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Nonlinear Integrodifferential Equations with Time Varying Delay

Kamalendra Kumar[a], Manoj Karnatak[b], Rakesh Kumar[b]

[a] *Department of Mathematics, SRMS College of Engineering and Technology, Bareilly, 243001, U.P., India.*
[b] *Department of Mathematics, Hindu College, Moradabad, U.P., India.*

Abstract

By practicing the manner of semigroup theory and Banach contraction theorem, the existence and uniqueness of mild and classical solutions of nonlinear integrodifferential equations with time varying delay in Banach spaces is showed. Certainly, an example is revealed to justify the abstract idea.

Keywords: Nonlinear integrodifferential equations; Time varying delay; Nonlocal condition; Mild and classical solution; Banach contraction theorem.
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1. Introduction

In this work, we examine the class of nonlinear integrodifferential equations with time varying delay of form:

$$x'(t) + Ax(t) = F_1\Big(t, x(\gamma_1(t)), ..., x(\gamma_n(t)), \int_{t_0}^{t} h_1(t, s, x(\gamma_{n+1}(s)))ds\Big)$$
$$+ F_2\Big(t, x(\eta_1(t)), ..., x(\eta_m(t)), \int_{t_0}^{t} h_2(t, s, x(\eta_{m+1}(s)))ds\Big), t \in (t_0, t_0 + b] \quad (1)$$

and

$$x(t_0) + g(x) = x_0, \quad (2)$$

Email addresses: kamlendra.14kumar@gmail.com (Kamalendra Kumar), karnatak.manoj@gmail.com (Manoj Karnatak), rakeshnaini1@gmail.com (Rakesh Kumar)

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in which $t_0 \geq 0, b > 0$. The infinitesimal generator that is expressed by $-A$, of a C_0 semigroup of operators on a Banach spaces. F_1, h_1, F_2, h_2 are functions which is stated in (1) and these functions gratifying some assumptions and $x_0 \in E$. Assigning the tool of semigroup, the existence of solutions for semilinear evolution equations is analyzed by Pazy [11]. The same classes of evolution equations as present in [11] with nonlocal condition are explored by Byszewskii [6]. During previous years, differential and integrodifferential system with time varying delay is considered by various investigators like [1] - [5], [7] - [10], [13], [14]. They have used different tools and techniques for discussing the outcomes.

2. Preliminaries

In this section, we give some definitions, notations and basic facts which are applied in the next sections.

Definition 2.1. *[11] A one parameter family $T(t), 0 \leq t < \infty$, of bounded linear operators from $E \to E$, where E is a Banach space, is said to be the semigroup of bounded linear operators on E if*

(i) *$T(0) = I$, the identity operator on E;*

(ii) *$T(t)T(s) = T(t+s); \forall \ t, s \geq 0$ (the semigroup property).*

A semigroup of bounded linear operators $T(t)$ is uniformly continuous

$$\lim_{t \to 0} \| T(t) - I\| = 0.$$

If the linear operator A explained by

$$D(A) = \left[y \in E : \lim_{t \to 0} \frac{T(t)y - y}{t} exists \right]$$

and

$$Ay = \lim_{t \to 0} \frac{T(t)y - y}{t}, y \in D(A)$$

is the infinitesimal generator of the semigroup $T(t)$. Here $D(A)$ is the domain of A.

Definition 2.2. *[11] A semigroup $T(t), 0 \leq t < \infty$, of bounded linear operators on E is a strongly continuous semigroup of bounded linear operators if*

$$\lim_{t \to 0} T(t)y = y, \quad \forall y \in E.$$

A strongly continuous semigroup of bounded linear operators on E will be termed as C_0-semigroup.

Theorem 2.3. *Suppose $T(t)$ be a C_0-semigroup. The constants $\Omega \geq 0$ and $M \geq 1$ exist such that*

$$\| T(t)\| \leq Me^{\Omega t}, 0 \leq t < \infty.$$

If $\Omega = 0$ then $T(t)$ is called uniformly bounded and if $M = 1$ it is said to be C_0-semigroup of contraction.

