1.0020073495530326
5	1.2022293827160495	1.0369432147599451	1.0083504715322693	1.0020084729074545
6	1.2019978995198903	1.0369240204872225	1.0083490733390914	1.0020083831449198
7	1.2020806290924595	1.0369288712600016	1.008349322776408	1.0020083944306721
8	1.2020461534066864	1.0369273647859891	1.0083492651087052	1.0020083924904215
9	1.202062241697395	1.0369279092382468	1.0083492812376988	1.0020083929100754
10	1.2020540518865783	1.0369276881785094	1.0083492760171904	1.0020083928018504

Table 7: Multilevel calculations using $m = 4$

r_{cut}	$\zeta(3)$	$\zeta(5)$	$\zeta(7)$	$\zeta(9)$
2	1.2717262369791664	1.0923579427083334	1.0441893446180557	1.0214235523897059
3	1.1985633152127895	1.0356742745855139	1.0079841317052431	1.0019195764945175
4	1.2024375479356739	1.0370061542162745	1.0083623466947453	1.0020102072594625
5	1.2019910332538626	1.0369188670224885	1.0083483137794671	1.0020083059971507
6	1.2020721642757761	1.0369292322644648	1.0083493903207716	1.0020083999827165
7	1.2020526373868923	1.0369274355743188	1.0083492591245904	1.0020083919662213
8	1.2020582339498642	1.0369278387167546	1.0083492811162826	1.0020083929622963
9	1.202056475862721	1.0369277300399995	1.0083492764688409	1.0020083927994301
10	1.2020570250395586	1.0369277634709004	1.0083492776383476	1.0020083928322392

Table 8: Multilevel calculations using $m = 6$

r_{cut}	$\zeta(3)$	$\zeta(5)$	$\zeta(7)$	$\zeta(9)$
2	1.5476839762369774	1.352475560709635	1.2450662279885922	1.1509198357077208
3	1.2035156936622424	1.0378958431217096	1.0087823666838591	1.0021596696011117
4	1.201790741461644	1.0368489154374176	1.0083321418955002	1.0020051460905239
5	1.202094889091571	1.0369348712440558	1.0083501347008073	1.0020085093799642
6	1.2020501652371047	1.0369268468958999	1.0083492305983703	1.0020083857860087
7	1.2020582914411686	1.0369279088938901	1.008349274832379	1.0020083934532036
8	1.2020566217703814	1.0369277226995695	1.0083492805728	1.0020083927515675
9	1.2020569279535709	1.0369277632907559	1.0083492757136803	1.0020083928370518
10	1.2020569373322751	1.0369277527897498	1.0083492781958088	1.0020083928241947

