Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 51 (1) (2022), 327 – 337
DOI : 10.15672/hujms.812606

Research Article

Two-sided sampling plan for exponential distribution under type II censored samples

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Abstract

Acceptance sampling plan paid the attention of many researchers in the last few years, their works focused basically on Bayesian sampling plans under one-sided decision function and different forms of censoring. In the present paper, a single variable sampling plan for exponential distribution based on type II censored samples under random decision function is developed. For a polynomial loss function, an explicit expression for the Bayes risk is determined. To obtain an approximation for the optimal sampling plan, a simple algorithm based on a discretization method is presented. Finally, an illustrative example and a simulation study followed by extensive tables for the proposed sampling plan are provided.

Mathematics Subject Classification (2020). 62D05, 62F15, 62N05

Keywords. Bayesian sampling, type II censored samples, random decision function

1. Introduction

Acceptance sampling plan is an important issue for quality control engineers, to determine whether the outcome of products or process are of high quality, the main purpose is to draw the optimal number of items and to determine as much as possible the quality of a batch. Different criteria can be applied for designing sampling plans, such as the decision theory approach and the operating characteristic curve (see e.g. [5, 7]). Decision theory approach is suited to quality control in the sense that the sampling plan is established by making an optimal decision. Extensive researches in different fields have been studied along with this approach, such as [10, 11, 18, 19].

In reliability analysis, the life test experiment is usually censored, which is a random phenomenon. Recently, considerable literature around acceptance sampling plans has been investigated by several authors, including [9, 13, 14, 16] derived the exact distribution of the maximum likelihood estimator (MLE) of the expected lifetime for the case when the lifetime of components follows exponential distribution based on type I and type II

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Received: 06.12.2020; Accepted: 17.11.2021

hybrid censored samples. Lin et al. [15] have discussed Bayesian sampling plans under the type I and type II hybrid censoring based on the results of [9]. Chen et al. [8] developed a Bayesian sampling plan for type II censoring based on a curtailed Bayes decision function. Yang et al. [20] proposed a modified type II hybrid censoring such as the life test experiment interrupted at the time $\tau = min\left\{max\left(X_{(m)}, t\right), X_{(n)}\right\}$. Prajapati et al. [17] developed a new shrinkage estimator for the expected lifetime of exponential distribution under type I censoring and type I hybrid censoring which always exists even if no failure occurs at the censoring time $\tau = min\left(X_{(m)}, t\right)$. In addition, Prajapati and his colleagues claimed that the construction of the Bayes decision function (as in [8], which is based on the posterior expectation) is quite difficult, when the loss function is a polynomial with a higher degree or if it is not a polynomial. Balamurali et al. [6] provided a mixed double sampling plan based on process capability index. The case of acceptance sampling plan under nonparametric distribution has been investigate by [3]. Aslam et al. [4] considered the designing of modified multiple dependent state sampling plan under Weibull and Birnbaum-Saunders distributions. Aslam [1, 2] have proposed acceptance sampling plan for variable and attribute using the neutrosophic statistics. Işık and Kaya [12] have developed double acceptance sampling plan for Binomial distribution based on neutrosophic statistics. The previous studies have focused on Bayesian sampling plan under one-sided decision function. However, the decision function may be one-sided or two-sided, the first original aspect of the problem is illustrated in Figure 1.

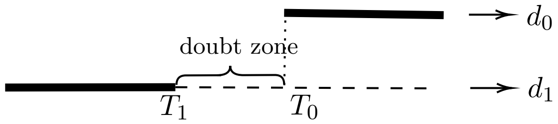

Figure 1. Schematic representation of a decision function with doubt zone.

where d_0 and d_1 represent respectively the decisions of accepting and rejecting the batch. T_0 and T_1 denote respectively the minimum acceptable and the maximum rejectable surviving time. The doubt zone is a family of regular functions that will be chosen according to the suitability of various models. In the classic case, we pass from d_1 to d_0 with a discontinuity i.e. $T_0 = T_1$. Nevertheless, it is crucial to be addressed to the situation when $T_1 < T_0$. With this goal, this paper provides an alternative approach to design a two-sided decision function with a linear doubt zone, such that the transition from d_1 to d_0 is done by a linear random function, as shown in Figure 2.

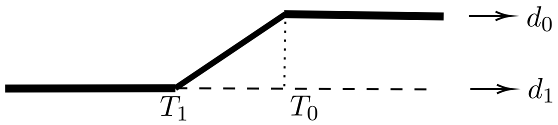

Figure 2. Schematic representation of a decision function with linear doubt zone.

Our aim is to develop a single variable sampling plan when the lifetime of product follows the exponential distribution. The sampling plan is achieved by using a two-sided decision function and under type II censoring. The remainder of this paper is organized as follows. In Section 2, we describe the proposed model and all necessary assumptions. Further, we obtain an explicit expression for the Bayes risk based on a polynomial loss function. In Section 3 we determine an upper bound of the sample size and we suggest an algorithm with an approximation method for finding a local optimal sampling plan. In Section 4, we introduce some numerical results followed by some extensive tables. Finally, Section 5 concludes with a summary.