Now, if E is assumed as a Banach space with norm $\| .\|$. Also, C_0-semigroup of operators on E is written by $\{T(t)\}_{t \geq 0}$. Throughout paper, infinitesimal generator is represented by $-A$ and the same is C_0-semigroup of operators on E. Here the domain of A is given by $D(A)$ and also $t_0 \geq 0, b > 0$,

$$J := [t_0, t_0 + b],$$

$$\Delta := \{(t,s) : t_0 \le s \le t \le t_0 + b\},$$

$$M := \sup_{t\in[0,b]} \| T(t)\|_{BL(E,E)},$$
$$X := C(J,E)$$

and $F_1 : J \times E^{n+1} \to E, h_1 : \Delta \times E \to E, F_2 : J \times E^{m+1} \to E, h_2 : \Delta \times E \to E, g : X \to E, \gamma_i : J \to J \ \ (i = 1, 2, ..., n+1), \eta_j : J \to J \ \ (j = 1, 2, ..., m+1)$ are stated functions and these functions meet some assumptions.
For the suitability, the operator norm $\| .\|_{BL(E,E)}$ will be indicated by $\| T(t)\|$.

The following two definitions will be mandatory for the mild and classical solutions of the system (1) - (2).

Definition 2.4. *The following integral equation is fulfilled by the function $x \in X$,*

$$\begin{aligned} x(t) &= T(t-t_0)x_0 - T(t-t_0)g(x) \\ &+ \int_{t_0}^{t} T(t-s)F_1\Big(s, x(\gamma_1(s)), ..., x(\gamma_n(s)), \int_{t_0}^{s} h_1(s,\tau,x(\gamma_{n+1}(\tau)))d\tau\Big)ds \\ &+ \int_{t_0}^{t} T(t-s)F_2\Big(s, x(\eta_1(s)), ..., x(\eta_m(s)), \int_{t_0}^{s} h_2(s,\tau,x(\eta_{m+1}(\tau)))d\tau\Big)ds, \\ & t \in (t_0, t_0+b] \end{aligned} \tag{3}$$

is remarked to be mild solution of the system (1) - (2) *on J.*

Definition 2.5. *A function $x : J \to E$ is termed as classical solution of the system* (1) - (2) *on J if:*

(*i*) *x is a continuous on J and is continuously differentiable on $J/\{t_0\}$.*

(*ii*) $x'(t) + Ax(t) = F_1\Big(t, x(\gamma_1(t)), ..., x(\gamma_n(t)), \int_{t_0}^{t} h_1(t,s,x(\gamma_{n+1}(s)))ds\Big)$
$+F_2\Big(t, x(\eta_1(t)), ..., x(\eta_m(t)), \int_{t_0}^{t} h_2(t,s,x(\eta_{m+1}(s)))ds\Big), t \in J/\{t_0\}$

(*iii*) $x(t_0) + g(x) = x_0$

3. Main Results

3.1. Existence of Mild Solution

The existence of mild solution is discussed by means of following theorem.

Theorem 3.1. *Presume that*

(*i*) *$-A$ is the infiniesimal generator of a C_0-semigroup $T(t), t \ge 0$ in E such that $\|T(t)\| \le M$, for some $M \ge 1$.*

(*ii*) *Here the function $F_1 : J \times E^{n+1} \to E$ and $F_2 : J \times E^{m+1} \to E$ are continuous. We take constants $M_1 > 0, M_2 > 0$ in such a manner that $\forall\, x_i, y_i \in E, i = 1, 2, ..., n+1$ and $\forall\, x_j, y_j \in E, j = 1, 2, ..., m+1$, we get*

$$\| F_1(t, x_1, x_2, ..., x_{n+1}) - F_1(t, y_1, y_2, ..., y_{n+1})\| \le M_1\bigg(\sum_{i=1}^{n+1} \| x_i - y_i\|\bigg) \tag{4}$$

and

$$\| F_2(t, x_1, x_2, ..., x_{m+1}) - F_2(t, y_1, y_2, ..., y_{m+1})\| \le M_2\bigg(\sum_{j=1}^{m+1} \| x_j - y_j\|\bigg) \tag{5}$$

(iii) *Next,* $h_1, h_2 : \Delta \times E \to E$ *are continuous functions and we consider constants* $H_1 > 0, H_2 > 0$ *in such a way that* $\forall\ x, y \in E$,

$$\| h_1(t,s,x) - h_1(t,s,y)\| \leq H_1 \| x - y\|$$

and

$$\| h_2(t,s,x) - h_2(t,s,y)\| \leq H_2 \| x - y\|$$

(iv) *The function* $g : X \to E$ *and there is a constant* $G > 0$ *such that*

$$\| g(u) - g(v)\| \leq G \| u - v\|_X, \forall\ u, v \in E$$

(v) *The functions* $\gamma_i \in C(J, J), i = 1, 2, ..., n+1$ *and the function* $\eta_j \in C(J, J), j = 1, 2, ..., m+1$.