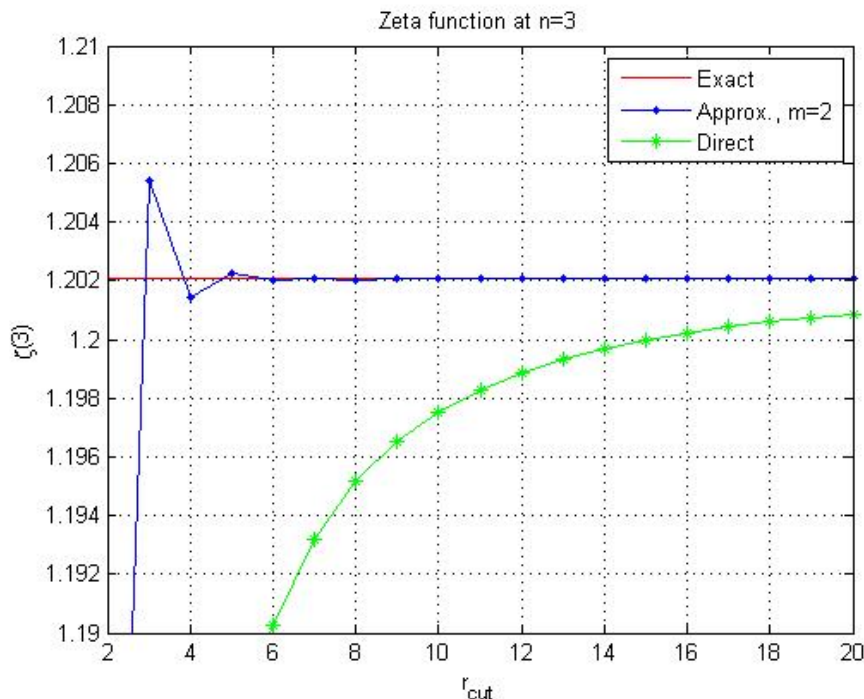


Figure 1: $\zeta(3)$ using Multilevel approach with $m=2$, the direct calculations, and the "exact" value

Table 9: The exact value of $\zeta(n)$ up to 12 digits and the r_{cut} used in (39)

n	$\zeta(n)$	r_{cut}
3	1.202056903159	more than 100000
5	1.036927755143	1173
7	1.008349277381	108
9	1.002008392826	32

4. Comparison with efficient Algorithms

In this section, numerical results of the multilevel method is compared with those of two well known methods that are used to efficiently calculate Zeta functions at positive integer variables. The two methods are listed below.

Method1 [46]: In this method, the Zeta function may be evaluated to any desired precision if m and $r_{cut} = p$ are chosen large enough in the Euler-Maclaurin formula:

$$\zeta(n) \approx \sum_{j=1}^{p-1} j^{-n} + \frac{p^{-n}}{2} + \frac{p^{1-n}}{n-1} + \sum_{k=1}^m T_{k,p}(n) \tag{40}$$

Table 10: The values of r_{cut} by multilevel method at which the values of the approximated zeta function in Table (9) are obtained

m	$r_{cut}(n=3)$	$r_{cut}(n=5)$	$r_{cut}(n=7)$	$r_{cut}(n=9)$
2	171	56	26	16
4	1038	68	18	14
6	824	24	32	12

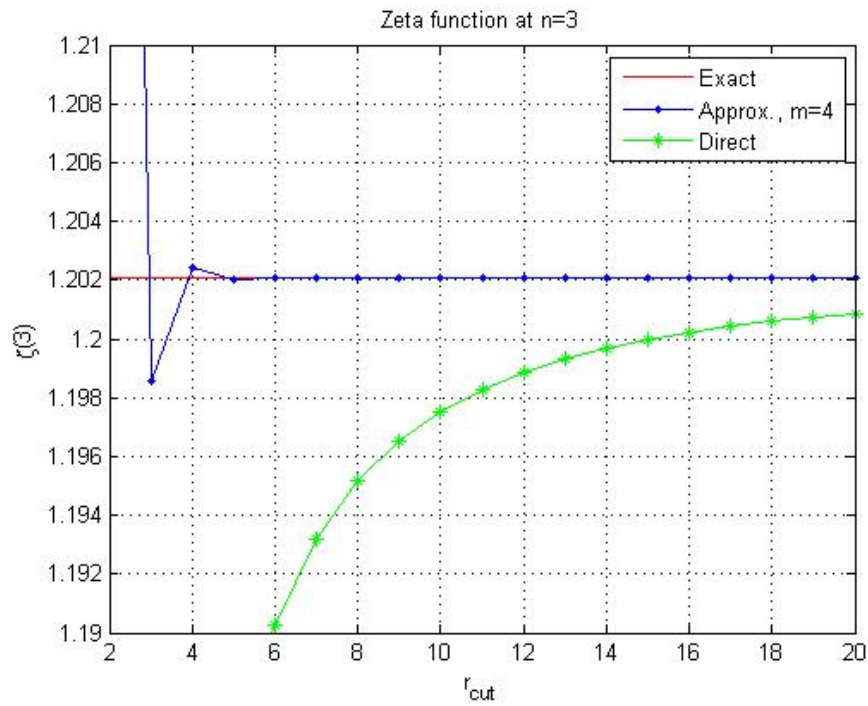


Figure 2: $\zeta(3)$ using Multilevel approach with $m=4$, the direct calculations, and the "exact" value

