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Some Relations for the Generalized $\widetilde{\mathcal{G}}_{\mathrm{n}}, \widetilde{\mathcal{P}}_{\mathrm{n}}$ Integral Transforms and Riemann-Liouville, Weyl Integral Operators

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Highlights

- This paper gives some new integral transforms and Parseval-Goldstein type relations.
- A number of interesting infinite integrals are presented.
- Theorems on generalized Riemann-Liouville and Weyl fractional integral operators are obtained.

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Abstract

In this paper, Parseval-Goldstein type theorems involving the $\tilde{\mathcal{G}}_{\mathrm{n}}$-integral transform which is modified from $\mathcal{G}_{2n}$-integral transform [7] and the $\tilde{\mathcal{P}}_{n}$-integral transform [8] are examined. Then, theorems in this paper are shown to yield a number of new identities involving several well-known integral transforms. Using these theorems and their corollaries, a number of interesting infinite integrals of elementary and special functions are presented. Generalizations of Riemann-Liouville and Weyl fractional integral operators are also defined. Some theorems relating generalized Laplace transform, generalized Widder Potential transform, generalized Hankel transform and generalized Bessel transform are obtained. Some illustrative examples are given as applications of these theorems and their results.

1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

Stieltjes and Widder Potential transforms are well-known and used in many areas such as mathematical analysis, mathematical physics, applied mathematics and engineering sciences. Fractional derivative and integral were firstly discussed on a letter sent by Leibniz to L'Hospital, in which he wrote about the meaning of D the derivative of half-order. In the following centuries, the theory of fractional derivatives and integrals were developed by different authors that emerged new fractional derivative operators and their applications such as Riemann, Liouville, Weyl and Caputo [1-3]. Recently, the relations between fractional integral operators and classical integral transforms were given. New Parseval-Goldstein type identities were obtained [4-6].

In this paper, new relations are obtained using $\tilde{\mathcal{G}}_{\mathrm{n}}$-integral transform which is modified from $\mathcal{G}_{2n}$-integral transform [7] and $\tilde{\mathcal{P}}_{n}$-integral transform [8]. Two generalizations of fractional integrals are defined. New identities for two new generalized fractional integrals and generalized integral transforms [4,8] are obtained. Some definitions will be given, before the main results. For the convergence of the mentioned integrals in this manuscript, the class of the functions were considered in the related cited manuscripts, respectively.

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The generalizations of the Widder-Potential transform and the Glasser transform where $\mathcal{P}_{2n}$ and $\mathcal{G}_{2n}$ which are defined in [7,8], are given as follows:

$$\tilde{\mathcal{P}}_n\{f(x); y\} = \mathcal{P}_{2n}\{f(x); y\} = \int_0^\infty \frac{x^{2n-1}f(x)}{x^{2n}+y^{2n}}dx, \tag{1}$$

$$\tilde{\mathcal{G}}_n\{f(x); y\} = \mathcal{G}_{2n}\{x^{n-1}f(x); y\} = \int_0^\infty \frac{x^{n-1}f(x)}{\sqrt{x^{2n}+y^{2n}}}dx. \tag{2}$$

Replacing n with $2n$ in (1) and (2), the following definitions of integral transforms are obtained,

$$\tilde{\mathcal{P}}_{2n}\{f(x); y\} = \int_0^\infty \frac{x^{4n-1}f(x)}{x^{4n}+y^{4n}}dx, \tag{3}$$

$$\tilde{\mathcal{G}}_{2n}\{f(x); y\} = \int_0^\infty \frac{x^{2n-1}f(x)}{\sqrt{x^{4n}+y^{4n}}}dx, \tag{4}$$

respectively. The $\tilde{\mathcal{G}}_{2n}$-integral transform is related to the Glasser transform and generalized Stieltjes transform by means of,

$$2n\tilde{\mathcal{G}}_{2n}\{f(x); y\} = \mathcal{G}\{f(x^{1/2n}); y^{2n}\}, \tag{5}$$

$$4n\tilde{\mathcal{G}}_{2n}\{f(x); y\} = \mathcal{S}_{1/2}\{x^{-1/2}f(x^{1/4n}); y^{4n}\}, \tag{6}$$

where generalized Stieltjes transform is defined in [9]. The Stieltjes integral transform and Widder-Potential transform and Glasser transform are defined by [9-11],

$$\mathcal{S}\{f(x); y\} = \int_0^\infty \frac{f(x)}{x+y}dx, \tag{7}$$

$$\mathcal{P}\{f(x); y\} = \int_0^\infty \frac{xf(x)}{x^2+y^2}dx, \tag{8}$$

$$\mathcal{G}\{f(x); y\} = \int_0^\infty \frac{f(x)}{\sqrt{x^2+y^2}}dx, \tag{9}$$

respectively. The Widder Potential transform and the Stieltjes transform are related by the following relation [9],

$$\mathcal{P}\{f(x); y\} = \frac{1}{2}\mathcal{S}\{f(x^{1/2}); y^2\}. \tag{10}$$

Another generalization of the Widder Potential transform of f(x) is defined for $\nu \in \mathbb{C}$, $n \in \mathbb{N}$ in [4], as follows:

$$\tilde{\mathcal{P}}_{\nu,n}\{f(x); y\} = \mathcal{P}_{\nu,2n}\{f(x); y\} = \int_0^\infty \frac{x^{2n-1}f(x)}{(x^{2n}+y^{2n})^\nu}dx. \tag{11}$$

The $\tilde{\mathcal{P}}_n$-transform and the Stieltjes transform, the $\mathcal{P}_{\nu,2n}$-transform and the generalized Stieltjes transform are related by, respectively,

$$\tilde{\mathcal{P}}_n\{f(x); y\} = \frac{1}{2n}\mathcal{S}\{f(x^{1/2n}); y^{2n}\}, \tag{12}$$

$$\tilde{\mathcal{P}}_{\nu,n}\{f(x); y\} = \frac{1}{2n}\mathcal{S}_\nu\{f(x^{1/2n}); y^{2n}\}. \tag{13}$$

The Laplace transform [1], the $\mathcal{L}_{2n}$-transform, the $\mathcal{L}_{4n}$-transform [8] are defined by

$$\mathcal{L}\{f(x); y\} = \int_0^\infty e^{-yx}f(x)dx, \tag{14}$$

$$\mathcal{L}_{2n}\{f(x); y\} = \int_0^\infty x^{2n-1}e^{-y^{2n}x^{2n}}f(x)dx, \tag{15}$$

$$\mathcal{L}_{4n}\{f(x); y\} = \int_0^\infty x^{4n-1}e^{-x^{4n}y^{4n}}f(x)dx, \tag{16}$$

respectively. The $\mathcal{L}_{4n}$-transform, the $\mathcal{L}_{2n}$-transform and the Laplace transform are related with the following relations [8]:

$$\mathcal{L}_{4n}\{f(x);y\} = \frac{1}{2}\mathcal{L}_{2n}\{f(x^{1/2});y^2\} = \frac{1}{4n}\mathcal{L}\{f(x^{1/2n});y^{2n}\}. \tag{17}$$

The Hankel transform of order ν [1] and generalized Hankel transform [4,9] are defined by

$$\mathcal{H}_\nu\{f(x);y\} = \int_0^\infty (xy)^{1/2}\mathcal{J}_\nu(xy)f(x)dx, \tag{18}$$

$$\mathcal{H}_{\nu,n}\{f(x);y\} = \int_0^\infty x^{n-1}(x^n y^n)^{1/2}\mathcal{J}_\nu(x^n y^n)f(x)dx, \tag{19}$$

where $\nu \in \mathbb{C}$, $n \in \mathbb{N}$, $\mathrm{Re}(\nu) > -1/2$ and $\mathcal{J}_\nu(x)$ is the Bessel function of the first kind of order ν [12,13] that has the following series representation,

$$\mathcal{J}_\nu(x) = \sum_{n=0}^\infty \frac{(-1)^n}{n!\Gamma(1+\nu+n)}\left(\frac{x}{2}\right)^{\nu+2n}, \quad \mathrm{Re}(\nu) > -1. \tag{20}$$

Replacing n with $2n$ in (19), the $\mathcal{H}_{\nu,2n}$-transform which is defined in [4] is obtained.
The Hankel transform and generalized Hankel transform are related by the following relation:

$$n\mathcal{H}_{\nu,n}\{f(x);y\} = \mathcal{H}_\nu\{f(x^{1/n});y^n\}. \tag{21}$$

The Bessel transform of order ν [1] and generalized Bessel transform [4] are defined by

$$\mathcal{K}_\nu\{f(x);y\} = \int_0^\infty (xy)^{1/2}\mathcal{K}_\nu(xy)f(x)dx, \tag{22}$$

$$\mathcal{K}_{\nu,n}\{f(x);y\} = \int_0^\infty x^{n-1}(x^n y^n)^{1/2}\mathcal{K}_\nu(x^n y^n)f(x)dx, \tag{23}$$

where $\nu \in \mathbb{C}$, $n \in \mathbb{N}$, and $\mathcal{K}_\nu(x)$ is the modified Bessel function of the second kind of order ν [12,13] and is defined as:

$$\mathcal{K}_\nu(x) = \frac{\pi}{2}\frac{\mathcal{I}_{-\nu}(x)-\mathcal{I}_\nu(x)}{\sin(\pi\nu)}, \tag{24}$$

$$\mathcal{I}_\nu(x) = \sum_{n=0}^\infty \frac{1}{n!\Gamma(1+\nu+n)}\left(\frac{x}{2}\right)^{\nu+2n}, \quad \mathrm{Re}(\nu) > -1. \tag{25}$$

The Bessel transform and generalized Bessel transform are related by the following relation:

$$n\mathcal{K}_{\nu,n}\{f(x);y\} = \mathcal{K}_\nu\{f(x^{1/n});y^n\}. \tag{26}$$

Also, the following relation could be obtained easily from (23) and the formula [13, p.10, Entry(42)]:

$$\mathcal{K}_{1/2,2n}\{f(x);y\} = \sqrt{\frac{\pi}{2}}\mathcal{L}_{2n}\{f(x);y\}. \tag{27}$$

