



Casimir Değişmezlerinden Üç Eksenli ve Kübik Terimler

Triaxiality and Cubic Terms From Casimir Invariants

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Özet. Etkileşen Bozon Modeli (IBM) Hamiltoniyen'in $O(6)$ simetrisini kıran kübik terimler, $U(6)$ ve alt gruplarının Casimir değişmezlerinden oluşturulmuştur. Bu yazıda, yukarıdaki Casimir değişmezlerinden oluşturulabilen kübik terimleri ele alıyor ve bu tür terimlerin kalıcı üç eksenli deformasyona yol açıp açamayacağını araştırılmıştır. Kübik terimlerin klasik sınırı elde edilmiştir ve hiçbirinin kalıcı üç eksenli deformasyona yol açmasını beklemediği gösterilmiştir.

Anahtar Kelimeler: Casimir değişmezliği, Birbirini etkileyen Boson Modeli (IBM), Triaxial deformasyon.

Abstract. Cubic terms breaking $O(6)$ symmetry of the Interacting Boson Model (IBM) Hamiltonian are constructed out of the Casimir invariants of $U(6)$ and its subgroups. The classical limit of these cubic terms is obtained, and it is shown that none of them expect one can give rise to permanent triaxial deformation.

Keywords: Casimir invariants, Interacting Boson Model (IBM), Triaxial deformation.

1. Introduction

The isotopes of Xenon and Barium are known [1] to be examples of the O (6) dynamical symmetry of IBM [2-4]. Triaxiality is also known to occur in this region of the periodic table and an intimate connection between the two is speculated. It has been shown that the conventional Interacting Boson Model (IBM) Hamiltonian in none of the limits U (5), SU (3) and O(6) can give rise to permanent triaxial deformation [5]. The classical limit of the O (6) Hamiltonian expressed in terms of the deformation parameters β . The nucleus in the O (6) limit is thus γ unstable and will, in general, execute large oscillations in γ . The average value of γ for such oscillations may not be zero. The situation is very similar to that in the γ unstable model of Wilets and Jean [6]. Even in their original work establishing O (6) symmetry in Xe and Ba isotopes Casten and von Brentano considered departures from the O (6) symmetry [7]. Van Isacker and Chen [8,9] introduced in the O (6) Hamiltonian terms involving three boson creation and destruction operators (the so called “cubic terms”) and showed that such terms can give rise to permanent triaxial deformation [10-12]. Such cubic terms, when added to the O (6) Hamiltonian improve agreement with experiments.

The IBM Hamiltonian in the three limits can be written in terms of the Casimir invariants of U (5), SU (3), O (6) and their subgroups. In this paper we consider cubic terms which can be constructed out of the above Casimir invariants and investigate whether such terms can give rise to permanent triaxial deformation. Such constructions will not only make the nature of symmetry breaking transparent, but it will also simplify the evaluation of the matrix elements of the cubic terms. Interacting boson model is defined in Sec.2. In Sec. 3 cubic terms are constructed out of the Casimir invariants and their classical limits are evaluated. This followed by conclusions in Sec. 4.

2. The interacting boson model

In this section we give a brief description of the IBM with particular emphasis on the version of the model which includes higher-order interactions between the bosons, A full account of the IBM is given in ref. [13].

2.1 The general Hamiltonian

The building blocks of the IBM are s and d bosons with angular moments $l = 0$ and $l = 2$. A nucleus is characterized by a constant total number of bosons N which

equals half the number of valence nucleons (particles or holes, whichever is smaller). In this paper no distinction is made between neutron and proton bosons, an approximation which is known as IBM-1.

Since the Hamiltonian of the IBM-1 conserves the total number of bosons, it can be written in terms of the 36 operators $b_{lm}^+ b_{l'm'}$ where b_{lm}^+ (b_{lm}) creates (annihilates) a boson with angular momentum l and zprojection m . It can be shown [13] that this set of 36 operators generates the Lie algebra U(6) of unitary transformations in six dimensions. A Hamiltonian that conserves the total number of bosons is of the generic form

$$\hat{H} = E_0 + \hat{H}^{(1)} + \hat{H}^{(2)} + \hat{H}^{(3)} + \dots, \quad (1)$$

where the index refers to the order of the order of the interaction in the generators of U(6). The first term E_0 is a constant which represents the binding energy of the core.