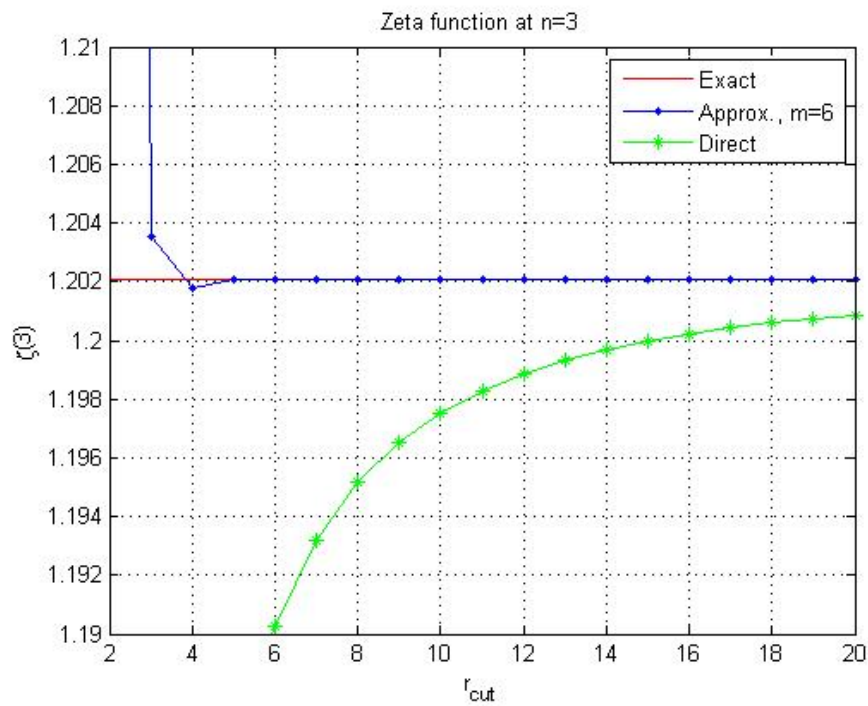


Figure 3: $\zeta(3)$ using Multilevel approach with $m=6$, the direct calculations, and the "exact" value

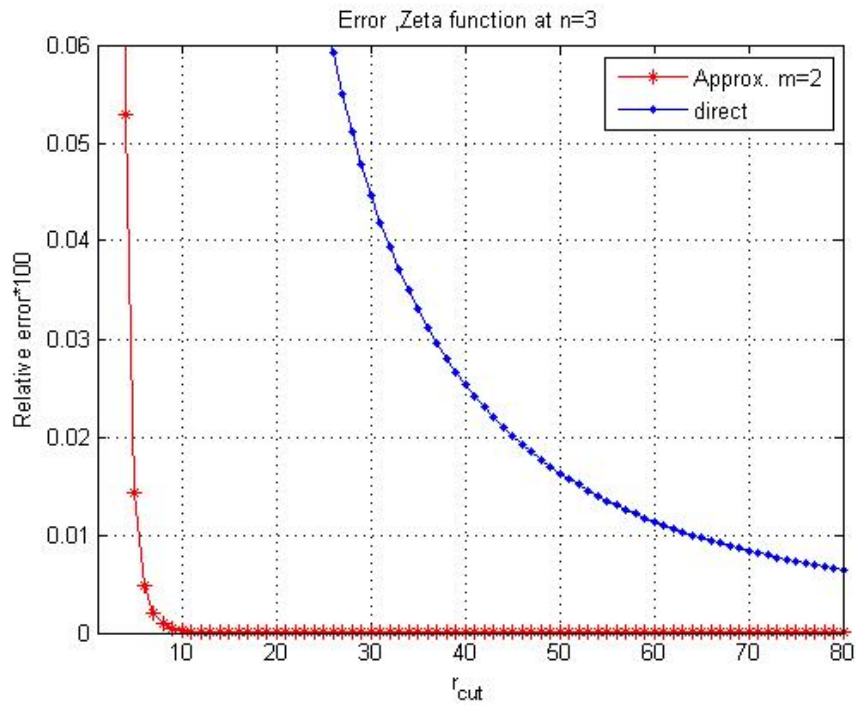


Figure 4: The error in calculating $\zeta(3)$ using $m=2$, and direct calculations