(vi) *finally*

$$M\big[G + M_1 b(n + H_1 b) + M_2 b(m + H_2 b)\big] < 1 \tag{6}$$

If all the above conditions are satisfied then the equations (1) - (2) *has a unique mild solution on* J.

Proof. By explaining a mapping ϕ on X by the formula

$$\begin{aligned}
&(\phi u)(t) = T(t-t_0)x_0 - T(t-t_0)g(u) \\
&+ \int_{t_0}^{t} T(t-s)F_1\Big(s, u(\gamma_1(s)), ..., u(\gamma_n(s)), \int_{t_0}^{s} h_1(s,\tau,u(\gamma_{n+1}(\tau)))d\tau\Big)ds \\
&+ \int_{t_0}^{t} T(t-s)F_2\Big(s, u(\eta_1(s)), ..., u(\eta_m(s)), \int_{t_0}^{s} h_2(s,\tau,u(\eta_{m+1}(\tau)))d\tau\Big)ds,
\end{aligned}$$

for $u \in X$ and $t \in J$.

It is simple to understand that $\phi : X \to X$.

Just now, we shall try to demonstrate that ϕ is a contraction on x. For this plan, make the difference

$$\begin{aligned}
(\phi u)(t) - (\phi v)(t) &= -T(t-t_0)[g(u) - g(v)] \\
&+ \int_{t_0}^{t} T(t-s)\bigg[F_1\Big(s, u(\gamma_1(s)), ..., u(\gamma_n(s)), \int_{t_0}^{s} h_1(s,\tau,u(\gamma_{n+1}(\tau)))d\tau\Big) \\
&- F_1\Big(s, v(\gamma_1(s)), ..., v(\gamma_n(s)), \int_{t_0}^{s} h_1(s,\tau,v(\gamma_{n+1}(\tau)))d\tau\Big)\bigg]ds \\
&+ \int_{t_0}^{t} T(t-s)\bigg[F_2\Big(s, u(\eta_1(s)), ..., u(\eta_m(s)), \int_{t_0}^{s} h_2(s,\tau,u(\eta_{m+1}(\tau)))d\tau\Big) \\
&- F_2\Big(s, v(\eta_1(s)), ..., v(\eta_m(s)), \int_{t_0}^{s} h_2(s,\tau,v(\eta_{m+1}(\tau)))d\tau\Big)\bigg]ds
\end{aligned}$$

Now, taking norm both sides, we obtain

$$\begin{aligned}
&\| (\phi u)(t) - (\phi v)(t)\| \leq \| T(t-t_0)\| \| g(u) - g(v)\| \\
&+ \int_{t_0}^{t} \| T(t-s)\| \bigg\| F_1\Big(s, u(\gamma_1(s)), ..., u(\gamma_n(s)), \int_{t_0}^{s} h_1(s,\tau,u(\gamma_{n+1}(\tau)))d\tau\Big) \\
&- F_1\Big(s, v(\gamma_1(s)), ..., v(\gamma_n(s)), \int_{t_0}^{s} h_1(s,\tau,v(\gamma_{n+1}(\tau)))d\tau\Big)\bigg\| ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_{t_0}^{t}\| T(t-s)\|\bigg\| F_2\Big(s,u(\eta_1(s)),...,u(\eta_m(s)),\int_{t_0}^{s}h_2(s,\tau,u(\eta_{m+1}(\tau)))d\tau\Big) \\
&-F_2\Big(s,v(\eta_1(s)),...,v(\eta_m(s)),\int_{t_0}^{s}h_2(s,\tau,v(\eta_{m+1}(\tau)))d\tau\Big)\bigg\|ds \qquad (7)\\
\le{}& MG\| u-v\|_X+MM_1\int_{t_0}^{t}\Bigg(\sum_{i=1}^{n}\Big\| u(\gamma_i(s))-v(\gamma_i(s))\Big\| \\
&+\int_{t_0}^{s}\Big\| h_1(s,\tau,u(\gamma_{n+1}(\tau)))-h_1(s,\tau,v(\gamma_{n+1}(\tau)))\Big\|d\tau\Bigg)ds \\
&+MM_2\int_{t_0}^{t}\Bigg(\sum_{j=1}^{m}\Big\| u(\eta_j(s))-v(\eta_j(s))\Big\| \\
&+\int_{t_0}^{s}\Big\| h_2(s,\tau,u(\eta_{m+1}(\tau)))-h_2(s,\tau,v(\eta_{m+1}(\tau)))\Big\|d\tau\Bigg)ds \\
\le{}& MG\| u-v\|_X+MM_1b\Big[n\| u-v\|_X+H_1b\| u-v\|_X\Big] \\
&\qquad +MM_2b\Big[m\| u-v\|_X+H_2b\| u-v\|_X\Big],\ \forall\ u,v\in J,t\in I.
\end{aligned}
$$