The second term is the one-body part

$$\hat{H}^{(1)} = \epsilon_s [s^+ x s]_{\{0\}} + \epsilon_d \sqrt{5} [d^+ x \check{d}]_{\{0\}} \equiv \epsilon_s \tilde{n}_s + \epsilon_d \tilde{n}_d, \quad (2)$$

where x refers to coupling in angular momentum (shown as an upper script in round brackets), $\tilde{b}_{lm} \equiv (-)^{l-m} b_{l-m}$ and the coefficients ϵ_s and ϵ_d are the energies of the s and d bosons. The third term in the Hamiltonian (1) represents the two-body interaction

$$\hat{H}^{(2)} = \sum_{l_1 \leq l_2, l'_1 \leq l'_2, L} \tilde{v}_{l_1 l_2 l'_1 l'_2}^L [[b_{l_1}^+ x b_{l_2}^+]^{(L)} x [\tilde{b}_{l'_2} x \tilde{b}_{l'_1}]^{(L)}]_{\{0\}}^{\{0\}}, \quad (3)$$

where the coefficients \tilde{v} are related to the interaction matrix elements between normalized two-boson states,

$$\langle l_1 l_2; LM | \hat{H}^{(2)} | l'_1 l'_2; LM \rangle = \sqrt{\frac{(1 + \delta_{l_1 l_2})(1 + \delta_{l'_1 l'_2})}{2L + 1}} \tilde{v}_{l_1 l_2 l'_1 l'_2}^L. \quad (4)$$

Since the boson are necessarily symmetrically coupled, allowed two-boson states are $s^2(L=0)$, $sd(L=2)$ and $d^2(L=0,2,4)$. Since for n states with a given angular momentum one has $n(n+1)/2$ interactions, seven independent two-body interactions v are found; three for $L=0$, three for $L=2$ and one for $L=4$.

This analysis can be extended to higher-order interactions. One may consider, for example, the three-body interactions $\langle l_1 l_2 l_3; LM | \hat{H}^{(3)} | l'_1 l'_2 l'_3; LM \rangle$. The allowed three-boson states are $s^3(L = 0), s^2 d(L = 2), sd^2(L = 0, 2, 4)$ and $d^3(L = 0, 2, 3, 4, 6)$, leading to $6+6+1+3+1=17$ independent three-body interactions for $L = 0, 2, 3, 4, 6$, respectively. Note that any three-boson state $s^i d^{3-i}$ is fully characterized by its angular momentum L ; this is no longer the case for higher boson numbers when additional labels must be introduced.

The number of possible interactions at each order n is summarized in table 1 for up to $n = 3$. Some of these interactions exclusively contribute to the binding energy and do not influence the excitation spectrum of a nucleus. To determine the number of such interactions, one notes that the Hamiltonian $\hat{N}\hat{H}^{(n-1)}$ for constant boson number (i.e., a single nucleus) essentially reduces to the $(n - 1)$ -body Hamiltonian $\hat{H}^{(n-1)}$. Consequently, of the N_n independent interactions of order n contained in $\hat{H}^{(n)}$, N_{n-1} terms of the type $\hat{N}\hat{H}^{(n-1)}$ must be discarded if one wishes to retain only those that influence the excitation energies. For example, given that there is one term of order zero (i.e., a constant), one of the two first-order terms (i.e., the combination \hat{N}) does not influence the excitation spectrum. Likewise, there are two first-order terms (i.e., \hat{n}_s and \hat{n}_d) and hence two of the two-body interactions do not influence the excitation spectrum. This argument leads to the numbers quoted in Table 1.

Table 1 Enumeration of n -body interactions in IBM-1 for $n \leq 3$.

Order	Number of interactions		
	total	Type I ^a	Type II ^b
$n = 0$	1	1	0
$n = 1$	2	1	1
$n = 2$	7	2	5
$n = 3$	17	7	10
^a Interaction energy is constant for all states with the same N			
^b Interaction energy varies from state to state.			