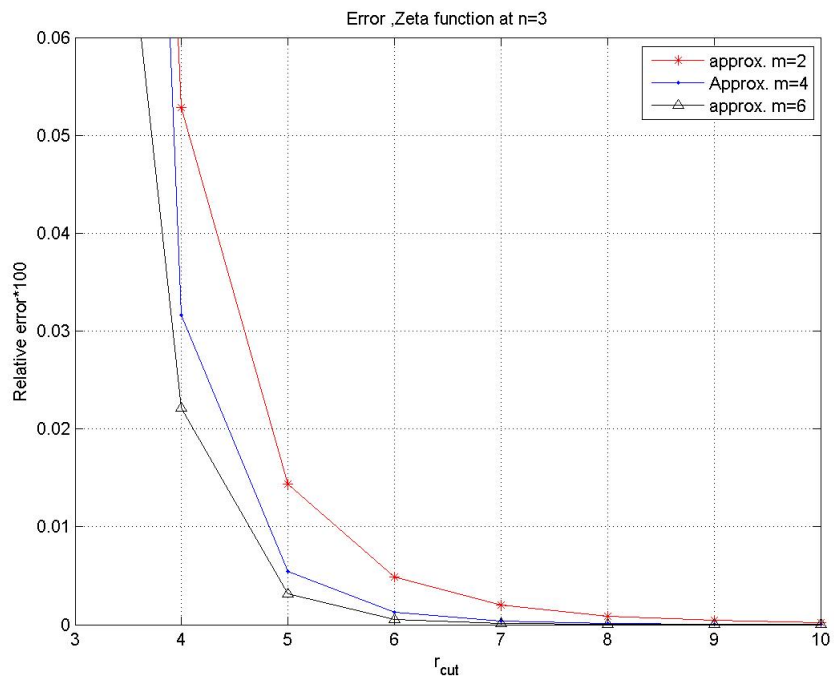


Figure 5: The error in calculating $\zeta(3)$ using $m=2,4,6$

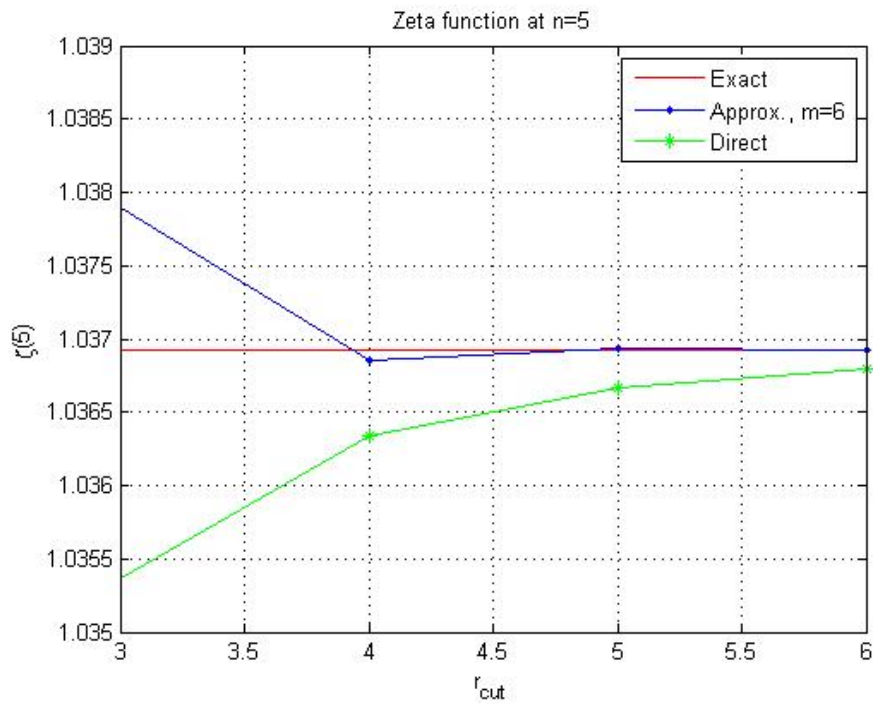


Figure 6: $\zeta(5)$ using Multilevel approach with $m=6$, the direct calculations, and the "exact" value