2. Model

Assume that we have a batch of components presented for life testing. The lifetime of each item is a random variable X which follows a exponential distribution $Exp(\lambda)$ with expected lifetime $1/\lambda$ and the probability density function (pdf)

$$f(x|\lambda) = \begin{cases} \lambda \exp(-\lambda x), & \text{for } x \geq 0, \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

with the scale parameter λ is unknown. We suppose also that λ has a prior distribution $\Gamma(\alpha, \beta)$, where α and β are known, with the pdf

$$g(\lambda) = \begin{cases} \lambda^{\alpha-1} \exp(-\beta\lambda)\beta^{\alpha}/\Gamma(\alpha), & \text{for } \lambda > 0, \\ 0, & \text{otherwise}. \end{cases} \tag{2.2}$$

A random sample $(X_1, X_2, \dots, X_n)$ of size n is taken from a batch for life testing. Let $(X_{(1)}, X_{(2)}, \dots, X_{(n)})$ be the order statistic of $(X_1, X_2, \dots, X_n)$. Assume that the type II censoring is adopted, the life test is interrupted after a certain number $m(\leq n)$ of failed items, where we only observe $\underline{X} = \left(X_{(1)}, X_{(2)}, \dots, X_{(m)}\right)$. Then, the MLE of the expected lifetime $\theta = 1/\lambda$ is given by

$$\hat{\theta} = \frac{\sum_{i=1}^{m} X_{(i)} + (n-m)X_{(m)}}{m}. \tag{2.3}$$

Remark 2.1. $\hat{\theta}$ defined by Equation (2.3) follows a $\Gamma(m, m\lambda)$ distribution.

Note that the proof of Remark 2.1. is existed in the literature. However, to make this paper a self value contain, the following proof is presented:
Let $(X_1, X_2, \dots, X_n)$ a sample is selected from $Exp(\lambda)$, and $(X_{(1)}, X_{(2)}, \dots, X_{(n)})$ is the order statistic, then the joint pdf of $(X_{(1)}, X_{(2)}, \dots, X_{(m)})$ is

$$g(x_{(1)}, x_{(2)}, \dots, x_{(m)}) = \frac{n!}{(n-m)!}\lambda^m \exp\left(-\lambda\left[\sum_{i=1}^{m} X_{(i)} + (n-m)X_{(m)}\right]\right).$$

According to the transformation $Y_1 = X_{(1)}$ $Y_i = X_{(i)} - X_{(i-1)}$ for $i = 2, \dots, n$ we have

$$\begin{cases} X_{(1)} = Y_1 \\ X_{(2)} = Y_1 + Y_2 \\ X_{(3)} = Y_1 + Y_2 + Y_3 \\ \vdots \\ X_{(m)} = Y_1 + Y_2 + Y_3 + \dots + Y_m. \end{cases}$$

Therefore,

$$\sum_{i=1}^{m} X_{(i)} + (n-m)X_{(m)} = \sum_{i=1}^{m}(n-i+1)Y_i$$

and

$$g(x_{(1)}, x_{(2)}, \dots, x_{(m)}) = f(y_1, y_2, \dots, y_m) = \frac{n!}{(n-m)!}\lambda^m \exp\left(-\lambda\sum_{i=1}^{m}(n-i+1)y_i\right)$$
$$= n\lambda e^{-\lambda n y_1}(n-1)\lambda e^{-\lambda(n-1)y_2} \dots (n-m+2)\lambda e^{-2\lambda y_{n-1}}(n-m+1)\lambda e^{-\lambda y_m}.$$

Thus, the $Y_1, \dots, Y_m$ are mutually independent, such that $Y_i \rightsquigarrow Exp((n-i+1)\lambda)$ for $(i = 1, \dots, m)$, and therefore,

$$\frac{\sum\limits_{i=1}^{m} X_{(i)} + (n-m)X_{(m)}}{m} \rightsquigarrow \Gamma(m, m\lambda).$$

The proof is completed.