The $\mathcal{F}_{c,2n}$- and $\mathcal{F}_{s,2n}$-integral transforms [8] are defined by

$$\mathcal{F}_{c,2n}\{f(x);y\} = \int_0^\infty x^{2n-1}\cos(x^{2n}y^{2n})f(x)dx, \tag{28}$$

$$\mathcal{F}_{s,2n}\{f(x);y\} = \int_0^\infty x^{2n-1}\sin(x^{2n}y^{2n})f(x)dx, \qquad (\forall n \in \mathbb{N}) \tag{29}$$

which are related to the Fourier-cosine and Fourier-sine transforms [1] by means of the following relations:

$$2n\mathcal{F}_{c,2n}\{f(x);y\} = \mathcal{F}_c\{f(x^{1/2n});y^{2n}\}, \tag{30}$$

$$2n\mathcal{F}_{s,2n}\{f(x);y\} = \mathcal{F}_s\{f(x^{1/2n});y^{2n}\}. \tag{31}$$

The following identity is easily obtained from (28) and known formula [14, p.7. Entry(1)],

$$\mathcal{F}_{c,2n}\{\mathcal{F}_{c,2n}\{f(x);y\};x\} = \frac{\pi}{8n^2}f(x). \tag{32}$$

Weyl fractional integral operator of order μ is defined as follows:

$$\mathcal{W}^{-\mu}\{f(x);y\}=\frac{1}{\Gamma(\mu)}\int_y^\infty (x-y)^{\mu-1}f(x)dx, \tag{33}$$

where $y\geq 0$, $\mu\in\mathbb{C}$, $\mathrm{Re}(\mu)>0$ [1-3].
Riemann-Liouville fractional integral operator of order μ is defined as follows:

$$\mathcal{D}^{-\mu}\{f(x);y\}=\frac{1}{\Gamma(\mu)}\int_0^y (y-x)^{\mu-1}f(x)dx, \tag{34}$$

where $y\geq 0$, $\mu\in\mathbb{C}$, $\mathrm{Re}(\mu)>0$ [1-3].

Now, two new fractional integrals called the n-generalized Weyl fractional integral and the n-generalized Riemann-Liouville fractional integral are introduced.

The generalized Weyl fractional integral is defined as follows:

$$\mathcal{W}_{\mu,2n}\{f(x);y\}=\frac{1}{\Gamma(\mu)}\int_y^\infty x^{2n-1}(x^{2n}-y^{2n})^{\mu-1}f(x)dx, \tag{35}$$

where $y\geq 0$, $\mu\in\mathbb{C}$, $\mathrm{Re}(\mu)>0$.
The generalized Riemann-Liouville fractional integral is defined as follows:

$$\mathcal{R}_{\mu,2n}\{f(x);y\}=\frac{1}{\Gamma(\mu)}\int_0^y x^{2n-1}(y^{2n}-x^{2n})^{\mu-1}f(x)dx, \tag{36}$$

where $y\geq 0$, $\mu\in\mathbb{C}$, $\mathrm{Re}(\mu)>0$.
Weyl fractional derivative of order α is defined as follows:

$$\mathcal{W}^{\alpha}f(x)=\frac{d^n}{dx^n}\mathcal{W}^{-(n-\alpha)}f(x), \tag{37}$$

where $n\in\mathbb{N}$, $\alpha\in\mathbb{C}$, $\mathrm{Re}(\alpha)>0$ and $n-1<\mathrm{Re}(\alpha)\leq n$ [1-3].
Riemann-Liouville fractional derivative of order α is defined as follows:

$${}_0\mathcal{D}_y^{\alpha}f(x)=\frac{d^n}{dx^n}\mathcal{D}^{-(n-\alpha)}f(x), \tag{38}$$

where $n\in\mathbb{N}$, $\alpha\in\mathbb{C}$, $\mathrm{Re}(\alpha)>0$ and $n-1<\mathrm{Re}(\alpha)\leq n$ [1-3].

In the formulas (37) and (38), fractional derivatives are defined by means of fractional integral operators.

In definitions (33)-(36), $\Gamma(z)$ is the Gamma Euler function given by the following formula [12],

$$\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}dt, \tag{39}$$

where $z\in\mathbb{C}$, $\mathrm{Re}(z)>0$.
Incomplete gamma and complementary incomplete gamma functions are defined as follows [13]:

$$\gamma(a,x)=\int_0^x t^{a-1}e^{-t}dt, \tag{40}$$

$$\Gamma(a,x)=\int_x^\infty t^{a-1}e^{-t}dt. \tag{41}$$

The error and the complementary error functions are defined as follows [13]:

$$\mathrm{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt, \tag{42}$$

$$\mathrm{erfc}(x)=\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt, \tag{43}$$

where we have

$$\mathrm{erf}(x)+\mathrm{erfc}(x)=1. \tag{44}$$

2. MAIN THEOREMS

In the theorems and lemmas of this section, Parseval-Goldstein type new identities which show the relationship between the known integral transforms and the newly defined integral transforms and integral operators are given.

Theorem 1. If the integrals involved converge absolutely and $n \in \mathbb{N}, -1 < \mathrm{Re}\nu < 1/2$, then the following identities hold true:

$$\tilde{\mathcal{G}}_{2n}\left\{u^{2n\nu+n}\mathcal{H}_{\nu,2n}\{x^n f(x); u\}; y\right\} = \frac{y^{2n\nu}}{n\sqrt{2\pi}}\mathcal{K}_{\nu+1/2,2n}\{f(x); y\}, \tag{45}$$

$$\mathcal{H}_{\nu,2n}\left\{u^{2n\nu+n}\tilde{\mathcal{G}}_{2n}\{f(x); u\}; y\right\} = \frac{y^{-n}}{n\sqrt{2\pi}}\mathcal{K}_{\nu+1/2,2n}\{x^{2n\nu}f(x); y\}, \tag{46}$$

$$\tilde{\mathcal{G}}_{2n}\left\{u^{2n\nu+n}\mathcal{H}_{\nu,2n}\{x^{2n\nu+n}f(x); u\}; y\right\} = y^{2n\nu+n}\mathcal{H}_{\nu,2n}\left\{u^{2n\nu+n}\tilde{\mathcal{G}}_{2n}\{f(x); u\}; y\right\}. \tag{47}$$

Proof. Using the definitions (4) of the $\tilde{\mathcal{G}}_{2n}$-transform, the definition of the $\mathcal{H}_{\nu,2n}$-transform obtained by replacing n with $2n$ in (19), and changing the order of integration, which is permissible by absolute convergence of the integrals involved, it is found that

$$\tilde{\mathcal{G}}_{2n}\left\{u^{2n\nu+n}\mathcal{H}_{\nu,2n}\{x^n f(x); u\}; y\right\} = \int_0^\infty x^{4n-1}f(x)\tilde{\mathcal{G}}_{2n}\{u^{2n\nu+2n}\mathcal{J}_\nu(x^{2n}u^{2n}); y\}dx. \tag{48}$$

Using the relation (5) and the following known formula [11, p.174, (h)],

$$\mathcal{G}\{u^{\nu+1}\mathcal{J}_\nu(xu); y\} = \sqrt{\frac{2}{\pi x}}y^{\nu+1/2}\mathcal{K}_{\nu+1/2}(xy), \quad -1 < \mathrm{Re}\nu < \frac{1}{2}, \tag{49}$$

it is obtained that

$$\tilde{\mathcal{G}}_{2n}\left\{u^{2n\nu+n}\mathcal{H}_{\nu,2n}\{x^n f(x); u\}; y\right\} = \frac{1}{n\sqrt{2\pi}}y^{2n\nu}\int_0^\infty x^{2n-1}x^n y^n\mathcal{K}_{\nu+1/2}(x^{2n}y^{2n})f(x)dx. \tag{50}$$

Now, the assertion (45) follows from definition (23) of the $\mathcal{K}_{\nu,2n}$-integral transform with replacing n with $2n$.

Similarly, the proof of (46) would be given using definitions (4) of the $\tilde{\mathcal{G}}_{2n}$-transform, the definition of the $\mathcal{H}_{\nu,2n}$-transform obtained by replacing n with $2n$ in (19), the known formula [11, p.174, (h)], and the definition of the $\mathcal{K}_{\nu,2n}$-integral transform obtained by replacing n with $2n$ in (23), respectively. The assertion (47) immediately follows from (45) and (46). Thus, the proof of theorem is completed.

Remark 2. If $\nu = 0$ is set into (45) and (46) and the relation (27) is used, then the following relations are obtained:

$$\tilde{\mathcal{G}}_{2n}\left\{u^n\mathcal{H}_{0,2n}\{x^n f(x); u\}; y\right\} = \frac{1}{2n}\mathcal{L}_{2n}\{f(x); y\}, \tag{51}$$

$$\mathcal{H}_{0,2n}\left\{u^n\tilde{\mathcal{G}}_{2n}\{f(x); u\}; y\right\} = \frac{y^{-n}}{2n}\mathcal{L}_{2n}\{f(x); y\}. \tag{52}$$

Corollary 3. The following identity

$$\tilde{\mathcal{G}}_{2n}\{u^{2n\nu+n}g(u); y\} = \frac{2\sqrt{2n}}{\sqrt{\pi}}y^{2n\nu}\mathcal{K}_{\nu+1/2,2n}\left\{x^{-n}\mathcal{H}_{\nu,2n}\{g(u); x\}; y\right\}, \tag{53}$$

holds true, provided that $n \in \mathbb{N}$, $\mathrm{Re}\nu > 0$ and the integrals involved converge absolutely.

Proof. Putting

$$\mathcal{H}_{\nu,2n}\{x^n f(x); u\} = g(u) \tag{54}$$

in (45) of Theorem 1 and using the relation obtained by replacing n with $2n$ in (21), we get

$$\mathcal{H}_{\nu,2n}\{x^n f(x); u\} = \frac{1}{2n}\mathcal{H}_\nu\{x^{1/2} f(x^{1/2n}); u^{2n}\} = g(u),$$

$$\mathcal{H}_\nu\{x^{1/2} f(x^{1/2n}); u\} = 2ng(u^{1/2n}). \tag{55}$$

Using the definition of inverse Hankel transform, it is derived that

$$x^{1/2} f(x^{1/2n}) = \mathcal{H}_\nu\{2ng(u^{1/2n}); x\},$$

$$f(x) = x^{-n}\mathcal{H}_\nu\{2ng(u^{1/2n}); x^{2n}\}. \tag{56}$$

Utilizing the relation obtained by replacing n with $2n$ in (21), it is found that

$$f(x) = 4n^2 x^{-n}\mathcal{H}_{\nu,2n}\{g(u); x\}. \tag{57}$$

Substituting (57) into (45), the assertion (53) is obtained.