Let $\lambda=M\big[G+M_1b(n+H_1b)+M_2b(m+H_2b)\big]$. Then, by (7) and by assumption (6), we have

$$\| (\phi u)(t)-(\phi v)(t)\| \le \lambda\| u-v\|_X, for \;\; u,v\in X$$

with $0<\lambda<1$. This shows that the operator ϕ is a contraction on X. □

3.2. Existence of Classical Solution

In this section, we shall study the existence of classical solution through the following theorem.

Theorem 3.2. *Suppose that*

(i) *The assumptions (i) and (iv) of Theorem 3.1 holds.*

(ii) *E is reflexive Banach space, $x_0\in D(A)$ and $g(x)\in D(A)$, where x reveals the unique mild solution of system* (1) - (2).

(iii) *There are constants $M_1>0, M_2>0$ in such way that*

$$\| F_1(t,x_1,x_2,...,x_{n+1})-F_1(s,\tilde{x}_1,\tilde{x}_2,...,\tilde{x}_{n+1})\| \le M_1\Big[|t-s|+\sum_{i=1}^{n+1}\| x_i-\tilde{x}_i\|\Big]$$

for $t,s\in J, x_i,\tilde{x}_i\in E, i=1,2,...,n+1$
and

$$\| F_2(t,x_1,x_2,...,x_{m+1})-F_2(s,\tilde{x}_1,\tilde{x}_2,...,\tilde{x}_{m+1})\| \le M_2\Big[|t-s|+\sum_{j=1}^{m+1}\| x_j-\tilde{x}_j\|\Big]$$

for $t,s\in J, x_j,\tilde{x}_j\in E, j=1,2,...,m+1$;

(iv) *There exist constants $H_1,H_2>0$ such that*

$$\| h_1(t_1,s,x)-h_1(t_2,s,\tilde{x})\| \le H_1\big[|t_1-t_2|+\| x-\tilde{x}\|\big]$$

and

$$\| h_2(t_1,s,x)-h_2(t_2,s,\tilde{x})\| \le H_2\big[|t_1-t_2|+\| x-\tilde{x}\|\big]$$

(v) *There are constants* $C_3 > 0, C_4 > 0$ *such that*

$$\| x(\gamma_i(t)) - x(\gamma_i(s))\| \le C_3 \| x(t) - x(s)\|, i = 1, 2, ..., n+1$$

and

$$\| x(\eta_j(t)) - x(\eta_j(s))\| \le C_4 \| x(t) - x(s)\|, j = 1, 2, ..., m+1$$

for $t, s \in J$.

If all the above assumptions are fulfil, then x *is the unique classical solution of the system* (1) - (2) *on* J.

Proof. The equation (1) - (2) possess a unique mild solution if all the conditions of Theorem 3.1 are satisfied, which is represented by x.