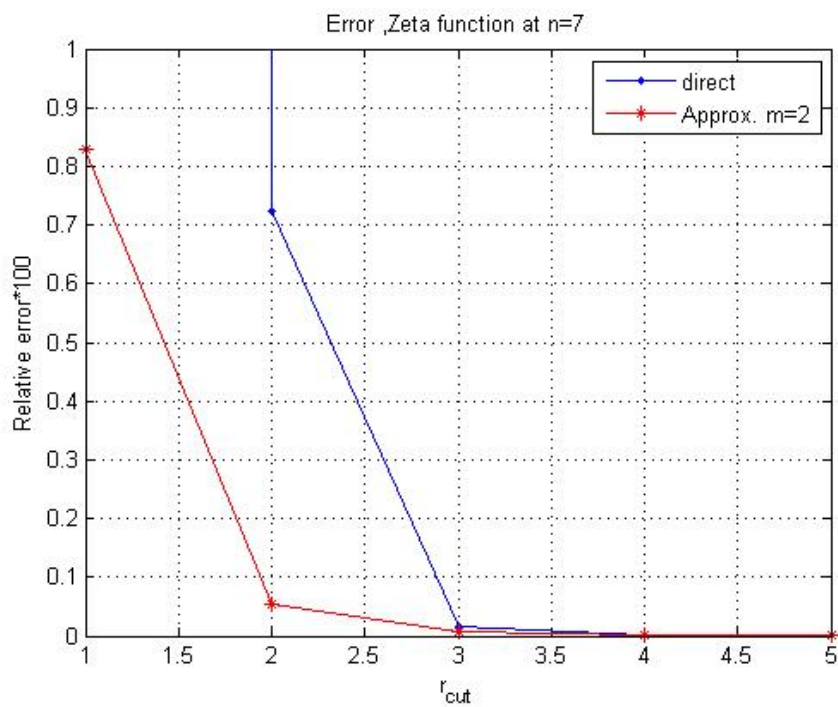


Figure 7: the error in calculating $\zeta(7)$ using $m=2$

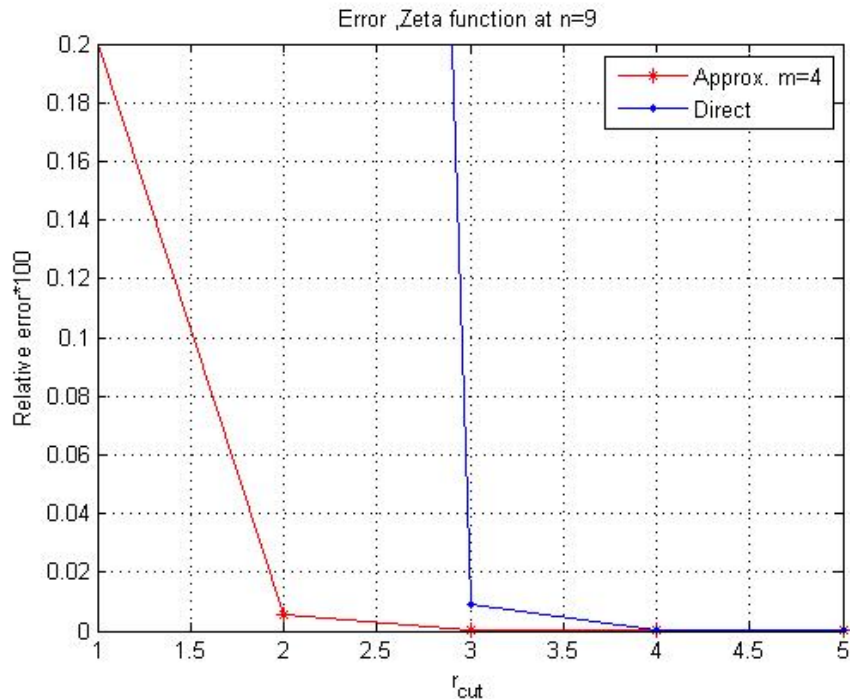


Figure 8: The error in calculating $\zeta(9)$ using $m=4$

where

$$T_{k,p}(n) = \frac{B_{2k}}{(2k)!} p^{1-n-2k} \prod_{j=0}^{2k-2} (n+j) \tag{41}$$

and B_{2k} are the Bernoulli numbers. Table(11) shows $\zeta(n)$ for $m = 100$ at integer values of r_{cut} between 1 and 10. Figure (9) shows the convergence of the multilevel method and method 1 for $\zeta(3)$. Comparing results in table (11) with those of the multilevel approach in tables (6),(7), and (8) shows the superiority of the multilevel method.

Method2 [8]: For $P_m(x) = \sum_{k=0}^m a_k x^k$, an arbitrary polynomial of degree m that does not vanish at -1 . If