Theorem 4. The Parseval-Goldstein type relations

$$\int_0^\infty u^{2n\nu+3n-1}\tilde{\mathcal{G}}_{2n}\{g(y); u\}\mathcal{H}_{\nu,2n}\{x^n f(x); u\}du = \frac{1}{n\sqrt{2\pi}}\int_0^\infty y^{2n\nu+2n-1} g(y)\mathcal{K}_{\nu+1/2,2n}\{f(x); y\}dy, \tag{58}$$

$$\int_0^\infty u^{2n\nu+3n-1}\tilde{\mathcal{G}}_{2n}\{g(y); u\}\mathcal{H}_{\nu,2n}\{x^n f(x); u\}du = \frac{1}{n\sqrt{2\pi}}\int_0^\infty x^{2n-1} f(x)\mathcal{K}_{\nu+1/2,2n}\{y^{2n\nu} g(y); x\}dx, \tag{59}$$

$$\int_0^\infty y^{2n\nu+2n-1} g(y)\mathcal{K}_{\nu+1/2,2n}\{f(x); y\}dy = \int_0^\infty x^{2n-1} f(x)\mathcal{K}_{\nu+1/2,2n}\{y^{2n\nu} g(y); x\}dx, \tag{60}$$

hold true, provided that each of the integrals involved converges absolutely.

Proof. Making use of definition (4) of the $\tilde{\mathcal{G}}_{2n}$-transform and changing the order of integration, we have

$$\int_0^\infty u^{2n\nu+3n-1}\tilde{\mathcal{G}}_{2n}\{g(y); u\}\mathcal{H}_{\nu,2n}\{x^n f(x); u\}du = \int_0^\infty y^{2n-1} g(y)\tilde{\mathcal{G}}_{2n}\{u^{2n\nu+n}\mathcal{H}_{\nu,2n}\{x^n f(x); u\}, y\}dy. \tag{61}$$

Utilizing the relation (45), we arrive at (58).

Similarly, using the definition of the $\mathcal{H}_{\nu,2n}$-transform obtained by replacing n with $2n$ in (19), changing the order integration and using the assertion (46) of Theorem 1, (59) is obtained. The assertion (60) follows from (58) and (59). This completes the proof of Theorem 4 under the hypothesis stated.

Corollary 5. If the conditions of Theorem 4 are satisfied, then identities

$$\int_0^\infty u^{2n\alpha+n-1}\mathcal{H}_{\nu,2n}\{x^n f(x); u\}du = \frac{2\sqrt{2}}{\Gamma\left(\frac{\alpha-\nu}{2}\right)\Gamma\left(\frac{1-\alpha+\nu}{2}\right)}\int_0^\infty y^{2n\alpha-1}\mathcal{K}_{\nu+1/2,2n}\{f(x); y\}dy, \tag{62}$$

$$\int_0^\infty u^{2n\alpha+n-1}\mathcal{H}_{\nu,2n}\{x^n f(x); u\}du = \frac{2^{\alpha-1}}{n}\frac{\Gamma\left(\frac{1+\alpha+\nu}{2}\right)}{\Gamma\left(\frac{1-\alpha+\nu}{2}\right)}\int_0^\infty x^{2n-1-2n\alpha} f(x)dx, \tag{63}$$

$$\int_0^\infty y^{2n\alpha-1}\mathcal{K}_{\nu+1/2,2n}\{f(x); y\}dy = \frac{2^{\alpha-5/2}}{n}\Gamma\left(\frac{1+\alpha+\nu}{2}\right)\Gamma\left(\frac{\alpha-\nu}{2}\right)\int_0^\infty x^{2n-1-2n\alpha} f(x)dx, \tag{64}$$

hold true, where $n \in \mathbb{N}$, $0 < \mathrm{Re}(\alpha-\nu) < 1$ and $\mathrm{Re}(\alpha) > |\mathrm{Re}(\nu+\frac{1}{2})| - \frac{1}{2}$.

Proof. Putting

$$g(y) = y^{2n(\alpha-\nu-1)}$$

in Theorem 4 and using the relation (5) and the known formula [11, p.174,Entry(m)], the following holds true for $0 < \mathrm{Re}(\alpha - \nu) < 1$,

$$\tilde{\mathcal{G}}_{2n}\left\{y^{2n(\alpha-\nu-1)}; u\right\} = \frac{u^{2n(\alpha-\nu-1)}}{4n} B\left(\frac{\alpha-\nu}{2}, \frac{1-\alpha+\nu}{2}\right). \tag{65}$$

Using relation (26) for $n = 2n$ and the known formula [9, p.127, 10.2(1)], we get for $\mathrm{Re}(\alpha) > |\mathrm{Re}(\nu + \frac{1}{2})| - \frac{1}{2}$,

$$\mathcal{K}_{\nu+1/2,2n}\left\{y^{2n(\alpha-1)}; x\right\} = \frac{2^{\alpha-3/2}}{2nx^{2n\alpha}} \Gamma\left(\frac{\alpha}{2} + \frac{\nu}{2} + \frac{1}{2}\right) \Gamma\left(\frac{\alpha}{2} - \frac{\nu}{2}\right). \tag{66}$$

Substituting (65) and (66) into (58), (59) and (60), respectively, we arrive at (62), (63) and (64). Thus, the proof is completed.

Remark 6. Setting $\nu = 0$ in (64), it is obtained that

$$\int_0^\infty y^{2n\alpha-1} \mathcal{K}_{1/2,2n}\{f(x); y\} dy = \frac{2^{\alpha-5/2}}{n} \Gamma\left(\frac{\alpha}{2} + \frac{1}{2}\right) \Gamma\left(\frac{\alpha}{2}\right) \int_0^\infty x^{2n-1-2n\alpha} f(x) dx. \tag{67}$$

Using the relation (27) and Legendre's duplication formula for the gamma function [12, p.5, (15)], it is derived that

$$\int_0^\infty y^{2n\alpha-1} \mathcal{L}_{2n}\{f(x); y\} dy = \frac{\Gamma(\alpha)}{2n} \int_0^\infty x^{2n-1-2n\alpha} f(x) dx. \tag{68}$$

Theorem 7. The identities

$$2n\mathcal{F}_{c,2n}\left\{\tilde{\mathcal{G}}_{2n}\{x^n f(x); u\}; y\right\} = y^{-n} \mathcal{K}_{0,2n}\{f(x); y\}, \tag{69}$$

$$\tilde{\mathcal{G}}_{2n}\{x^n f(x); u\} = \frac{4n}{\pi} \mathcal{F}_{c,2n}\left\{y^{-n} \mathcal{K}_{0,2n}\{f(x); y\}; u\right\}, \tag{70}$$

$$2n\tilde{\mathcal{G}}_{2n}\left\{\mathcal{F}_{c,2n}\{x^n f(x); u\}; y\right\} = y^{-n} \mathcal{K}_{0,2n}\{f(x); y\}, \tag{71}$$

$$\tilde{\mathcal{G}}_{2n}\{f(x); u\} = \frac{4n}{\pi} u^{-n} \mathcal{K}_{0,2n}\left\{y^{-n} \mathcal{F}_{c,2n}\{f(x); y\}; u\right\}, \tag{72}$$

hold true, provided that the integrals involved converge absolutely.

Proof. From the definition (28) of the generalized Fourier-cosine transform and definition (4) of the $\tilde{\mathcal{G}}_{2n}$-transform, we have

$$2n\mathcal{F}_{c,2n}\left\{\tilde{\mathcal{G}}_{2n}\{x^n f(x); u\}; y\right\} = 2n \int_0^\infty u^{2n-1} cos(u^{2n} y^{2n}) \left(\int_0^\infty \frac{x^{2n-1} x^n f(x)}{\sqrt{x^{4n} + u^{4n}}} dx\right) du. \tag{73}$$

Changing the order of integration, which is permissible by absolute convergence of the integrals involved and using definition (4) once again, it follows from (73) that

$$\begin{aligned} 2n\mathcal{F}_{c,2n}\left\{\tilde{\mathcal{G}}_{2n}\{x^n f(x); u\}; y\right\} &= 2n \int_0^\infty x^{3n-1} f(x) \left(\int_0^\infty \frac{u^{2n-1} cos(u^{2n} y^{2n})}{\sqrt{x^{4n} + u^{4n}}} dx\right) du \\ &= 2n \int_0^\infty x^{3n-1} f(x) \tilde{\mathcal{G}}_{2n}\{cos(u^{2n} y^{2n}); x\} dx. \end{aligned} \tag{74}$$

Using the relation (5) and the known following formula [11, p.174,(b)]

$$\mathcal{G}\{cos(ax); y\} = \mathcal{K}_0(ay), \tag{75}$$

we have

$$\begin{aligned} 2n\mathcal{F}_{c,2n}\left\{\tilde{\mathcal{G}}_{2n}\{x^n f(x); u\}; y\right\} &= \int_0^\infty x^{3n-1} f(x) \mathcal{G}\{cos(uy^{2n}); x^{2n}\} dx \\ &= y^{-n} \int_0^\infty x^{2n-1} x^n y^n \mathcal{K}_0(x^{2n} y^{2n}) f(x) dx. \end{aligned} \tag{76}$$

Making use of the definition of the $\mathcal{K}_{\nu,2n}$-transform obtained by replacing n with $2n$ (23), we arrive at (69). Applying the $\mathcal{F}_{c,2n}$-integral transform to both sides of (69) and using the relation (32), (70) is obtained. Using the definitions of the $\tilde{\mathcal{G}}_{2n}$-transform and the $\mathcal{F}_{c,2n}$-transform and changing the order of integration, which is permissible under the hypothesis of Theorem 7, we get

$$2n\tilde{\mathcal{G}}_{2n}\{\mathcal{F}_{c,2n}\{x^n f(x); u\}; y\} = 2n\int_0^\infty x^{2n-1}x^n f(x)\tilde{\mathcal{G}}_{2n}\{\cos(x^{2n}u^{2n}); y\}dx. \tag{77}$$

Using the relations (5) and (75) and the definition of the $\mathcal{K}_{\nu,2n}$-transform obtained by replacing n with $2n$ in (23), we arrive at (71). Setting

$$g(u) = 2n\mathcal{F}_{c,2n}\{x^n f(x); u\} \tag{78}$$

in (71) and applying $\mathcal{F}_{c,2n}$-transform to both sides of (78), it is found that

$$\mathcal{F}_{c,2n}\{g(u); x\} = 2n\mathcal{F}_{c,2n}\{\mathcal{F}_{c,2n}\{x^n f(x); u\}; x\}. \tag{79}$$

Using the relation (32), the following is derived:

$$\frac{4n}{\pi}x^{-n}\mathcal{F}_{c,2n}\{g(u); x\} = f(x). \tag{80}$$

Now, the assertion (72) of Theorem 7 easily follows upon inserting (78) and (80) into (71). Then, it is found that

$$\tilde{\mathcal{G}}_{2n}\{g(u); y\} = \frac{4n}{\pi}y^{-n}\mathcal{K}_{0,2n}\{x^{-n}\mathcal{F}_{c,2n}\{g(u); x\}; y\}. \tag{81}$$

Replacing the variables u by x, y by u, x by y and the function $g(u)$ by f(x), we arrive at (72). Thus, the proof of Theorem 7 is completed.