2.1. Decision role

Based on the observed data $\underline{x} = (x_{(1)}, x_{(2)}, \ldots, x_{(m)})$, a decision function $\delta(\underline{x})$ is made. We consider the following two-sided decision function

$$\delta(\underline{x}) = \begin{cases} d_0, & \text{for } \hat{\theta} \geq T_0, \\ \begin{cases} d_1, & \text{with probability } p_{\hat{\theta}} \\ d_0, & \text{with probability } 1 - p_{\hat{\theta}} \end{cases} & \text{for } T_1 \leq \hat{\theta} < T_0, \\ d_1, & \text{for } \hat{\theta} < T_1, \end{cases} \tag{2.4}$$

where, $p_{\hat{\theta}} = \dfrac{T_0 - \hat{\theta}}{T_0 - T_1}$. The loss to make a decision between d_0 and d_1, and select the sampling plan (n, m, T_0, T_1) is defined as follows:

$$L(\lambda, \delta(\underline{x})) = \begin{cases} nC_s - (n-m)v_s + \tau C_t + \sum\limits_{i=0}^{k} a_i \lambda^i, & \text{if } \delta(\underline{x}) = \delta_0, \\ nC_s - (n-m)v_s + \tau C_t + C_r, & \text{if } \delta(\underline{x}) = \delta_1, \end{cases} \tag{2.5}$$

where, the random variable $X_{(m)}$ is the censoring time and m is the number of failure, the parameters C_s, C_t and C_r are positive constants and represent respectively the unit inspection cost, the cost per unit of time used for the test and the loss due to rejection of the batch, the quantity $a_0 + a_1\lambda + \cdots + a_k\lambda^k$ denotes the loss of accepting the batch and be positive and increasing in λ. When the life test was finished, the unfailure items can be reused and therefore have the salvage value v_s, where $0 \leq v_s < C_s$.

To derive the sampling plan (n, m, T_0, T_1) based on the decision function $\delta(\underline{x})$, we carried out the following procedure:

(1) Select a random sample of size n from the batch for life testing based on type II censored samples.
(2) Interrupt the test until the m-th failure is observed with $m \leq n$, and record the value of $x_{(1)}, x_{(2)}, \ldots, x_{(m)}$.
(3) Compute the quantity $\hat{\theta}$ as

$$\hat{\theta} = \frac{\sum\limits_{i=1}^{m} x_{(i)} + (n-m)x_{(m)}}{m}.$$

(4) Accept the batch if $\hat{\theta} \geq T_0$, and reject the batch if $\hat{\theta} < T_1$. If $T_1 \leq \hat{\theta} < T_0$ the batch is rejected and accepted with probability $p_{\hat{\theta}} = (T_0 - \hat{\theta})/(T_0 - T_1)$ and $1 - p_{\hat{\theta}}$ respectively.

Theorem 2.2. *The Bayes risk can be described by the following equation:*

$$R(n, m, T_0, T_1) = nC_s - (n-m)v_s + C_t \tfrac{\beta}{\alpha} \tbinom{n}{m-1} \sum_{j=0}^{m-1} (-1)^j \tbinom{m-1}{j} \tfrac{n-m+1}{(n+j-m+1)^2} + C_r$$
$$+ \sum_{i=0}^{k} A_i \tfrac{\Gamma(\alpha+i)}{\beta^i \Gamma(\alpha)} \Big\{ 1 - I_{q_1}(m, \alpha+i) - \tfrac{T_0}{T_0 - T_1} \left[I_{q_0}(m, \alpha+i) - I_{q_1}(m, \alpha+i) \right]$$
$$+ \tfrac{\beta}{(\alpha+i-1)(T_0-T_1)} \left[I_{q_0}(m+1, \alpha+i-1) - I_{q_1}(m+1, \alpha+i-1) \right] \Big\},$$

where $q_i = mT_i/(mT_i + \beta)$ for $i = 0, 1$, and $B_x(a, b)$, $I_x(a, b)$ denote the incomplete Beta function and the incomplete Beta ratio respectively.

$$A_i = \begin{cases} a_0 - C_r, & \textit{for } i = 0, \\ a_i, & \textit{for } i = 1, \ldots, k \end{cases}$$

Proof. Based on the loss function defined in Equation (2.5), the expression of the risk can be computed as follows:

$$\begin{aligned} R\left(n,m,T_0,T_1\right) &= E\{E\left[L(\lambda,\delta(\underline{x}))\right]\} \\ &= E\left\{E\left[nC_s + C_t\tau - (n-m)v_s + \left(1-d_0\right)C_r + d_0\sum_{i=0}^{k} a_i\lambda^i|\lambda\right]\right\} \\ &= nC_s - (n-m)v_s + C_tE\left\{E\left[X_{(m)}\right]|\lambda\right\} + C_r + r(n,m|d_0) \end{aligned}$$