$$C_j = (-1)^j \left(\sum_{k=0}^j (-1)^k a_k - P_m(-1) \right) \tag{42}$$

then

$$\zeta(n) \approx \frac{-1}{(1-2^{1-n})p_m(-1)} \sum_{j=0}^{n-1} \frac{C_j}{(1+j)^n} \tag{43}$$

The suggested polynomial in [8] is $p(x) = x^n(1-x)^n$. Table (12) shows the values of Zeta function at different values of the polynomial degree m in (43). It is easy to see that the multilevel results for $m = 2$ is better than those of method 2 using at least $m = 10$. Of course increasing m in (43) will increase the accuracy but this increases the computational work, while the multilevel approach is of very low computational work for small values of m and r_{cut} .

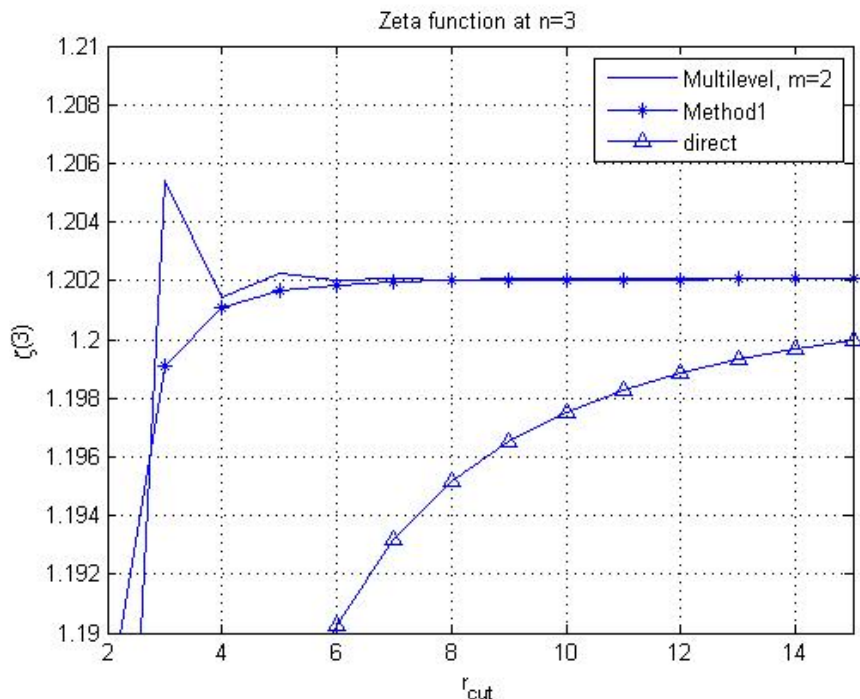


Figure 9: $\zeta(3)$ by Multilevel multilevel approach with $m=2$, the direct calculations, and method 1