We conclude that, in the nucleus-by-nucleus fits that will be performed in this work, there is a single one-boson energy of relevance, as well as five two-body and ten three-body interactions. This number of independent terms is too high for practical applications and simplifications must be sought on the basis of physical, empirical or formal arguments. Some of them are based on the classical limit of the IBM-1.

3. Cubic terms from Casimir invariants and their classical limits

The Casimir invariants of U (5) and its subgroups can be conveniently written in terms of the following operators.

$$P = \frac{1}{2}(\tilde{d}\tilde{d} - \tilde{s} \cdot \tilde{s}) \quad (5)$$

$$L = \sqrt{10}[d^+ \times d]^{(1)} \quad (6)$$

$$Q = [d^+ \times \tilde{s} + \tilde{s}^+ \times d]^{(2)} - \frac{\sqrt{7}}{2}[d^+ \times \tilde{d}]^{(2)} \quad (7)$$

$$U = [d^+ \times \tilde{d}]^{(3)} \quad (8)$$

$$V = [d^+ \times \tilde{d}]^{(4)} \quad (9)$$

Expressed in terms of these operators, the Casimir operators of U (6) and its subgroups in the three chains are given below [5]:

$$C_2(U6) = N(N + 5) \quad (10)$$

$$C_1(U5) = n_d \quad (11)$$

$$C_2(SU3) = \frac{2}{3} \left[2Q \cdot Q + \frac{3}{4} L \cdot L \right] \quad (12)$$

$$C_2(O6) = 2 [N(N + 4) - 4P^+ \cdot P] \quad (13)$$

$$C_2(O5) = 4 \left[\frac{1}{10} L \cdot L + U \cdot U \right] \quad (14)$$

$$C_2(O3) = 2 [L \cdot L] \quad (15)$$

Where $C_p(G)$ represents the Casimir operator of order p of the group G. The subgroup O (2) has been omitted because unless the nucleus is in a magnetic field, the Hamiltonian does not depend on the magnetic quantum number M. The O (6) limit of

the IBM Hamiltonian can be expressed as a linear combination of the linear and quadratic Casimir operators of U (6) and its subgroups.

$$H = e_0 + e_1 C_1(U6) + e_2 C_2(U6) + \beta C_2(O5) + \gamma C_2(O3) + \eta C_2(O6) \quad (16)$$

The following cubic terms can be constructed out of the Casimir invariants given equations (10) – (15):

$$[C_1(U5)]^3 = n_d^3 \quad (17)$$

$$C_2(O5)C_1(U5) = 4 \left[\frac{1}{10} L \cdot L + U \cdot U \right] n_d \quad (18)$$

$$C_2(O3)C_1(U5) = 2L^2 n_d \quad (19)$$

$$C_2(O6)C_1(U5) = 2 [N(N + 4) - 4P^+ \cdot P] n_d \quad (20)$$

$$C_2(SU_3) C_1(U5) = \frac{2}{3} [2Q \cdot Q + \frac{3}{4} L \cdot L] n_d \quad (21)$$

The classical limit of all the above terms have been calculated using the techniques of Van isacker and Chen[8]. Considering a coherent state.

$$|N_{23}\rangle = \left[\left(S^+ + 2 \cos 3d_0^+ + \frac{1}{\sqrt{2}} 2 \sin 3(d_{+2}^+ + d_{-2}^+) \right)^N \right] |10\rangle \quad (22)$$

Where N is the boson number and β and γ are the usual deformation parameters, the classical limit of an operator O is given by

$$O_{cl} = \frac{\langle N_{23} | O | N_{23} \rangle}{\langle N_{23} | N_{23} \rangle} \quad (23)$$

For a general three body operator of the form

$$A_3 = \sum_{ijkl > N} \langle ijkl >_N b_i^+ \cdot b_j^+ \cdot b_k^+ \cdot b_l \cdot b_{>} \cdot b_N A_3 = \sum_{ijkl > N} \langle b_i^+ b_j^+ b_k^+ b_l b_{>} b_N \rangle \quad (24)$$