Such as

$$\begin{aligned} r(n,m,|d_0) &= E\left\{E\left[d_0\left(\sum_{i=0}^{k} a_i\lambda^i - C_r\right)|\lambda\right]\right\} \\ &= E\left\{E\left[\sum_{i=0}^{k} A_i\lambda^i(1_{\ \hat{\theta}>T_1} - p_{\hat{\theta}}1_{T_1\le\hat{\theta}<T_0})|\lambda\right]\right\} \\ &= \sum_{i=0}^{k} A_i\tfrac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty e^{-\beta\lambda}\lambda^{\alpha+i-1}\left[\int_0^{T_1} f_{\hat{\theta}}\left(y\right)dy + \int_{T_1}^{T_0}\tfrac{T_0-y}{T_0-T_1}f_{\hat{\theta}}\left(y\right)dy\right]d\lambda \\ &= \sum_{i=0}^{k} A_i\int_0^\infty \tfrac{m^m\beta^\alpha}{\Gamma(\alpha)\Gamma(m)}\lambda^{m+\alpha+i-1}\left\{\int_{T_1}^{\infty} e^{-(my+\beta)\lambda}y^{m-1}dy - \int_{T_1}^{T_0}\tfrac{T_0-y}{T_0-T_1}e^{-(my+\beta)\lambda}y^{m-1}dy\right\}d\lambda \\ &= \sum_{i=0}^{k} A_i\tfrac{m^m\beta^\alpha\Gamma(m+\alpha+i)}{\Gamma(\alpha)\Gamma(m)}\left\{\int_{T_1}^{\infty}\tfrac{y^{m-1}}{(my+\beta)^{m+\alpha+i}}dy - \int_{T_1}^{T_0}\tfrac{T_0-y}{T_0-T_1}\tfrac{y^{m-1}}{(my+\beta)^{m+\alpha+i}}dy\right\}. \end{aligned}$$

For $z = \dfrac{my}{my+\beta}$, therefore we have

$$\begin{aligned} r(n,m|d_0) &= \sum_{i=0}^{k} A_i\tfrac{\Gamma(m+\alpha+i)}{\Gamma(\alpha)\Gamma(m)\beta^i}\int_{q_1}^{1} z^{m-1}\left(1-z\right)^{\alpha+i-1}dz - \int_{q_1}^{q_0}\frac{T_0-\frac{\beta z}{m(1-z)}}{T_0-T_1}z^{m-1}\left(1-z\right)^{\alpha+i-1}dz \\ &= \sum_{i=0}^{k} A_i\tfrac{\Gamma(\alpha+i)}{\beta^i\Gamma(\alpha)}\Big\{1 - I_{q_1}(m,\alpha+i) - \tfrac{T_0}{T_0-T_1}\left[I_{q_0}(m,\alpha+i) - I_{q_1}(m,\alpha+i)\right] \\ &\quad + \tfrac{\beta}{(\alpha+i-1)(T_0-T_1)}\left[I_{q_0}(m+1,\alpha+i-1) - I_{q_1}(m+1,\alpha+i-1)\right]\Big\}, \end{aligned}$$

and, by a standard computation, the expression of the expected censoring time is given by

$$E\left\{E\left[X_{(m)}\right]|\lambda\right\} = \tfrac{\beta}{\alpha}\tbinom{n}{m-1}\sum_{j=0}^{m-1}(-1)^j\tbinom{m-1}{j}\tfrac{n-m+1}{(n+j-m+1)^2},$$

therefore

$$\begin{aligned} R\left(n,m,T_0,T_1\right) &= nC_s - (n-m)v_s + C_t\tfrac{\beta}{\alpha}\tbinom{n}{m-1}\sum_{j=0}^{m-1}(-1)^j\tbinom{m-1}{j}\tfrac{n-m+1}{(n+j-m+1)^2} + C_r \\ &\quad + \sum_{i=0}^{k} A_i\tfrac{\Gamma(\alpha+i)}{\beta^i\Gamma(\alpha)}\Big\{1 - I_{q_1}(m,\alpha+i) - \tfrac{T_0}{T_0-T_1}\left[I_{q_0}(m,\alpha+i) - I_{q_1}(m,\alpha+i)\right] \\ &\quad + \tfrac{\beta}{(\alpha+i-1)(T_0-T_1)}\left[I_{q_0}(m+1,\alpha+i-1) - I_{q_1}(m+1,\alpha+i-1)\right]\Big\}. \end{aligned}$$

Thus, the proof is completed. □

3. Algorithm and numerical approximation

The expression of $R\left(n,m,T_0,T_1\right)$ is quite complicated, so we cannot evaluate it analytically, based on an approximation method by considering $T_1 = lT_0$, where $0 < l < 1$, we can obtain local optimal sampling plan numerically. By using the Lagrange polynomials for the interpolation points $x_0 = 0.5$, $x_1 = 1.0$ and $x_2 = 1.5$, the derivative of $R(n, m, T_0, lT_0)$ with respect to T_0 reduces to a quadratic equation which is given by

$$\sum_{i=0}^{2} \left[h(x_i) \prod_{j=0, j\neq i}^{2} \frac{T_0 - x_j}{x_i - x_j} \right] = 0, \tag{3.1}$$

where $h(T_0) = \frac{\partial R(n,r,T_0,lT_0)}{\partial T_0}$.

3.1. An upper bound for the optimal sample size

In order to obtain the optimal sampling plan, we suggest an upper bound for the optimal sample size i.e. we can get a local optimal sampling plan in a finite number of search steps.