Theorem 8. The Parseval-Goldstein type relations

$$\int_0^\infty u^{2n-1}\tilde{\mathcal{G}}_{2n}\{g(y); u\}\mathcal{F}_{c,2n}\{x^n f(x); u\}du = \frac{1}{2n}\int_0^\infty y^{n-1}g(y)\mathcal{K}_{0,2n}\{f(x); y\}dy, \tag{82}$$

$$\int_0^\infty u^{2n-1}\tilde{\mathcal{G}}_{2n}\{g(y); u\}\mathcal{F}_{c,2n}\{x^n f(x); u\}du = \frac{1}{2n}\int_0^\infty x^{2n-1}f(x)\mathcal{K}_{0,2n}\{y^{-n}g(y); x\}dx, \tag{83}$$

$$\int_0^\infty y^{n-1}g(y)\mathcal{K}_{0,2n}\{f(x); y\}dy = \int_0^\infty x^{2n-1}f(x)\mathcal{K}_{0,2n}\{y^{-n}g(y); x\}dx, \tag{84}$$

hold true, provided that each of the integrals involved converges absolutely.

Proof. Using the definition (4) of the $\tilde{\mathcal{G}}_{2n}$-transfrom and changing the order of integration, it is found that

$$\int_0^\infty u^{2n-1}\tilde{\mathcal{G}}_{2n}\{g(y); u\}\mathcal{F}_{c,2n}\{x^n f(x); u\}du = \int_0^\infty y^{2n-1}g(y)\tilde{\mathcal{G}}_{2n}\{\mathcal{F}_{c,2n}\{x^n f(x); u\}; y\}dy. \tag{85}$$

Using the relation (71) of Theorem 7, (82) is obtained. The proof of (83) is similar. The proof of assertion (84) follows from relations (82) and (83). This completes the proof of Theorem 8 under the hypothesis stated.

Corollary 9. The following identity

$$s^{2n}\tilde{\mathcal{P}}_{2n}\{u^{-2n}\tilde{\mathcal{G}}_{2n}\{g(y); u\}; s\} = \mathcal{L}_{2n}\{x^{-n}\mathcal{K}_{0,2n}\{y^{-n}g(y); x\}; s\}, \tag{86}$$

holds true, provided that the integrals involved converge absolutely.

Proof. Substituting

$$f(x) = x^{-n}e^{-s^{2n}x^{2n}} \tag{87}$$

into (83) of Theorem 8, we get

$$\int_0^\infty u^{2n-1}\tilde{\mathcal{G}}_{2n}\{g(y);u\}\mathcal{F}_{c,2n}\{e^{-s^{2n}x^{2n}};u\}du=\frac{1}{2n}\int_0^\infty x^{2n-1}x^{-n}e^{-s^{2n}x^{2n}}\mathcal{K}_{0,2n}\{y^{-n}g(y);x\}dx. \quad (88)$$

Using the relation (30) and the formula [14, p.14, Entry(1)], we have

$$\mathcal{F}_{c,2n}\{e^{-s^{2n}x^{2n}};u\}=\frac{1}{2n}\frac{s^{2n}}{s^{4n}+u^{4n}}. \quad (89)$$

Setting (89) into (88), it is found that

$$\int_0^\infty u^{2n-1}\frac{s^{2n}}{s^{4n}+u^{4n}}\tilde{\mathcal{G}}_{2n}\{g(y);u\}du=\int_0^\infty x^{n-1}e^{-s^{2n}x^{2n}}\mathcal{K}_{0,2n}\{y^{-n}g(y);x\}dx. \quad (90)$$

Using the definitions (3) and (15) in the relation (90), we arrive at the assertion (86).

Corollary 10. The following identities for $0<\mathrm{Re}\mu<1$

$$\int_0^\infty u^{2n\mu-1}\tilde{\mathcal{G}}_{2n}\{g(y);u\}du=\frac{B\left(\frac{\mu}{2},\frac{1-\mu}{2}\right)}{4n}\int_0^\infty y^{2n\mu-1}g(y)dy, \quad (91)$$

$$\int_0^\infty u^{2n\mu-1}\tilde{\mathcal{G}}_{2n}\{g(y);u\}du=\frac{2^\mu\Gamma\left(\frac{\mu}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{1-\mu}{2}\right)}\int_0^\infty x^{-2n\mu+n-1}\mathcal{K}_{0,2n}\{y^{-n}g(y);x\}dx, \quad (92)$$

$$\int_0^\infty y^{2n\mu-1}g(y)dy=\frac{2^{\mu+2}n}{\left[\Gamma\left(\frac{1-\mu}{2}\right)\right]^2}\int_0^\infty x^{-2n\mu+n-1}\mathcal{K}_{0,2n}\{y^{-n}g(y);x\}dx \quad (93)$$

hold true, provided that each of the integrals involved converges absolutely.

Proof. Setting

$$f(x)=x^{-n-2n\mu},\qquad 0<Re\mu<1 \quad (94)$$

in Theorem 8, then it is obtained that

$$\int_0^\infty u^{2n-1}\tilde{\mathcal{G}}_{2n}\{g(y);u\}\mathcal{F}_{c,2n}\{x^{-2n\mu};u\}du=\frac{1}{2n}\int_0^\infty y^{n-1}g(y)\mathcal{K}_{0,2n}\{x^{-n-2n\mu};y\}dy, \quad (95)$$

$$\int_0^\infty u^{2n-1}\tilde{\mathcal{G}}_{2n}\{g(y);u\}\mathcal{F}_{c,2n}\{x^{-2n\mu};u\}du=\frac{1}{2n}\int_0^\infty x^{n-1-2n\mu}\mathcal{K}_{0,2n}\{y^{-n}g(y);x\}dx, \quad (96)$$

$$\int_0^\infty y^{n-1}g(y)\mathcal{K}_{0,2n}\{x^{-n-2n\mu};y\}dy=\int_0^\infty x^{n-1-2n\mu}\mathcal{K}_{0,2n}\{y^{-n}g(y);x\}dx. \quad (97)$$

Using the relations obtained by replacing n with $2n$ in (26) and (30) and the formulas [14, p.10, Entry(1)], [9, p.127, Entry(1)] for $0<\mathrm{Re}\mu<1$, we have

$$\mathcal{F}_{c,2n}\{x^{-2n\mu};u\}=\frac{\pi}{4n}\frac{1}{\Gamma(\mu)}\sec\left(\frac{\pi\mu}{2}\right)u^{2n\mu-2n}, \quad (98)$$

$$\mathcal{K}_{0,2n}\{x^{-n-2n\mu};y\}=\frac{1}{2n}2^{-\mu-1}y^{2n\mu-n}\left[\Gamma\left(\frac{1-\mu}{2}\right)\right]^2. \quad (99)$$

Using the known identities for Gamma function [12, p.3, Entry(7)] and [12, p.5, Entry(15)]

$$\pi\sec\left(\frac{\pi\mu}{2}\right)=\Gamma\left(\frac{1-\mu}{2}\right)\Gamma\left(\frac{1+\mu}{2}\right),$$

$$\Gamma(\mu)=2^{\mu-1}\pi^{-1/2}\Gamma\left(\frac{\mu}{2}\right)\Gamma\left(\frac{\mu+1}{2}\right),$$

and substituting (98), (99) into the Equations (95), (96), (97), respectively, we get (91)-(93).

Theorem 11. If each of the integrals involved converges absolutely for $|y=arg(y^{4n})|<\pi, a=Rex^{4n}>0$, then we have the following identities hold true:

$$\tilde{\mathcal{G}}_{2n}\{\mathcal{L}_{4n}\{f(t);x\};y\} = \frac{1}{4n}\int_0^\infty t^{4n-1} e^{\frac{t^{4n}x^{4n}}{2}} \mathcal{K}_0\left(\frac{t^{4n}y^{4n}}{2}\right) f(t)dt, \tag{100}$$

$$\tilde{\mathcal{G}}_{2n}\{x^{2n}\mathcal{L}_{4n}\{f(t);x\};y\} = \frac{\sqrt{\pi}}{4n}\int_0^\infty t^{2n-1} e^{t^{4n}x^{4n}} erfc(t^{2n}y^{2n}) f(t)dt. \tag{101}$$

Proof. Using the definitions (4) of the $\tilde{\mathcal{G}}_{2n}$-transform, (16) of the $\mathcal{L}_{4n}$-transform and changing the order of integration, which is permissible by absolute convergence of the integrals involved, we have

$$\tilde{\mathcal{G}}_{2n}\{\mathcal{L}_{4n}\{f(t);x\};y\} = \int_0^\infty t^{4n-1} f(t)\tilde{\mathcal{G}}_{2n}\{e^{-t^{4n}x^{4n}};y\}dt. \tag{102}$$

Using the relation (6) and the following formula [9, p.233, Entry(11)]

$$\mathcal{S}_{1/2}\{t^{-1/2}e^{-at};y\} = e^{ay/2}\mathcal{K}_0(ay/2) \tag{103}$$

for $|y = \arg(y^{4n})| < \pi$, $a = \mathrm{Re}x^{4n} > 0$, we arrive at (100). Using definitions (4) of the $\tilde{\mathcal{G}}_{2n}$-transform, (16) of the $\mathcal{L}_{4n}$-transform and changing the order of integration, which is permissible by absolute convergence of the integrals involved, we have

$$\tilde{\mathcal{G}}_{2n}\{x^{2n}\mathcal{L}_{4n}\{f(t);x\};y\} = \int_0^\infty t^{4n-1} f(t)\tilde{\mathcal{G}}_{2n}\{x^{4n}e^{-t^{4n}x^{4n}};y\}dt. \tag{104}$$

Using the relation (6) and the following formula [9, p.233, Entry(10)]

$$\mathcal{S}_{1/2}\{e^{-at};y\} = a^{-1/2}e^{ay}\Gamma(1/2,ay) \tag{105}$$

for $|y = arg(y^{4n})| < \pi, a = Rex^{4n} > 0$ and considering definitions (43) and (41), we arrive at (101).
By the following Lemma, a relation between the generalized Weyl fractional integral and the generalized Laplace transform is given.