Next, we want to manifest that x is the unique classical solution of (1) - (2) on J. At this stage, we initiate

$$C_5 = \max_{s \in J} \left\| F_1\Big(s, x(\gamma_1(s)), ..., x(\gamma_n(s)), \int_{t_0}^{s} h_1(s, \tau, x(\gamma_{n+1}(\tau)))d\tau\Big) \right\|,$$

$$C_6 = \max_{s \in J} \left\| F_2\Big(s, x(\eta_1(s)), ..., x(\eta_m(s)), \int_{t_0}^{s} h_2(s, \tau, x(\eta_{m+1}(\tau)))d\tau\Big) \right\|,$$

$$C_7 = \max_{t,s \in \Delta} \Big\| h_1(t, s, x(\gamma_{n+1}(s))) \Big\|, \text{and } C_8 = \max_{t,s \in \Delta} \Big\| h_2(t, s, x(\eta_{n+1}(s))) \Big\|.$$

For this purpose, consider the difference

$$
\begin{aligned}
& x(t+h)-x(t) \\
&= [T(t+h-t_0)x_0 - T(t-t_0)x_0] - [T(t+h-t_0)g(x) - T(t-t_0)g(x)] \\
&\quad + \int_{t_0}^{t_0+h} T(t+h-s)F_1\Big(s, x(\gamma_1(s)),...,x(\gamma_n(s)), \int_{t_0}^{s} h_1(s,\tau,x(\gamma_{n+1}(\tau)))d\tau\Big)ds \\
&\quad + \int_{t_0+h}^{t+h} T(t+h-s)F_1\Big(s, x(\gamma_1(s)),...,x(\gamma_n(s)), \int_{t_0}^{s} h_1(s,\tau,x(\gamma_{n+1}(\tau)))d\tau\Big)ds \\
&\quad - \int_{t_0}^{t} T(t-s)F_1\Big(s, x(\gamma_1(s)),...,x(\gamma_n(s)), \int_{t_0}^{s} h_1(s,\tau,x(\gamma_{n+1}(\tau)))d\tau\Big)ds \\
&\quad + \int_{t_0}^{t_0+h} T(t+h-s)F_2\Big(s, x(\eta_1(s)),...,x(\eta_m(s)), \int_{t_0}^{s} h_2(s,\tau,x(\eta_{m+1}(\tau)))d\tau\Big)ds \\
&\quad + \int_{t_0+h}^{t+h} T(t+h-s)F_2\Big(s, x(\eta_1(s)),...,x(\eta_m(s)), \int_{t_0}^{s} h_2(s,\tau,x(\eta_{m+1}(\tau)))d\tau\Big)ds \\
&\quad - \int_{t_0}^{t} T(t-s)F_2\Big(s, x(\eta_1(s)),...,x(\eta_m(s)), \int_{t_0}^{s} h_2(s,\tau,x(\eta_{m+1}(\tau)))d\tau\Big)ds \\
&= T(t-t_0)[T(h)-I]x_0 - T(t-t_0)[T(h)-I]g(x) \\
&\quad + \int_{t_0}^{t_0+h} T(t+h-s)F_1\Big(s, x(\gamma_1(s)),...,x(\gamma_n(s)), \int_{t_0}^{s} h_1(s,\tau,x(\gamma_{n+1}(\tau)))d\tau\Big)ds \\
&\quad + \int_{t_0}^{t} T(t-s)\times \\
&\quad \Bigg[F_1\Big(s+h, x(\gamma_1(s+h)),...,x(\gamma_n(s+h)), \int_{t_0}^{s+h} h_1(s+h,\tau,x(\gamma_{n+1}(\tau)))d\tau\Big) \\
&\quad - F_1\Big(s, x(\gamma_1(s)),...,x(\gamma_n(s)), \int_{t_0}^{s} h_1(s,\tau,x(\gamma_{n+1}(\tau)))d\tau\Big)\Bigg]ds \\
&\quad + \int_{t_0}^{t_0+h} T(t+h-s)F_2\Big(s, x(\eta_1(s)),...,x(\eta_m(s)), \int_{t_0}^{s} h_2(s,\tau,x(\eta_{m+1}(\tau)))d\tau\Big)ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{t_0}^{t} T(t-s)\times \\
&\bigg[F_2\Big(s+h, x(\eta_1(s+h)),...,x(\eta_m(s+h)), \int_{t_0}^{s+h} h_2(s+h,\tau,x(\eta_{m+1}(\tau)))d\tau\Big) \\
&- F_2\Big(s, x(\eta_1(s)),...,x(\eta_m(s)), \int_{t_0}^{s} h_2(s,\tau,x(\eta_{m+1}(\tau)))d\tau\Big)\bigg]ds \\
\leq\ & Mh\|\ Ax_0\| + Mh\|\ Ag(x)\| + hMC_5 \\
&+\int_{t_0}^{t} MM_1\bigg[h+\sum_{i=1}^{n}\Big\|\ x(\gamma_i(s+h))-x(\gamma_i(s))\Big\| + \int_{t_0}^{s} H_1|s+h-s|d\tau \\
&+\int_{s}^{s+h} C_7 d\tau\bigg]ds + hMC_6 + \int_{t_0}^{t} MM_2\bigg[h+\sum_{j=1}^{m}\Big\|\ x(\eta_j(s+h))-x(\eta_j(s))\Big\| \\
&+\int_{t_0}^{s} H_2|s+h-s|d\tau + \int_{s}^{s+h} C_8 d\tau\bigg]ds \\
\leq\ & Mh\|\ Ax_0\| + Mh\|\ Ag(x)\| + hMC_5 + MM_1bh \\
&+ MM_1\int_{t_0}^{t}\sum_{i=1}^{n}\|\ x(\gamma_i(s+h))-x(\gamma_i(s))\|ds + MM_1hbH_1 + MM_1C_7hb \\
&+ hMC_6 + MM_2hb + MM_2\int_{t_0}^{t}\sum_{j=1}^{m}\|\ x(\eta_j(s+h))-x(\eta_j(s))\|ds \\
&+ MM_2H_2hb + MM_2C_8hb \\
\leq\ & Mh\|\ Ax_0\| + Mh\|\ Ag(x)\| + hMC_5 + MM_1bh \\
&+ MM_1nC_3\int_{t_0}^{t}\|\ x(s+h)-x(s)\big\|ds + MM_1bhH_1 + MM_1C_7bh + hMC_6 \\
&+ MM_2bh + MM_2mC_4\int_{t_0}^{t}\|\ x(s+h)-x(s)\big\|ds + MM_2bhH_2 + MM_2C_8bh \\
\leq\ & Mh\Big[\|\ Ax_0\| + \|\ Ag(x)\| + C_5 + M_1b + M_1bH_1 + M_1C_7b + C_6 + M_2b \\
&+ M_2bH_2 + M_2C_8b\Big] + \Big[MM_1nC_3 + MM_2mC_4\Big]\int_{t_0}^{t}\|\ x(s+h)-x(s)\big\|ds \\
\leq\ & Qh + M\Big[M_1nC_3 + M_2mC_4\Big]\int_{t_0}^{t}\|\ x(s+h)-x(s)\big\|ds
\end{aligned} \tag{8}
$$