Theorem 3.1. *The optimal size of the sample is bounded by*

$$N = \min\left\{ \left[\frac{C_r}{C_s - v_s} \right], \left[\frac{\sum_{i=0}^{k} a_i \gamma_i}{C_s - v_s} \right] \right\}, \tag{3.2}$$

where $[x]$ is the integer part of x, and γ_i represent the i-th moment of λ.

Proof. Let $(0, 0, 0, 0)$ and $(0, 0, \infty, \infty)$ be the sampling plans that accepts and rejects the batch at no sampling case. For (n', m', T_0', T_1') an optimal sampling plan, then $R(n', m', T_0', T_1') \leq R(0, 0, 0, 0) = \sum_{i=0}^{k} a_i \gamma_i$. and $R(n', m', T_0', T_1') \leq R(0, 0, \infty, \infty) = C_r$. As $n(C_s - v_s) \leq R(n', m', T_0', T_1')$, therefore we have

$$n(C_s - v_s) \leq \min\left\{ C_r, \sum_{i=0}^{k} a_i \gamma_i \right\}$$
$$n \leq \min\left\{ \left[\frac{C_r}{C_s - v_s} \right], \left[\frac{\sum_{i=0}^{k} a_i \gamma_i}{C_s - v_s} \right] \right\}.$$

Thus, the proof is completed. □

3.2. Derivation of the optimal sampling plan

In this subsection we determine local optimal sampling plans for two situations, when T_1 is fixed in the interval $(0, T_0)$ for fixed l, we denote its related local optimal sampling plan by (n_0, m_0, T_0, T_1), and when T_1 is flexible in the interval $(0, T_0)$, the correspondent local optimal sampling plan is (n_B, m_B, T_0, T_1). To derive the local optimal sampling plan under the two situations, the following two schemes are proposed

Scheme 1:

(1) Start with $(n, m) = (0, 0)$, compute N from Equation (3.2), and compute $R(0, 0, T_0(n, m), T_1(n, m)) = min\left\{ R(0, 0, \infty, \infty) = C_r, R(0, 0, 0, 0) = \sum_{i=0}^{k} a_i \gamma_i \right\}$.

(2) For each $n = 1, \ldots, N$ and $m = 1, \ldots, n$, compute the optimal $T_0(n, m)$ and $T_1(n, m) = lT_0(n, m)$ such that $T_0(n, m) = \underset{1\leq i\leq 2}{min} \left\{ R\left(n, m, T_0^{(i)}, lT_0^{(i)}\right) | T_0^{(i)} > 0 \right\}$, $T_0^{(i)}$ is the i-th solution of Equation (3.1).

(3) By comparison, choose (n_0, m_0, T_0, T_1) which corresponds to the smallest value of the Bayes risks $R(n, m, T_0(n, m), T_1(n, m))$.

Scheme 2:

(1) Start with $(n, m) = (0, 0)$, compute N from Equation (3.2), and compute $R(0, 0, T_0(n, m), T_1(n, m)) = min\left\{ R(0, 0, \infty, \infty) = C_r, R(0, 0, 0, 0) = \sum_{i=0}^{k} a_i \gamma_i \right\}$.

(2) For each $n = 1, \ldots, N$ and $m = 1, \ldots, n$, compute the optimal $T_0(n,m)$ such that $T_0(n,m) = \underset{1 \le i \le 2}{min} \left\{ R\left(n, m, T_0^{(i)}, lT_0^{(i)}\right) |T_0^{(i)} > 0 \right\}$, $T_0^{(i)}$ is the i-th solution Equation (3.1). Compute $T_1(n,m)$ using grid search method with grid size $T_0(n,m)/1000$.

(3) By comparison, choose (n_B, m_B, T_0, T_1) which corresponds to the smallest value of the Bayes risks $R(n, m, T_0(n,m), T_1(n,m))$.

4. Numerical results

To illustrate the proposed model, we assume that the loss is a quadratic function with $(k = 2)$. Various numerical examples are depicted in Tables 1-6. In each table we denote $(n_B, m_B, T_0, T_1) \equiv S_B$ and $(n_0, m_0, T_0, T_1) \equiv S_0$, and their minimum Bayes risk respectively $R(n_B, m_B, T_0, T_1) \equiv R_B$ and $R(n_0, m_0, T_0, T_1) \equiv R_0$. To examine the behavior of the Bayes risk function, we vary one(two) parameter(s) or coefficient(s) while the others keep fixed. So, as the true values of parameters and coefficients of the model for which we made the calculations, we take: $\alpha = 2.5$, $\beta = 1$, $a_0 = a_1 = a_2 = 5$, $C_s = 0.5$, $v_s = 0.2$, $C_t = 2$ and $C_r = 50$. For the sampling plan (n_0, m_0, T_0, T_1) we take $l = 2/3$. Furher, the standard values are indicated by '*', and the corresponding local optimal sampling plan and the minimum Bayes risk are in bold. The efficiency values in Tables 1-6 defined by eff$= 100R_0/R_B$ is the ratio of the minimum Bayes risks R_0 and R_B.