Lemma 12. The following identity

$$\mathcal{W}_{\mu,2n}\{\mathcal{L}_{2n}\{f(x);u\};y\} = \frac{1}{2n}\mathcal{L}_{2n}\{x^{-2n\mu}f(x);y\}, \tag{106}$$

holds true, provided that the integrals involved converge absolutely, where $\mathrm{Re}(\mu) > 0$, $\mu \in \mathbb{C}$.

Proof. Using definitions (35) and (15), we get

$$\mathcal{W}_{\mu,2n}\{\mathcal{L}_{2n}\{f(x);u\};y\} = \frac{1}{\Gamma(\mu)}\int_y^\infty u^{2n-1}(u^{2n}-y^{2n})^{\mu-1} \times \left[\int_0^\infty x^{2n-1}e^{-u^{2n}x^{2n}}f(x)dx\right]du. \tag{107}$$

Making the change of variable $u^{2n} - y^{2n} = t^{2n}$ in (107), we have

$$\mathcal{W}_{\mu,2n}\{\mathcal{L}_{2n}\{f(x);u\};y\} = \frac{1}{\Gamma(\mu)}\int_0^\infty t^{2n-1}t^{2n(\mu-1)}\left[\int_0^\infty x^{2n-1}e^{-(t^{2n}+y^{2n})x^{2n}}f(x)dx\right]dt.$$

Changing the order of integration that is permissible by absolute convergence of the integrals involved, and using the definition (15) and the relation (17), (106) is obtained.

Theorem 13. The following Parseval-Goldstein type relation

$$\int_0^\infty t^{2n-1}\mathcal{R}_{\mu,2n}\{f(x);t\}\mathcal{L}_{2n}\{g(u);t\}dt = \frac{1}{2n}\int_0^\infty f(x)x^{2n-1}\mathcal{L}_{2n}\{g(u)u^{-2n\mu};x\}dx, \tag{108}$$

holds true, provided that the integrals involved converge absolutely, where $Re\mu > 0, \mu \in \mathbb{C}$.

Proof. By the definition (36), we have

$$\int_0^\infty t^{2n-1}\mathcal{R}_{\mu,2n}\{f(x);t\}\mathcal{L}_{2n}\{g(u);t\}dt$$

$$= \int_0^\infty t^{2n-1}\left[\frac{1}{\Gamma(\mu)}\int_0^t x^{2n-1}(t^{2n}-x^{2n})^{\mu-1}f(x)dx\right]\mathcal{L}_{2n}\{g(u);t\}dt$$

$$= \int_0^\infty x^{2n-1} f(x) \left[\frac{1}{\Gamma(\mu)} \int_x^\infty t^{2n-1}(t^{2n} - x^{2n})^{\mu-1} \mathcal{L}_{2n}\{g(u); t\} dt\right] dx.$$

Then, by using the definition (35) and Lemma 12, we get

$$\int_0^\infty t^{2n-1} \mathcal{R}_{\mu,2n}\{f(x); t\} \mathcal{L}_{2n}\{g(u); t\} dt = \frac{1}{2n} \int_0^\infty x^{2n-1} f(x) \mathcal{L}_{2n}\{u^{-2n\mu} g(u); x\} dx.$$

Corollary 14. The following Parseval-Goldstein type relation

$$\mathcal{L}_{2n}\left\{u^{-n} \mathcal{R}_{1,2n}\left\{f(x); \frac{1}{u}\right\}; y\right\} = \frac{\sqrt{\pi}}{2ny^n} \int_0^\infty x^{2n-1} f(x) erf\left(\frac{y^n}{x^n}\right) dx, \tag{109}$$

holds true, provided that each of the integrals involved converges absolutely.

Proof. Setting $\mu = 1$ and $g(u) = sin(2u^n y^n)$ in (108), it is obtained that

$$\int_0^\infty t^{2n-1} \mathcal{R}_{1,2n}\{f(x); t\} \mathcal{L}_{2n}\{sin(2u^n y^n); t\} dt = \frac{1}{2n} \int_0^\infty f(x) x^{2n-1} \mathcal{L}_{2n}\{u^{-2n} sin(2u^n y^n); x\} dx. \tag{110}$$

Using the relation (17) and the formulas [14, p.153,Entry(32)], [14, p.154, Entry(34)], we get

$$\mathcal{L}_{2n}\{sin(2u^n y^n); t\} = \frac{1}{2n} \mathcal{L}\left\{sin\left(2\sqrt{u} y^n\right); t^{2n}\right\} = \frac{\sqrt{\pi}}{2n} y^n t^{-3n} e^{-y^{2n}/t^{2n}} \tag{111}$$

$$\mathcal{L}_{2n}\{u^{-2n} sin(2u^n y^n); x\} = \frac{1}{2n} \mathcal{L}\left\{u^{-1} sin\left(2\sqrt{u} y^n\right); x^{2n}\right\} = \frac{\pi}{2n} erf\left(\frac{y^n}{x^n}\right). \tag{112}$$

Substituting (111) and (112) into (110), it is found that

$$\int_0^\infty t^{2n-1} \mathcal{R}_{1,2n}\{f(x); t\} t^{-3n} e^{-y^{2n}/t^{2n}} dt = \frac{\sqrt{\pi}}{2ny^n} \int_0^\infty x^{2n-1} erf\left(\frac{y^n}{x^n}\right) f(x) dx. \tag{113}$$

Finally, by making change of variable $t = u^{-1}$ on the left-hand side of the Equation (113) and using the definition (15), we arrive at (109).

Corollary 15. The following Parseval-Goldstein type relation

$$\int_0^\infty t^{2n-1} e^{y^{2n} t^{2n}} E_1(y^{2n} t^{2n}) \mathcal{R}_{\mu,2n}\{f(x); t\} dt = \frac{\Gamma(1-\mu)}{2ny^{2\mu n}} \int_0^\infty x^{2n-1} e^{y^{2n} x^{2n}} \Gamma(\mu, y^{2n} x^{2n}) f(x) dx, \tag{114}$$

holds true, provided that each of the integrals involved converges absolutely for $|argy| < \frac{\pi}{2n}, 0 < Re\mu < 1$ and $E_1(x)$ is the exponential integral function which is defined by the following identity [12,13],

$$E_1(x) = \int_x^\infty \frac{e^{-u}}{u} du. \tag{115}$$

Proof. Setting $g(u) = (y^{2n} + u^{2n})^{-1}$ into (108), we have

$$\int_0^\infty t^{2n-1} \mathcal{R}_{\mu,2n}\{f(x); t\} \mathcal{L}_{2n}\{(y^{2n} + u^{2n})^{-1}; t\} dt$$
$$= \frac{1}{2n} \int_0^\infty f(x) x^{2n-1} \mathcal{L}_{2n}\{u^{-2\mu n}(y^{2n} + u^{2n})^{-1}; x\} dx. \tag{116}$$

Using the relation (17) and the formulas [14, p.137, Entry(4) and Entry(7)], we get

$$\mathcal{L}_{2n}\{(y^{2n} + u^{2n})^{-1}; t\} = \frac{e^{y^{2n} t^{2n}}}{2n} E_1(y^{2n} t^{2n}), \tag{117}$$

$$\mathcal{L}_{2n}\{u^{-2\mu n}(y^{2n} + u^{2n})^{-1}; x\} = \frac{\Gamma(1-\mu)}{2ny^{2\mu n}} e^{y^{2n} x^{2n}} \Gamma(\mu, y^{2n} x^{2n}). \tag{118}$$

Substituting (117) and (118) into (116), we arrive at (114).

Theorem 16. The following identity

$$\tilde{\mathcal{P}}_{2n}\{t^{-2n}\mathcal{R}_{1,2n}\{f(x);t\};y\} = \frac{1}{2ny^{2n}}\int_0^\infty x^{2n-1}f(x)arctan\left(\frac{y^{2n}}{x^{2n}}\right)dx, \tag{119}$$

holds true, provided that each of the integrals involved converges absolutely, where $Re(t^{2n}) > |Im(y^{2n})|$.

Proof. By setting $\mu = 1$ and $g(u) = sin(u^{2n}y^{2n})$ in (108), we get

$$\int_0^\infty t^{2n-1}\mathcal{R}_{1,2n}\{f(x);t\}\mathcal{L}_{2n}\{sin(u^{2n}y^{2n});t\}dt = \frac{1}{2n}\int_0^\infty f(x)x^{2n-1}\mathcal{L}_{2n}\{u^{-2n}sin(u^{2n}y^{2n});x\}dx. \tag{120}$$

Using the relation (17) and the formulas [14, p.150, Entry(1)], [14, p.152, Entry(16)], we have

$$\mathcal{L}_{2n}\{sin(u^{2n}y^{2n});t\} = \frac{1}{2n}\frac{y^{2n}}{y^{4n}+t^{4n}}, \tag{121}$$

$$\mathcal{L}_{2n}\{u^{-2n}sin(u^{2n}y^{2n});x\} = \frac{1}{2n}arctan\left(\frac{y^{2n}}{x^{2n}}\right). \tag{122}$$

Substituting (121) and (122) into (120) and using the definition (3), we arrive at (119).

Lemma 17. The identity for $Re\mu < 1, Re(t^{2n}) > -Re(s^{2n})$, and $Re(x^{2n}) > -Re(s^{2n})$,

$$\tilde{\mathcal{P}}_n\{\mathcal{R}_{\mu,2n}\{f(x);t\};s\}3 = \frac{\Gamma(1-\mu)}{2n}\tilde{\mathcal{P}}_{1-\mu,n}\{f(x);s\}, \tag{123}$$

holds true, provided that each of the integrals involved converges absolutely.

Proof. Setting $\mathrm{g(u) = e^{-s^{2n}u^{2n}}}$ in (108), we get

$$\int_0^\infty t^{2n-1}\mathcal{R}_{\mu,2n}\{f(x);t\}\mathcal{L}_{2n}\{e^{-s^{2n}u^{2n}};t\}dt = \frac{1}{2n}\int_0^\infty x^{2n-1}f(x)\mathcal{L}_{2n}\{u^{-2n\mu}e^{-s^{2n}u^{2n}};x\}dx. \tag{124}$$

Using the relation (17) and the formulas [14, p.143, Entry(1)], [14, p.144, Entry(3)], we have

$$\mathcal{L}_{2n}\{e^{-s^{2n}u^{2n}};t\} = \frac{1}{2n}\frac{1}{s^{2n}+t^{2n}}, \quad Re(t^{2n}) > -Re(s^{2n}), \tag{125}$$

$$\mathcal{L}_{2n}\{u^{-2n\mu}e^{-s^{2n}u^{2n}};x\} = \frac{1}{2n}\frac{\Gamma(1-\mu)}{(x^{2n}+s^{2n})^{1-\mu}}, \quad Re(x^{2n}) > -Re(s^{2n}). \tag{126}$$

Substituting results (125) and (126) into (124) and using the definitions (1) and (11), we arrive at (123).