for $t \in [t_0, t_0+h), h > 0$ and $t+h \in (t_0, t_0+b]$, where

$$
\begin{aligned}
Q := M\Big[&\|\ Ax_0\| + \|\ Ag(x)\| + C_5 + M_1b(1+H_1) + M_1C_7b + C_6 \\
&+ M_2b(1+H_2) + M_2C_8b\Big]
\end{aligned}
$$

With the use of Gronwall′s inequality and use of (8), we have

$$
\big\|\ x(t+h)-x(t)\big\| \leq Qh\exp^{bM[M_1nC_3+M_2mC_4]}
$$

for $t \in [t_0, t_0+h), h > 0$ and $t+h \in (t_0, t_0+b]$. Hence, x is Lipschitz continuous on J.

The Lipschitz continuity of x on J and inequalities (4), (5) imply that the function

$$t \in J \to Z(t) := F_1\Big(t, x(\gamma_1(t)), ..., x(\gamma_n(t)), \int_{t_0}^{t} h_1(t, s, x(\gamma_{n+1}(t)))ds\Big)$$
$$+ F_2\Big(t, x(\eta_1(t)), ..., x(\eta_m(t)), \int_{t_0}^{t} h_2(t, s, x(\eta_{m+1}(t)))ds\Big) \in E$$

is Lipschitz continuous on J. This property of $t \to z(t)$ along with assumptions of Theorem 3.2 suggested by Theorem 1 given in of [12] and by Theorem 3.1 together with equation (3), we conclude that the linear Cauchy problem