The local optimal sampling plan for the standard values of parameters and coefficients mentioned above for $(n_B, m_B, T_0, T_1) = (6, 4, 0.4860, 0.3611)$, which means: we put 6 items for life testing and the life test terminates after the 4-th failure. We may accept the batch if the estimator of the average lifetime $\hat{\theta}$ is greater than 0.4860. For $\hat{\theta}$ is between 0.4860 and 0.3611, the batch is rejected and accepted with probability $p_{\hat{\theta}} = \frac{0.4860 - \hat{\theta}}{0.4860 - 0.3611}$ and $1 - p_{\hat{\theta}}$, with the Byes risk $R_B = 43.1853$. Further, corresponding to $(\alpha, \beta, a_0, a_1, a_2, C_s, v_s, C_t, C_r) = (2.5, 1, 5, 5, 5, 0.5, 0.2, 2, 50)$, the local optimal sampling plan (n_0, m_0, T_0, T_1) is given by $(6, 4, 0.5638, 0.3759)$, the correspondent Bayes risk is $R_0 = 43.3182$.

Table 1. Local optimal sampling plans and their minimum Bayes risks for α and β
vary.

α	β	n_B	m_B	T_0	T_1	R_B	n_0	m_0	T_0	T_1	R_0	eff(%)
1.5	0.2	2	1	0.9941	0.9135	49.4396	3	2	0.9495	0.6330	49.4692	100.06
1.5	0.4	5	3	0.5381	0.5375	45.3279	5	3	0.7794	0.5196	45.8101	101.06
1.5	0.6	6	3	0.4883	0.3862	40.5936	7	4	0.5702	0.3801	40.7736	100.44
1.5	0.8	7	3	0.4532	0.2624	36.1097	7	3	0.3671	0.2448	36.3404	100.64
2.0	0.2	0	0	∞	∞	50.0000	0	0	∞	∞	50.0000	100.00
2.0	0.4	3	2	0.9503	0.6063	49.2325	3	2	0.9425	0.6284	49.2339	100.00
2.0	0.6	6	4	0.4974	0.4591	45.8774	5	3	0.7731	0.5154	46.3920	101.12
2.0	0.8	7	4	0.4864	0.3629	42.0000	7	4	0.5670	0.3780	42.1410	100.34
0.0	0.0	0	0	∞	∞	50.0000	0	0	∞	∞	50.0000	100.00
2.5	0.6	3	2	0.9425	0.6088	49.3219	3	2	0.9367	0.6245	49.3227	100.00
2.5	0.8	6	4	0.4977	0.4579	46.4831	5	3	0.7676	0.5118	47.0093	101.13
2.5*	**1.0***	**6**	**4**	**0.4860**	**0.3611**	**43.1853**	**6**	**4**	**0.5638**	**0.3759**	**43.3182**	**100.3**
3.0	0.8	3	2	0.9355	0.6109	49.4788	3	2	0.9316	0.6211	49.4792	100.00
3.0	1.0	6	4	0.4979	0.4565	47.0288	4	3	0.7628	0.5085	47.5362	101.08
3.0	1.2	6	4	0.4867	0.3597	44.1567	6	4	0.5608	0.3738	44.2800	100.28
0.0	0.0	0	0	∞	∞	50.0000	0	0	∞	∞	50.0000	100.00
3.5	1.0	3	2	0.9294	0.6125	49.6417	3	2	0.9271	0.6181	49.6418	100.00
3.5	1.2	6	4	0.4980	0.4552	47.5008	4	3	0.7585	0.5057	47.9645	100.98

In Table 1, it is easy to see that R_B is less than R_0 for each variation of (α, β) and we observe that the difference between R_B and R_0 becomes small for α is fixed and β decreases. Further, for $(\alpha, \beta) = (2.0, 0.2), (2.5, 0.4), (3.5, 0.8)$, the batch is rejected without take any sample cost, with $S_B = S_0 = (0, 0, \infty, \infty)$ and $R_B = R_0 = 50$. On the other hand, n_B and m_B in most cases are greater than or equal to n_0 and m_0 respectively, so by the sampling plan (n_B, m_B, T_0, T_1) we can observe more information about the expected lifetime $1/\lambda$ of the items set in life testing. So, based on these data, the decision function $\delta(\underline{x})$ can be an appropriate decision with a smaller risk. In Tables 2-4, we observe that, the Bayes risk for both cases is nondecreasing in a_0 ,a_1 and a_2. As expected, in each table, we can see that R_B is always less than R_0, while in most cases the efficiency values are close to 1 and in general less than 101%. Furthermore, there are some cases where the sampling plans S_B and S_0 occur at no sampling case, as can be seen from Table 4, when $a_2 = 2$, the optimal sampling plans take the form $(0, 0, 0, 0)$, which means the batch is accepted without take any sample cost with the Bayes risk $R_B = R_0 = 35$. In Table 5, it can be seen that, the number of observed failures is equal to the optimal sample size when C_t closes to 0. Otherwise, when C_t increases, the number of observed failures decreases and the optimal sample size increases, and this is due to the effect of C_t value on behavior of the Bayes risk. In Table 6, as expected the Bayes risk of S_B and S_0 is increasing when C_r increase with $R_B < R_0$.