Table 11: Method 1 calculations using $m = 100$ in (40)

r_{cut}	$\zeta(3)$	$\zeta(5)$	$\zeta(7)$	$\zeta(9)$
1	1	0.75	0.6666666666666666296	0.625
2	1.1875	1.03125	1.0065104166666667407	1.00146484375
3	1.201099537037036	1.0363940329218106	1.0082697473708275	1.001997579605497
4	1.201099537037036	1.0368300700874486	1.008340955053119	1.0020077449606910
5	1.2016620370370370	1.0369017888374485	1.0083478491937440	1.00200832096069
6	1.2018657407407407	1.0369189904835391	1.0083489408947043	1.00200838099697
7	1.2019534742468414	1.036924262451592	1.0083491785482936	1.002008390263502
8	1.2019961810617103	1.0369261826240727	1.0083492432392977	1.002008392146524
9	1.2020189541241246	1.0369269777483585	1.0083492640253153	1.002008392615642
10	1.2020319856741932	1.0369273413446937	1.0083492716168238	1.00200839275240

Table 12: Method 2 calculations

m	$\zeta(3)$	$\zeta(5)$	$\zeta(7)$	$\zeta(9)$
2	1.19849537037037	1.03636509773662	1.008269385893646	1.0019980802743242
3	1.20238194444444	1.036964969650205	1.008352807698766	1.0020086824153739
4	1.20202423342849	1.036924732657259	1.008349060014510	1.0020083798584031
5	1.20206036952552	1.036928030701190	1.008349293562950	1.0020083935874248
6	1.20205652264768	1.036927728142071	1.008349276024229	1.0020083927729590
7	1.20205694590271	1.036927757920244	1.008349277505425	1.0020083928302507
8	1.20205689827836	1.036927754847860	1.008349277370025	1.0020083928257261
9	1.20205690372393	1.036927755175618	1.008349277383118	1.0020083928261145
10	1.2020569030937	1.036927755139782	1.008349277381798	1.0020083928260792

5. Conclusions and Discussion

Accurate evaluation of general Dirichlet series is important in many problems in science and engineering. For this purpose, a multilevel approach is developed in this paper and is successfully applied. In the frame of this approach, the function (6) is split into two parts. The first of which is a local part (9) comprising of short range influences, and it can therefore be computed in a constant number of operations. The second part, (10), satisfies a well-defined smoothness criterion and is computed in a cheap and efficient way. The calculation of (10) is carried out recursively for increasingly coarser grids, the distance between each two neighboring grid points at each coarse level is doubled in comparison to the finer one. Theoretically, The recursion proceeds till infinity which leads to a convergent geometric series (30). The sum of it is constant. In fact, the Dirichlet series is split into a sum of several parts each of which is local at some level. A self contribution (20) is added and subtracted from (19) which completes the calculations of the smooth part. The convergence of the method is tested on a kind of Dirichlet series, the Riemann Zeta function $\zeta(n)$ for $n = 3, 5, 7,$ and 9 . The method converges independently of n . In comparison with direct calculations, the convergence becomes faster and the accuracy becomes better by decreasing n . For small values of n , the effect of the long range terms becomes larger and the method works more efficiently. This fact comes from the methodology of the method designed in order to calculate infinite summations with long range properties. Hence, this procedure can be applied in physical problems involving simulations of long range potentials which will be an alternative method of the well known Ewald's technique. The technique is compared with well known efficient methods. The comparison shows that the multilevel technique is distinctive. As an application of the method, Monte Carlo simulations of systems containing long range potentials can be performed in faster and efficient way. This will be the base of our future plan.

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