$$v'(t) + Av(t) = z(t), t \in J/\{t_0\}$$
$$v(t_0) = x_0 - g(x)$$

has a unique classical solution v in such a manner

$$v(t) = T(t - t_0)x_0 - T(t - t_0)g(x) + \int_{t_0}^{t} T(t - s)z(s)ds$$

$$= T(t - t_0)x_0 - T(t - t_0)g(x)$$
$$+ \int_{t_0}^{t} T(t - s)F_1\Big(s, x(\gamma_1(s)), ..., x(\gamma_n(s)), \int_{t_0}^{s} h_1(s, \tau, x(\gamma_{n+1}(\tau)))d\tau\Big)ds$$
$$+ \int_{t_0}^{t} T(t - s)F_2\Big(s, x(\eta_1(s)), ..., x(\eta_m(s)), \int_{t_0}^{s} h_2(s, \tau, x(\eta_{m+1}(\tau)))d\tau\Big)ds$$
$$= x(t), t \in J$$

As a result, $x(t)$ is the unique classical solution of the initial value problem (1) - (2) on J. This completes the proof of Theorem 3.2. □

3.3. Applications

Now, we discuss two examples in favour of our results.

(1) We assume the following partial integrodifferential equation of the form:

$$\frac{\partial z(t,x)}{\partial t} - \frac{\partial^2 z(t,x)}{\partial x^2}$$
$$= f_1\Big(t, z(\gamma_1(t), x), ..., z(\gamma_n(t), x), \int_{t_0}^{t} H_1(t, s, z(\gamma_{n+1}(s), x))ds\Big)$$
$$+ f_2\Big(t, z(\eta_1(t), x), ..., z(\eta_m(t), x), \int_{t_0}^{t} H_2(t, s, z(\eta_{m+1}(s), x))ds\Big),$$
$$0 < x < \pi, t \geq 0, \tag{9}$$

with initial and boundary conditions

$$z(0, t) = z(\pi, t) = 0, t \geq 0 \tag{10}$$

$$z_0(x) = z(t_0, x) + \sum_{p=1}^{k} c_p z(t_p, x), x \in [0, \pi]. \tag{11}$$

In continuation $E = L^2[0, \pi]$ and $A : D(A) \subset E \to E$ is the operator $Az = z''$ with domain $D(A) = \{z \in E : z, z'$ are absolutely contiuous, $z'' \in E, z(0) = z(\pi) = 0\}$.

It is well known that A is the infinitesimal generator of C_0-semigroup $\{T(t)\}_{t\geq 0}$ on E. It is assumed that for certain constants $N_i > 0, i = 1, 2, 3, 4, 5$, the following conditions are satisfied:

$$\| f_1(t, y_1, y_2, ..., y_{n+1}) - f_1(t, z_1, z_2, ..., z_{n+1})\| \leq N_1 \sum_{i=1}^{n+1} \| y_i - z_i\|$$

$$\| f_2(t, y_1, y_2, ..., y_{m+1}) - f_2(t, z_1, z_2, ..., z_{m+1})\| \leq N_2 \sum_{j=1}^{m+1} \| y_j - z_j\|$$

$$\| H_1(t, s, y) - H_1(t, s, z)\| \leq N_3(\| y - z\|)$$

$$\| H_2(t, s, y) - H_2(t, s, z)\| \leq N_4(\| y - z\|)$$

$$\| G(s_1) - G(s_2)\| \leq N_5(\| s_1 - s_2\|)$$

where $(Gz)(x) = \sum_{p=1}^{k} c_p z(t_p, x)$.

Define the function $F_1 : J \times E^{n+1} \to E;\quad F_2 : J \times E^{m+1} \to E;\quad h_1, h_2 : J \times J \times E \to E$ and $G : X \to E$ as follows

$$F_1(t, x_1(t), ..., x_{n+1}(t))(x) = f_1(t, x_1(x, t), ..., x_{n+1}(x, t))$$

$$F_2(t, x_1(t), ..., x_{m+1}(t))(x) = f_2(t, x_1(x, t), ..., x_{m+1}(x, t))$$

$$h_1(t, s, x_1(t))(x) = H_1(t, s, x_1(x, t))$$

$$h_2(t, s, x_1(t))(x) = H_2(t, s, x_1(x, t))$$

for $t \in J$ and $0 < x < \pi$. Then the above problem (9) -(11) can be formulated in (1) -(2). Since all the hypothesis of Theorem 3.1 are satisfied. Consequently, Theorem 3.1 can be applied for the equations (9) -(11).