Theorem 18. The following Parseval-Goldstein type identity

$$\int_0^\infty t^{2n-1}\tilde{\mathcal{P}}_n\{f(x);t\}\mathcal{R}_{\mu,2n}\{g(u);t\}dt = \frac{\Gamma(1-\mu)}{2n}\int_0^\infty x^{2n-1}f(x)\tilde{\mathcal{P}}_{1-\mu,n}\{g(u);x\}dx, \tag{127}$$

holds true, provided that each of the integrals involved converges absolutely, where $Re\mu < 1, Re(t^{2n}) > -Re(s^{2n}), and\ Re(x^{2n}) > -Re(s^{2n}), \mu, t \in \mathbb{C}$.

Proof. Applying the definition (1) and then changing the order of integration under the absolute convergence condition, we get

$$\int_0^\infty t^{2n-1}\tilde{\mathcal{P}}_n\{f(x);t\}\mathcal{R}_{\mu,2n}\{g(u);t\}dt$$

$$= \int_0^\infty t^{2n-1}\left(\int_0^\infty \frac{x^{2n-1}f(x)}{x^{2n}+t^{2n}}dx\right)\mathcal{R}_{\mu,2n}\{g(u);t\}dt$$

$$= \int_0^\infty x^{2n-1}f(x)\tilde{\mathcal{P}}_n\{\mathcal{R}_{\mu,2n}\{g(u);t\};x\}dx. \tag{128}$$

Using Lemma 17 on the right-hand side of (128), the relation (127) is obtained.

Corollary 19. The following equation holds true under the hyphothesis of Theorem 18,

$$\Gamma(1-\mu)\mathcal{L}_{2n}\{\tilde{\mathcal{P}}_{1-\mu,n}\{g(u);x\};y\} = \int_0^\infty t^{2n-1}e^{y^{2n}t^{2n}}E_1(y^{2n}t^{2n})\mathcal{R}_{\mu,2n}\{g(u);t\}dt, \tag{129}$$

where $0 < Re\mu < 1$ and $E_1(x)$ as defined in (115).

Proof. Setting $f(x) = e^{-x^{2n}y^{2n}}$ in Equation (127), and using the definition (15), we have

$$\int_0^\infty t^{2n-1}\tilde{\mathcal{P}}_n\{e^{-x^{2n}y^{2n}};t\}\mathcal{R}_{\mu,2n}\{g(u);t\}dt$$

$$= \frac{\Gamma(1-\mu)}{2n}\int_0^\infty x^{2n-1}e^{-x^{2n}y^{2n}}\tilde{\mathcal{P}}_{1-\mu,n}\{g(u);x\}dx$$

$$= \frac{\Gamma(1-\mu)}{2n}\mathcal{L}_{2n}\{\tilde{\mathcal{P}}_{1-\mu,n}\{g(u);x\};y\}. \tag{130}$$

Using relation (12) and the formulas [9, p.217, Entry(11)], we get

$$\tilde{\mathcal{P}}_n\{e^{-x^{2n}y^{2n}};t\} = \frac{1}{2n}\mathcal{S}\{e^{-x^{2n}y};t^{2n}\} = \frac{1}{2n}e^{y^{2n}t^{2n}}E_1(y^{2n}t^{2n}). \tag{131}$$

Substituting (131) into (130), we arrive at (129).

Remark 20. Using relations (129) and (114), it is derived that

$$\mathcal{L}_{2n}\{\tilde{\mathcal{P}}_{1-\mu,n}\{g(u);x\};y\} = \frac{1}{2ny^{2\mu n}}\int_0^\infty u^{2n-1}e^{y^{2n}u^{2n}}\Gamma(\mu,y^{2n}u^{2n})g(u)du, \tag{132}$$

which holds true, provided that each of the integrals involved converges absolutely for $|argy| < \frac{\pi}{2n}, 0 < Re\mu < 1$.

Corollary 21. The following identity

$$\mathcal{L}_n\{t^n\mathcal{R}_{\mu,2n}\{g(u);t\};y\} = \frac{\Gamma(1-\mu)}{\pi}\mathcal{F}_{s,n}\{x^n\tilde{\mathcal{P}}_{1-\mu,n}\{g(u);x\};y\}, \tag{133}$$

holds true, provided that each of the integrals involved converges absolutely.

Proof. Taking $f(x) = sin(x^ny^n)$ in Theorem 18, the following identity is found:

$$\int_0^\infty t^{2n-1}\tilde{\mathcal{P}}_n\{sin(x^ny^n);t\}\mathcal{R}_{\mu,2n}\{g(u);t\}dt = \frac{\Gamma(1-\mu)}{2n}\int_0^\infty x^{2n-1}\sin(x^ny^n)\tilde{\mathcal{P}}_{1-\mu,n}\{g(u);x\}dx. \tag{134}$$

Using relation (12) and the formula [9, p.219, Entry(36)], we get

$$\tilde{\mathcal{P}}_n\{\sin(x^ny^n);t\} = \frac{\pi}{2n}e^{-y^nt^n}. \tag{135}$$

Setting (135) into (134) and using the definitions (19), (23), we arrive at (133).

Corollary 22. The following identity

$$\frac{2(-1)^k}{\Gamma(1-\mu)}\mathcal{K}_{v,n}\left\{t^{n\left(v+2k+\frac{1}{2}\right)}\mathcal{R}_{\mu,2n}\{g(u);t\};y\right\} = \mathcal{H}_{v,n}\left\{x^{n\left(v+2k+\frac{1}{2}\right)}\tilde{\mathcal{P}}_{1-\mu,n}\{g(u);t\};y\right\}, \tag{136}$$

holds true, provided that each of the integrals involved converges absolutely, where $y > 0, -k-1 < Rev < -2k+3/2, Re\mu < 1$.

Proof. Taking $f(x) = x^{nv+2nk}\mathcal{J}_v(x^ny^n)$ in Theorem 18, the following identity is found:

$$\int_0^\infty t^{2n-1}\tilde{\mathcal{P}}_n\{x^{nv+2nk}\mathcal{J}_v(x^ny^n);t\}\mathcal{R}_{\mu,2n}\{g(u);t\}dt$$

$$= \frac{\Gamma(1-\mu)}{2n}\int_0^\infty x^{2n-1}x^{nv+2nk}\mathcal{J}_v(x^ny^n)\tilde{\mathcal{P}}_{1-\mu,n}\{g(u);x\}dx. \tag{137}$$

Using relation (12) and the formula [9, p.225, Entry(10)], we get

$$\tilde{\mathcal{P}}_n\{x^{n\nu+2nk}\mathcal{J}_\nu(x^ny^n);t\}=\frac{2(-1)^k}{2n}t^{n\nu+2nk}\mathcal{K}_\nu(y^nt^n). \tag{138}$$

Setting (138) into (137) and using the definitions (19) and (23), we arrive at (136).

3. SOME ILLUSTRATIVE EXAMPLES

Example 23. It is shown that for $Rea^{2n}>0, Re\nu>-1, -1<Re\alpha<0$,

$$\int_0^\infty u^{2n\alpha+2n-1}\mathcal{J}_{\nu/2}(a^{2n}u^{2n})\mathcal{K}_{\nu/2}(a^{2n}u^{2n})du=\frac{B\left(-\frac{\alpha}{2},\frac{1+\alpha}{2}\right)}{8na^{2n\alpha+2n}}\frac{\Gamma\left(\frac{1+\alpha+\nu}{2}\right)}{\Gamma\left(\frac{1-\alpha+\nu}{2}\right)}, \tag{139}$$

where $\mathcal{J}_\nu$ [13, p.5, Entry(12)] and $\mathcal{K}_\nu$ [13, p.5, Entry(13)] are Modified Bessel functions of first and second kind of order ν, respectively and $B(x,y)$ is the beta function [12, p.9, Entry(1)].

Demonstration. Setting

$$f(x)=\frac{1}{x^{2n}(x^{4n}+4a^{4n})^{1/2}} \tag{140}$$

in relation (63) of Corollary 5, and using the relationship obtained by replacing n with $2n$ in (21) and the known formula [9, p.23, Entry(11)], for $Rea^{2n}>0, Re\nu>-1$, we have

$$\mathcal{H}_{\nu,2n}\left\{\frac{1}{x^n(x^{4n}+4a^{4n})^{1/2}};u\right\}=\frac{u^n}{2n}\mathcal{J}_{\nu/2}(a^{2n}u^{2n})\mathcal{K}_{\nu/2}(a^{2n}u^{2n}). \tag{141}$$

Using relationship (5) and the known formula [11, p.174, Entry(m)], for $-1<Re\alpha<0$, we get

$$\tilde{\mathcal{G}}_{2n}\{x^{-2n-2n\alpha};2^{1/2n}a\}=\frac{a^{-2n\alpha-2n}}{2^{\alpha+3}n}B\left(-\frac{\alpha}{2},\frac{1+\alpha}{2}\right). \tag{142}$$

Substituting (140), (141) and (142) into (63), assertion (139) is obtained.

Example 24. It is shown that for $Re\nu>0$,

$$\mathcal{K}_{\nu,2n}\{x^{-2n(\nu-1/2)}(x^{4n}-a^{4n})^{\nu-1}H(x^{2n}-a^{2n});y\}$$

$$=\frac{2^{\nu-2}\Gamma(\nu)}{n}y^{-2n(\nu-1/2)}\mathcal{K}_0(a^{2n}y^{2n}), \tag{143}$$

where H is the Heaviside function [1, p.15, Entry(2.3.9)].