Where b_i^+ (b_i) are boson creation (destruction) operators, the classical limit is given by:

$$\langle A_3 \rangle_{cl} = \frac{N(N-1)(N-2)}{K_6} \sum_{ijkl > N} K_i \cdot K_j \cdot K_k \cdot K_l \cdot K_{>} \cdot K_N \alpha_{ijkl > N} \quad (25)$$

Where α_i ' s are the coefficients of the boson creation operators in eqn. (25) and $\alpha^2 = \sum_i K_i^2$. Using equations (21 to 23) the following classical limits have been obtained:

$$\langle n_d^3 \rangle_{cl} = \frac{N(N-1)(N-2)}{(1+2^2)^3} 2^6 + \frac{N(N-1)}{(1+2^2)} 2^4 + \frac{N2}{(1+2^2)} \quad (26)$$

$$\langle L^2 n_d \rangle_{cl} = 2N(N-1) \frac{2^2}{(1+2^2)} + 2N \frac{2^2}{(1+2^2)} \quad (27)$$

$$\langle (U \cdot U) n_d \rangle_{cl} = 8N(N-1) \frac{2^4}{(1+2^2)} + 2N \frac{2^2}{(1+2^2)} \quad (28)$$

$$\langle (P^+ \cdot P) n_d \rangle_{cl} = \frac{1}{4} \frac{N(N-1)(N-2)}{(1+2^2)} [2^6 + 32^4 + 2^2] + \frac{1}{2} \frac{N(N-1)}{(1+2^2)} [2^4 + 2^2] \quad (29)$$

$$\begin{aligned} \langle (Q \cdot Q) n_d \rangle_{cl} = & \frac{1}{2} \frac{N(N-1)(N-2)}{(1+2^2)} [2^6 - 4\sqrt{22^5} \cos 33 + 82^4] + \frac{1}{2} \frac{N(N-1)}{(1+2^2)} [(2 + \cos^2 3) - \\ & 2\sqrt{22^3} (10 \cos 3 - 12 \cos^3 3) + 142^2 + 22^4] + \frac{1}{2} \frac{N-2}{2(1+2^2)} [(2 + \cos^2 3) - 4\sqrt{2} \cos 3 + \\ & 22] \end{aligned} \quad (30)$$

The operators in eqns. (26) – (30) are related to the cubic terms given in (17) – (21) Sec. 4.

4. Concluding Remarks

Permanent triaxial deformation will occur if the minima of the energy functional $E(N, \boldsymbol{\beta}, Y)$ with respect to $\boldsymbol{\beta}$ and Y occur for any value of Y other than 0^0 and 60^0 . The classical limit of the O(6) Hamiltonian is independent of Y and the classical limits given by eqns. (24) – (28) show that only the cubic term $n_d = \frac{3}{4} [\mathbf{C}_2(\mathbf{SU3}) - \frac{1}{4} \mathbf{C}_2(\mathbf{O3})] \mathbf{C}_1(\mathbf{U5})$ is capable of producing permanent triaxial deformation. Van Isacker and Chen considered cubic terms of the form $\mathbf{d}(\mathbf{1}, \mathbf{k}, \mathbf{3}) = [(\mathbf{d}^+ \mathbf{d}^+)^1 \mathbf{d}^+]^3 [(\mathbf{d} \mathbf{d})^k \mathbf{d}]^3$ which for $1=K$ has the classical limit $N(N-1)(N-2) \frac{2^6}{(1+2^2)^3} (\mathbf{A} + \mathbf{B} \cos^3 33)$. Minimization with respect to $\boldsymbol{\gamma}$ shows that such a cubic term is capable of producing a permanent triaxial deformation corresponding to $\gamma = 30^0$. This leads to the somewhat unnatural conclusion that all triaxial nuclei must have the same triaxial deformation corresponding to $\boldsymbol{\gamma} = \mathbf{30}^0$. Minimization of the right hand side of (30) with respect to $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ on the other hand, will yield values of γ which depend on $\boldsymbol{\beta}$. Thus, both the deformation parameters $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ will be different for different nuclei -a scenario more likely to be found in nature.

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