(2) Consider the another partial integrodifferential equation of the form:

$$\begin{aligned} &\frac{\partial w(t, y)}{\partial t} - \frac{\partial^2 w(t, y)}{\partial y^2} \\ &= c_1(t)w(\sin t, y) + c_2(t) \sin w(t, y) + \frac{1}{t^2 + 1} \int_{t_0}^{t} c_3(s)w(\sin s, y)ds \\ &+ \tilde{c}_1(t)w(\sin t, y) + \tilde{c}_2(t) \sin w(t, y) + \frac{1}{t^2 + 1} \int_{t_0}^{t} \tilde{c}_3(s)w(\sin s, y)ds, \end{aligned} \tag{12}$$

$$w(t, 0) = w(t, \pi) = 0; \tag{13}$$

$$w(0, y) + \sum_{p=1}^{k} C_p w(t_p, y) = w_0(y) \tag{14}$$

where we state the conditions as follows:

(*i*) The function $c_j(.)$ and $\tilde{c}_j(.), j = 1,2,3$ are continuous on $[0,1]$ with condition

$$l_j = \sup_{0\le s\le 1} \|c_j(s)| < 1, j = 1,2,3$$

and

$$\tilde{l}_j = \sup_{0\le s\le 1} \|\tilde{c}_j(s)| < 1, j = 1,2,3$$

(*b*) The function $C_p \in R, p = 1,2,...,k$.

Let us consider that $E = L^2[0,\pi]$. Explain $A = D(A) \subset E \to E$ is linear operator which is described by $Aw = w''$ with domain $D(A) = \{w \in E : w, w'$ are absolutely continuous, $w'' \in E, w(0) = w(\pi) = 0\}$ Then operator A can be expressed

$$Aw = \sum_{n=1}^{\infty} n^2(w,w_n)w_n, w \in D(A)$$

where $w_n(y) = \left(\frac{\sqrt{2}}{\pi}\right)\sin ny, n = 1,2,...$ is the orthogonal set of eigenvalues of A. Further, for $w \in E$, we have

$$T(t)w = \sum_{n=1}^{\infty} exp\left(\frac{-n^2 t}{1+n^2}\right)(w,w_n)w_n.$$

It common that A is the infinitesimal generator of C_0-semigroup $\{T(t)\}_{\ge 0}$ on E.

To solve this system, we will define the operators $F_1, F_2 : J \times E \times E \to E;\ h_1, h_2 : J \times J \times E \to E;\ g : X \to E$ by

$$F_1\left(t, w(\alpha(t)), \int_0^t h_1(t,s,w(\alpha(t)))ds\right)(y)$$
$$= c_1(t)w(\sin t, y) + c_2(t)\sin w(t,y) + \frac{1}{t^2+1}\int_{t_0}^t c_3(s)w(\sin s, y)ds;$$

$$F_2\left(t, w(\alpha(t)), \int_0^t h_2(t,s,w(\alpha(t)))ds\right)(y)$$
$$= \tilde{c}_1(t)w(\sin t, y) + \tilde{c}_2(t)\sin w(t,y) + \frac{1}{t^2+1}\int_{t_0}^t \tilde{c}_3(s)w(\sin s, y)ds;$$

$$\int_0^t h_1(t,s,w(\alpha(t)))ds = \frac{1}{t^2+1}\int_{t_0}^t c_3(s)w(\sin s, y)ds;$$

$$\int_0^t h_2(t,s,w(\alpha(t)))ds = \frac{1}{t^2+1}\int_{t_0}^t \tilde{c}_3(s)w(\sin s, y)ds$$

$$g(w)(y) = \sum_{p=1}^{k} t_p w(t_p, y)$$

Then system (12) - (14) yields the abstract form (1) - (2). With the choice of the above functions it is clear that all the conditions of the Theorem 3.1 are fulfilled. Thus with the help of Theorem 3.1, we assume that the system (12) - (14) has a mild solution on J.

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