Demonstration. Putting

$$g(u)=u^{-2n(\nu+1/2)}cos(a^{2n}u^{2n}) \tag{144}$$

in identity (53) of Corollary 3 and using relation (5) and the known formula [11, p.174, Entry(b)], we have

$$\tilde{\mathcal{G}}_{2n}\{cos(a^{2n}u^{2n});y\}=\frac{\mathcal{K}_0(a^{2n}y^{2n})}{2n}. \tag{145}$$

Using the relationship obtained by replacing n with $2n$ in (21) and the formula [9, p.37,Entry(30)] for $Re\nu>-1/2$, it is found that

$$\mathcal{H}_{\nu,2n}\{u^{-2n(\nu+1/2)}cos(a^{2n}u^{2n});x\}=\frac{1}{2n}\frac{\sqrt{\pi}(x^{4n}-a^{4n})^{\nu-1/2}}{2^\nu x^{2n\nu-n}\Gamma(\nu+1/2)}H(x^{2n}-a^{2n}), \tag{146}$$

where H is the Heaviside function. Substituting (144), (145) and (146) into (53) and setting $\nu=\nu-1/2$, we arrive at (143).

Example 25. It is shown that for $-1/2<Re\nu<0$,

$$\tilde{\mathcal{G}}_{2n}\{u^{2n(\nu+1)}[\mathcal{J}_\nu(a^{2n}u^{2n})-\boldsymbol{L}_\nu(a^{2n}u^{2n})];y\}$$

$$=\sqrt{\frac{\pi}{2}}\frac{y^{2n\nu+n}}{2na^n sin(\pi\nu)}\left[Y_{\nu+1/2}(a^{2n}y^{2n})-\boldsymbol{H}_{\nu+1/2}(a^{2n}y^{2n})\right] \tag{147}$$

where $\mathcal{I}_\nu$ is the modified Bessel function [9, p.5, Entry(12)], $\boldsymbol{L}_\nu$ is the Modified Struve function [13, p.38, Entry(52)], Y_ν is Bessel function of order of ν [12, p.4, Entry(4)] and $\boldsymbol{H}_\nu$ is the Struve function [13, p.38, Entry(55)].

Demonstration. Setting

$$f(x) = x^{-2n\nu-2n}(x^{4n}+a^{4n})^{-1} \tag{148}$$

in identity (45) of Theorem 1 and using the relation obtained by replacing n with $2n$ in (21) and known formula [9, p.23,(14)] for $Re(\nu) > -1/2, Rea^{2n} > 0$, we have

$$\mathcal{H}_{\nu,2n}\left\{\frac{x^{-2n\nu-n}}{x^{4n}+a^{4n}}; u\right\} = \frac{1}{2n}\frac{\pi}{2}a^{-2n\nu-2n}u^n[\mathcal{I}_\nu(a^{2n}u^{2n}) - \boldsymbol{L}_\nu(a^{2n}u^{2n})]. \tag{149}$$

Using the relation obtained by replacing n with $2n$ in (26) and the formula [9, p.128,Entry(7)] for $\mathrm{Re}\nu < 0$, $\mathrm{Re}a^{2n} > 0$, $\mathrm{Re}y^{2n} > 0$, we get

$$\mathcal{K}_{\nu+1/2,2n}\left\{\frac{x^{-2n\nu-2n}}{x^{4n}+a^{4n}}; y\right\} = \frac{\pi^2}{8n}\frac{y^n\csc(\pi\nu)}{a^{2n\nu+n}}\left[Y_{\nu+1/2}(a^{2n}y^{2n}) - \mathbf{H}_{\nu+1/2}(a^{2n}y^{2n})\right]. \tag{150}$$

Setting (148), (149) and (150) in (45), we arrive at (147).

Example 26. It is shown that

$$\mathcal{H}_{\nu,2n}\left\{u^{2n(\nu+1/2)}e^{-a^{2n}u^{2n}}; y\right\} = \sqrt{\frac{2}{\pi}}\frac{a^{2n}}{2n}\frac{\left(2y^{2n}\right)^{\nu+1/2}}{(y^{4n}+a^{4n})^{\nu+3/2}}\Gamma\left(\nu+\frac{3}{2}\right), \tag{151}$$

where $Re\nu > -3/2, Rey^{2n} > |Ima^{2n}|$.

Demonstration. Putting

$$f(x) = x^{2n}\mathcal{J}_0(a^{2n}x^{2n}) \tag{152}$$

in relation (46) of Theorem 1 and using (5) and the formula [11, p.174, Entry(f)], we have

$$\tilde{\mathcal{G}}_{2n}\{x^{2n}\mathcal{J}_0(a^{2n}x^{2n}); u\} = \frac{e^{-a^{2n}u^{2n}}}{2na^{2n}}. \tag{153}$$

Using the relation obtained by replacing n with $2n$ in (26) and the known formula [9, p.137, Entry(17)] for $Re\nu > -3/2$, we get

$$\mathcal{K}_{\nu+1/2,2n}\{x^{2n\nu+2n}\mathcal{J}_0(a^{2n}x^{2n}); y\} = \frac{1}{2n}\frac{2^{\nu+1/2}y^{2n\nu+2n}}{(y^{4n}+a^{4n})^{\nu+3/2}}\Gamma\left(\nu+\frac{3}{2}\right). \tag{154}$$

Substituting (152), (153) and (154) into (46) of Theorem 1, the assertion (151) is obtained.

Example 27. It is shown that

$$\int_0^\infty \frac{y^{2n(\nu+\alpha+1)-1}}{(y^{4n}+a^{4n})^{\nu+3/2}}dy = \frac{a^{2n(\alpha-\nu-2)}}{4n}\frac{\Gamma\left(\frac{\alpha}{2}+\frac{\nu}{2}+\frac{1}{2}\right)\Gamma\left(1-\frac{\alpha}{2}+\frac{\nu}{2}\right)}{\Gamma\left(\nu+\frac{3}{2}\right)}, \tag{155}$$

where $0 < Re(2-\alpha+\nu) < 3/2$ and $\mathrm{Re}\nu > -3/2$.

Demonstration. Setting

$$f(x) = x^{2n\nu+2n}\mathcal{J}_0(a^{2n}x^{2n}) \tag{156}$$

in relation (64) of Corollary 5 and using the formula (154), we have

$$\int_0^\infty \frac{y^{2n(\nu+\alpha+1)-1}}{(y^{4n}+a^{4n})^{\nu+3/2}}dy = \frac{2^{\alpha-\nu-2}\Gamma\left(\frac{\alpha}{2}+\frac{\nu}{2}+\frac{1}{2}\right)\Gamma\left(\frac{\alpha}{2}-\frac{\nu}{2}\right)}{\Gamma\left(\nu+\frac{3}{2}\right)}\mathcal{M}\{\mathcal{J}_0(a^{2n}x^{2n}); 2n(2-\alpha+\nu)\}, \tag{157}$$

where $\mathcal{M}$ is the Mellin Transform [14, p.305]. Using the formulas [14, p.307 Entry(5)] for $h = 2n$ and [14, p.326, Entry(1)], we get

$$\mathcal{M}\{\mathcal{J}_0(a^{2n}x^{2n}); 2n(2-\alpha+\nu)\} = \frac{1}{n}\frac{2^{\nu-\alpha}\Gamma\left(1-\frac{\alpha}{2}+\frac{\nu}{2}\right)}{a^{2n(2-\alpha+\nu)}\Gamma\left(\frac{\alpha}{2}-\frac{\nu}{2}\right)}, \tag{158}$$

where $0 < Re(2-\alpha+\nu) < 3/2$. Substituting (158) into (157), (155) is obtained.

Example 28. It is shown that

$$\int_0^\infty y^{2n\mu-1}cos(a^{2n}y^{2n})dy = \frac{a^{-2n\mu}}{2n\pi}\Gamma(\mu)cos\left(\frac{\mu\pi}{2}\right), \tag{159}$$

where $0 < Re\mu < 1$.

Demonstration. Putting

$$g(y) = cos(a^{2n}y^{2n}) \tag{160}$$

in relation (93) of Corollary 10, we get

$$\int_0^\infty y^{2n\mu-1}cos(a^{2n}y^{2n})dy = \frac{2^{\mu+2}n}{\left[\Gamma\left(\frac{1-\mu}{2}\right)\right]^2}\int_0^\infty x^{-2n\mu+n-1}\mathcal{K}_{0,2n}\{y^{-n}cos(a^{2n}y^{2n}); x\}dx. \tag{161}$$

Using the relation obtained by replacing n with $2n$ in (26), the formulas [13, p.79, Entry(15)] and [9, p.137, Entry(17)], we have

$$\mathcal{K}_{0,2n}\{y^{-n}cos(a^{2n}y^{2n}); x\} = \frac{\pi}{4n}\frac{x^n}{(x^{4n}+a^{4n})^{1/2}}. \tag{162}$$

Setting (162) in (161), it is found that

$$\int_0^\infty y^{2n\mu-1}cos(a^{2n}y^{2n})dy = \frac{2^\mu\pi}{\left[\Gamma\left(\frac{1-\mu}{2}\right)\right]^2}\tilde{\mathcal{G}}_{2n}\{x^{-2n\mu}; a\}. \tag{163}$$

Using the relation (65) for $0 < Re(\mu) < 1, \alpha = 1, \nu = \mu$ and $u = a$, (159) is obtained.

Example 29. It is shown that for $Re(\nu) > -1/2$ and $-Re2\nu < Re\mu < 3/2$

$$\int_0^\infty u^{2n\mu-1}\mathcal{I}_\nu(a^{2n}u^{2n})\mathcal{K}_\nu(a^{2n}u^{2n})du = \frac{B\left(\frac{\mu}{2},\frac{1-\mu}{2}\right)\Gamma\left(\nu+\frac{\mu}{2}\right)}{8na^{2n\mu}\Gamma\left(1+\nu-\frac{\mu}{2}\right)}, \tag{164}$$

where $\mathcal{I}_\nu$ [13, p.5, Entry(12)] and $\mathcal{K}_\nu$ [13, p.5, Entry(13)] are Modified Bessel functions of first and second kind of order ν, respectively.

Demonstration. Putting

$$g(y) = \mathcal{J}_{2\nu}(2a^{2n}y^{2n}) \tag{165}$$

in relation (91) of Corollary 10 and using relation (5) and the formula [11, p.174, Entry(g)] for $Re(\nu) > -1/2$, we get

$$\int_0^\infty u^{2n\mu-1}\mathcal{I}_\nu(a^{2n}u^{2n})\mathcal{K}_\nu(a^{2n}u^{2n})du = \frac{B\left(\frac{\mu}{2},\frac{1-\mu}{2}\right)}{2}\mathcal{M}\{\mathcal{J}_{2\nu}(2a^{2n}y^{2n}); 2n\mu\}. \tag{166}$$

Using the formulas [14, p.307 Entry(5)] for $h = 2n$ and [14, p.326 Entry(1)], it is found that

$$\mathcal{M}\{\mathcal{J}_{2\nu}(2a^{2n}y^{2n}); 2n\mu\} = \frac{1}{4n}\frac{\Gamma\left(\nu+\frac{\mu}{2}\right)}{a^{2n\mu}\Gamma\left(1+\nu-\frac{\mu}{2}\right)}, \tag{167}$$

where $-Re2\nu < Re\mu < 3/2$. Setting (167) in (166), (164) is obtained.