Table 2. Local optimal sampling plans and their minimum Bayes risks for a_0 varies.

a_0	n_B	m_B	T_0	T_1	R_B	n_0	m_0	T_0	T_1	R_0	eff(%)
0	6	4	0.4736	0.2884	40.7881	5	3	0.4317	0.2878	41.0116	100.55
1	6	4	0.4770	0.3020	41.2778	6	4	0.4348	0.2899	41.3476	100.17
2	6	4	0.4792	0.3158	41.7626	6	4	0.4700	0.3133	41.7655	100.01
3	6	4	0.4813	0.3302	42.2422	6	4	0.5031	0.3354	42.2557	100.03
4	6	4	0.4847	0.3456	42.7165	6	4	0.5343	0.3562	42.7810	100.15
5*	**6**	**4**	**0.4860**	**0.3611**	**43.1853**	**6**	**4**	**0.5638**	**0.3759**	**43.3182**	**100.31**
6	6	4	0.4889	0.3779	43.6483	6	4	0.5918	0.3945	43.8531	100.47
8	6	4	0.4931	0.4137	44.5566	6	4	0.6435	0.4290	44.8853	100.74
10	5	3	0.4988	0.4898	45.4289	5	3	0.7155	0.4770	45.7791	100.77
15	5	3	0.6685	0.6678	47.5853	4	2	0.8579	0.5720	47.6792	100.20
20	4	2	0.9183	0.6897	49.0483	4	2	0.9443	0.6295	49.0575	100.02
25	0	0	∞	∞	∞	0	0	∞	∞	50.0000	100.00

Table 3. Local optimal sampling plans and their minimum Bayes risks for a_1 varies.

a_1	n_B	m_B	T_0	T_1	R_B	n_0	m_0	T_0	T_1	R_0	eff(%)
0	6	4	0.4429	0.1873	39.2967	2	1	0.1493	0.0995	43.7406	111.31
1	6	4	0.4559	0.2197	40.1363	2	1	0.3334	0.2222	42.7147	106.42
2	6	4	0.4660	0.2530	40.9466	4	2	0.3758	0.2505	41.8472	102.20
3	6	4	0.4735	0.2874	41.7254	5	3	0.4300	0.2867	41.8943	100.40
4	6	4	0.4806	0.3235	42.4716	6	4	0.4880	0.3253	42.4733	100.00
5*	**6**	**4**	**0.4860**	**0.3611**	**43.1853**	**6**	**4**	**0.5638**	**0.3759**	**43.3182**	**100.31**
6	6	4	0.4915	0.4011	43.8664	6	4	0.6259	0.4173	44.1644	100.68
8	6	4	0.4989	0.4875	45.1330	5	3	0.7454	0.4970	45.5426	100.91
10	6	4	0.6242	0.6236	46.4597	5	3	0.8163	0.5442	46.5692	100.24
15	4	2	0.9444	0.7546	48.3618	4	2	0.9831	0.6554	48.3910	100.06
20	4	2	0.9944	0.9606	49.4869	4	2	1.0698	0.7132	49.6149	100.26
25	0	0	∞	∞	50.0000	0	0	∞	∞	50.0000	100.00

Table 4. Local optimal sampling plans and their minimum Bayes risks for a_2 varies.

a_2	n_B	m_B	T_0	T_1	R_B	n_0	m_0	T_0	T_1	R_0	eff(%)
2	0	0	0.0000	0.0000	35.0000	0	0	0.0000	0.0000	35.0000	100.00
3	5	3	0.3906	0.1160	39.8685	0	0	0.0000	0.0000	43.7500	109.74
4	6	4	0.4637	0.2513	41.6877	4	2	0.3617	0.2411	42.4664	101.87
5*	**6**	**4**	**0.4860**	**0.3611**	**43.1853**	**6**	**4**	**0.5638**	**0.3759**	**43.3182**	**100.31**
6	6	4	0.4979	0.4675	44.4318	6	4	0.7057	0.4705	44.9243	101.11
7	6	4	0.6141	0.6135	45.6829	5	3	0.8154	0.5436	45.9398	100.56
8	5	3	0.8648	0.5976	46.6689	5	3	0.8732	0.5822	46.6700	100.00
9	5	3	0.9065	0.6309	47.2430	5	3	0.9157	0.6105	47.2452	100.00
10	5	3	0.9327	0.6725	47.7162	5	3	0.9484	0.6323	47.7250	100.02
12	5	3	0.9646	0.7630	48.4671	5	3	0.9960	0.6640	48.5151	100.10
15	5	3	0.9899	0.9028	49.2940	5	3	1.0425	0.6950	49.4732	100.36
20	0	0	∞	∞	50.0000	0	0	∞	∞	50.0000	100.00

Table 5. Local optimal sampling plans and their minimum Bayes risks for C_t varies.