Remark 30. Example 29 could be obtained easily by setting $\alpha = \mu - 1$ and $\nu = 2\nu$ in Example 23.

Example 31. It is shown that

$$\int_0^\infty x^{-2n(\mu+\nu)-1}{}_2F_1\left(\frac{\nu+1}{2},\frac{\nu+1}{2};\mu+1;-\frac{a^{4n}}{x^{4n}}\right)dx=\frac{\Gamma(\nu+1)\Gamma\left(\frac{\nu}{2}+\frac{\mu}{2}\right)\left[\Gamma\left(\frac{1-\mu}{2}\right)\right]^2}{4na^{2n(\mu+\nu)}\Gamma\left(1+\frac{\nu}{2}-\frac{\mu}{2}\right)\left[\Gamma\left(\frac{\nu+1}{2}\right)\right]^2} \quad (168)$$

where $Re\nu>-1,-Re\nu<Re\mu<3/2,Rex^{2n}>|Ima^{2n}|$.

Demonstration. Setting

$$g(y)=\mathcal{J}_\nu(a^{2n}y^{2n}) \quad (169)$$

in relation (93) of Corollary 10, we get

$$\mathcal{M}\{\mathcal{J}_\nu(a^{2n}y^{2n});2n\mu\}=\frac{2^{\mu+2}n}{\left[\Gamma\left(\frac{1-\mu}{2}\right)\right]^2}\int_0^\infty x^{-2n\mu+n-1}\mathcal{K}_{0,2n}\{y^{-n}\mathcal{J}_\nu(a^{2n}y^{2n});x\}dx, \quad (170)$$

where $-Re\nu<Re\mu<3/2$. Using the relation obtained by replacing n with $2n$ in (26) and the formula [9, p.137, Entry(17)] for $Re\nu>-1$, we have

$$\mathcal{K}_{0,2n}\{y^{-n}\mathcal{J}_\nu(a^{2n}y^{2n});x\}=\frac{1}{2n}\frac{a^{2n\nu}x^{-2n\nu-n}\left[\Gamma\left(\frac{\nu}{2}+\frac{1}{2}\right)\right]^2}{2\Gamma(\nu+1)}{}_2F_1\left(\frac{\nu}{2}+\frac{1}{2},\frac{\nu}{2}+\frac{1}{2};\nu+1;-\frac{a^{4n}}{x^{4n}}\right), \quad (171)$$

where $Re\nu>-1,Rex^{2n}>|Ima^{2n}|$. Substituting the relation (167) for $\nu=\nu/2,a=2^{1/2n}a$ and (171) into (170), (168) is obtained.

Example 32. It is shown that

$$\tilde{\mathcal{P}}_{2n}\{u^{-2n}\mathcal{K}_0(a^{2n}u^{2n});s\}=\frac{1}{2n}\frac{\pi^2}{4s^{2n}}[\boldsymbol{H}_0(a^{2n}s^{2n})-Y_0(a^{2n}s^{2n})], \quad (172)$$

where $\mathcal{K}_\nu$ is Modified Bessel functions of second kind of order ν [13, p.5, Entry(13)], Y_ν is Bessel function of second kind order of ν [12, p.4, Entry(4)], $\boldsymbol{H}_\nu$ is the Struve function [13, p.38, Entry(55)].

Demonstration. Setting

$$g(y)=cos(a^{2n}y^{2n}) \quad (173)$$

in (86) of Corollary 9 and using (17) and the formula [14, p.138, Entry(10)], we get

$$\mathcal{L}_{2n}\left\{\frac{1}{(x^{4n}+a^{4n})^{1/2}};s\right\}=\frac{1}{2n}\frac{\pi}{2}[\boldsymbol{H}_0(a^{2n}s^{2n})-Y_0(a^{2n}s^{2n})]. \quad (174)$$

Substituting (145), (162), (173) and (174) into (86), we arrive at the assertion (172).

Example 33. The following equation holds true under the hypothesis of Corollary 14,

$$\int_0^\infty x^{-n(\nu-1)-1}erf\left(\frac{y^n}{x^n}\right)dx=\frac{y^{n(1-\upsilon)}}{\sqrt{\pi}n(1-\upsilon)}\Gamma\left(\frac{\nu}{2}\right),(0<Re\nu). \quad (175)$$

Demonstration. Setting $f(x)=x^{-n(\upsilon+1)}$ in (109), we get

$$\mathcal{L}_{2n}\left\{u^{-n}\mathcal{R}_{1,2n}\left\{x^{-n(\upsilon+1)};\frac{1}{u}\right\};y\right\}=\frac{\sqrt{\pi}}{2ny^n}\int_0^\infty x^{-n(\nu-1)-1}erf\left(\frac{y^n}{x^n}\right)dx. \quad (176)$$

Using definitions (36) and (15) on the left-hand side of (109), we have

$$\mathcal{L}_{2n}\left\{u^{-n}\left[\int_0^{1/u}x^{-n(\upsilon+1)-1}dx\right];y\right\}=\frac{y^{-n\upsilon}}{2n^2(1-\upsilon)}\Gamma\left(\frac{\nu}{2}\right). \quad (177)$$

Substituting (177) into (176), (175) is obtained.

Example 34. The following equation holds true under the hypothesis of Theorem 16,

$$\int_0^\infty x^{n(v+1)-1} arctan\left(\frac{y^{2n}}{x^{2n}}\right) dx = \frac{-\pi y^{n(v+1)}}{2n(v+1)} csc\left(\frac{\pi(v-1)}{4}\right), \qquad (-3 < Rev < 1). \quad (178)$$

Demonstration. Setting $f(x) = x^{n(v-1)}$ on the left-hand side of Equation (119) and using the formula [9, p.216, Entry(5)], we have

$$\tilde{\mathcal{P}}_{2n}\{t^{-2n}\mathcal{R}_{1,2n}\{x^{n(v+1)}; t\}; y\} = \frac{1}{n(v+1)}\tilde{\mathcal{P}}_{2n}\{t^{n(v-1)}; y\} = \frac{-\pi y^{n(v-1)}}{4n^2(v+1)} csc\left(\frac{\pi(v-1)}{4}\right). \quad (179)$$

Finally, by setting (179) into Equation (119), (178) is obtained.

Example 35. The following equations hold true under the hypotheses of Theorem 18,

$$\int_0^\infty t^{3n+\alpha-2} e^{y^{2n}t^{2n}} E_1(y^{2n}t^{2n}) dt = \frac{\Gamma\left(\frac{\alpha-1}{2n}+\frac{3}{2}\right)}{2n} \frac{\pi}{sin\left(\pi\left(\frac{\alpha-1}{2n}+\frac{3}{2}\right)\right)} \frac{1}{y^{3n+\alpha-1}}, \quad (180)$$

$$\int_0^\infty u^{2n+\alpha-2} e^{y^{2n}u^{2n}} \Gamma(1/2; y^{2n}u^{2n}) du = \frac{\sqrt{\pi}}{sin\left(\pi\left(\frac{\alpha-1}{2n}+\frac{3}{2}\right)\right)} \frac{\Gamma\left(\frac{\alpha-1}{2n}+\frac{3}{2}\right)}{y^{2n+\alpha-1}} \quad (181)$$

where $-\frac{3}{2} < Re\left(\frac{\alpha-1}{2n}\right) < -\frac{1}{2}$.

Demonstration. Setting $\mu = 1/2$ and $g(u) = u^{\alpha-1}$ into (129) and (132), we have

$$\Gamma(1/2)\mathcal{L}_{2n}\{\tilde{\mathcal{P}}_{1/2,n}\{u^{\alpha-1}; x\}; y\} = \int_0^\infty t^{2n-1} e^{y^{2n}t^{2n}} E_1(y^{2n}t^{2n}) \mathcal{R}_{1/2,2n}\{u^{\alpha-1}; t\} dt. \quad (182)$$

Using the relations (13) and (17) and formulas [14, p.133,Entry(3)] and [9, p.233, Entry(8)], we get

$$\tilde{\mathcal{P}}_{1/2,n}\{u^{\alpha-1}; x\} = \frac{1}{2n} \frac{\Gamma\left(\frac{\alpha-1}{2n}+1\right)\Gamma\left(\frac{1}{2}-\frac{\alpha-1}{2n}\right)}{\sqrt{\pi}} x^{n+\alpha-1}, \quad (-1 < Re\left(\frac{\alpha-1}{2n}\right) < -\frac{1}{2}), \quad (183)$$

$$\mathcal{L}_{2n}\{x^{n+\alpha-1}; y\} = \frac{1}{2n} \frac{\Gamma\left(\frac{3}{2}+\frac{\alpha-1}{2n}\right)}{y^{3n+\alpha-1}} (-\frac{3}{2} < Re\left(\frac{\alpha-1}{2n}\right)). \quad (184)$$

In addition, using the definition of (35), we have

$$\mathcal{R}_{1/2,2n}\{u^{\alpha-1}; t\} = \frac{1}{\sqrt{\pi}} \int_0^t u^{2n-1}(t^{2n} - u^{2n})^{-1/2} u^{\alpha-1} du. \quad (185)$$

Changing the variable in (185) to $\mathrm{u} = \mathrm{t}\mathrm{y}^{1/2\mathrm{n}}$ and using the definition of beta function, we get

$$\mathcal{R}_{1/2,2n}\{u^{\alpha-1}; t\} = \frac{t^{n+\alpha-1}}{2n} \frac{\Gamma\left(\frac{\alpha-1}{2n}+1\right)}{\Gamma\left(\frac{\alpha-1}{2n}+\frac{3}{2}\right)}. \quad (186)$$

Substituting (183), (184) and (186) into (129) and (132), respectively, we arrive at (180) and (181).

4. CONCLUSION

It could be concluded that there are many other infinite integrals which could be evaluated in this manner by applying the Lemma, the theorems and its corollaries considered here. Also, the new equalities in this manuscript could be used in many other topics. Fractional calculus is frequently applied in other fields of sciences and medicine, as it could be seen that it was used for modeling in various areas [15-20]. We believe that the relations in this manuscript could be used in different disciplines in the future.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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