C_t	n_B	m_B	T_0	T_1	R_B	n_0	m_0	T_0	T_1	R_0	eff(%)
0.1	5	5	0.4859	0.3489	41.4131	5	5	0.5436	0.3624	41.5049	100.22
0.5	4	4	0.4860	0.3611	42.0130	5	5	0.5436	0.3624	42.1138	100.24
1.0	5	4	0.4860	0.3611	42.4742	5	4	0.5638	0.3759	42.6071	100.31
1.5	6	4	0.4860	0.3611	42.8686	6	4	0.5638	0.3759	43.0015	100.31
2.0*	**6**	**4**	**0.4860**	**0.3611**	**43.1853**	**6**	**4**	**0.5638**	**0.3759**	**43.3182**	**100.31**
2.5	7	4	0.4860	0.3611	43.4845	7	4	0.5638	0.3759	43.6174	100.31
3.0	7	4	0.4860	0.3611	43.7377	7	4	0.5638	0.3759	43.8706	100.30
4.0	6	3	0.4878	0.3824	44.1684	6	3	0.5919	0.3946	44.3425	100.39
5.0	7	3	0.4878	0.3824	44.5223	7	3	0.5919	0.3946	44.6965	100.39
10.0	9	3	0.4878	0.3824	45.9504	9	3	0.5919	0.3946	46.1245	100.38
15.0	9	2	0.4910	0.4261	46.8565	9	2	0.6359	0.4239	47.0473	100.41
20.0	10	2	0.4910	0.4261	47.6102	10	2	0.6359	0.4239	47.8010	100.40

Table 6. Local optimal sampling plans and their minimum Bayes risks for C_r varies.

C_r	n_B	m_B	T_0	T_1	R_B	n_0	m_0	T_0	T_1	R_0	eff(%)
30	0	0	∞	∞	30.0000	0	0	∞	∞	30.0000	100.00
35	4	2	0.9183	0.6897	34.0483	4	2	0.9443	0.6295	34.0575	100.03
40	5	3	0.6685	0.6678	37.5853	4	2	0.8579	0.5720	37.6792	100.25
45	5	3	0.4988	0.4898	40.4289	5	3	0.7155	0.4770	40.7791	100.87
50*	**6**	**4**	**0.4860**	**0.3611**	**43.1853**	**6**	**4**	**0.5638**	**0.3759**	**43.3182**	**100.31**
55	6	4	0.4736	0.2884	45.7881	5	3	0.4317	0.2878	46.0116	100.49
60	6	4	0.4574	0.2282	48.2709	4	2	0.2697	0.1798	50.7146	105.06
65	7	5	0.4404	0.1797	50.6478	2	1	0.0229	0.0153	61.1168	120.67
70	7	5	0.4129	0.1362	52.9360	0	0	0.0000	0.0000	61.2500	115.71
75	7	5	0.3739	0.0946	55.2441	0	0	0.0000	0.0000	61.2500	110.87
80	8	6	0.3169	0.0621	57.9032	0	0	0.0000	0.0000	61.2500	105.78
85	0	0	0.0000	0.0000	61.2500	0	0	0.0000	0.0000	61.2500	100.00

5. Conclusion

With the Bayesian approach, we have determined single sampling plans, for exponential distribution with type II censoring, we proposed a two-sided decision function with a random doubt zone for which we suggest an approach to decide the quality of a batch with a suitable probability. Explicit expression for the Bayes risk is obtained for the polynomial loss function which includes the unit inspection cost, the time-cost, the rejection cost, the salvage value and the after-sales cost. To evaluate the Bayes risk, we assume that the loss function is a quadratic function, such as the computations can be done in a similar way for higher degree. Furthermore, after determining an upper bound for the optimal size of sample, we developed a finite algorithm which allowed to simulate the risk function numerically based on the grid search method. Also we introduce a comparison performance and some optimal sampling plans followed by related Bayes risks for different values of parameters. From the numerical presented above, we can see that the Bayes risk based on the proposed random decision function have robust behavior with considering to changes in the parameters and coefficients in the proposed sampling plan.

In some real cases, the quality characteristics data is derived from a complex process or from an uncertain environment, so the use of the neutrosophic statistics can model the uncertainty and handle the human's assessments. To sum up, the current study can be extended using the neutrosophic statistics based on appropriate approach for the doubt zone in the decision function.

Acknowledgment

We are grateful to the Editor and the two referees for their careful reading of the manuscript, their insightful suggestions and for their comments as well that greatly improved the presentation of this work. The paper is partially supported by DGRSDT Grant CD656701 in Algeria.

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