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Research Article

Maps that preserve left (right) K-Cauchy sequences

Olivier Olela Otafudu

School of Mathematical and Statistical Sciences North-West University, Potchefstroom Campus, Potchefstroom 2520, South Africa

Abstract

It is well-known that on quasi-pseudometric space (X, q), every q^s-Cauchy sequence is left (or right) K-Cauchy sequence but the converse does not hold in general. In this article, we study a class of maps that preserve left (right) K-Cauchy sequences that we call left (right) K-Cauchy sequentially-regular maps. Moreover, we characterize totally bounded sets on a quasi-pseudometric space in terms of maps that preserve left K-Cauchy and right K-Cauchy sequences and uniformly locally semi-Lipschitz maps.

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1. Introduction

Let $f : (X, d) \to (Y, d')$ be a map between two metric spaces (X, d) and (Y, d'). Then f is called *Cauchy sequentially-regular* (or *Cauchy continuous*) if $(f(x_n))$ is a Cauchy sequence on (Y, d') whenever (x_n) is a Cauchy sequence on (X, d). In [12], Snipes investigated the class of Cauchy sequentially-regular maps. He proved that a map $f : (X, d) \to (Y, d')$ is uniformly continuous if and only if f preserves parallel sequences. Moreover, He characterized Cauchy sequentially-regular maps in terms of maps that preserve equivalent sequences.

In addition, Snipes observed that the class of Cauchy sequentially-regular maps from the metric space (X, d) into the metric space (Y, d') sits between the class of uniformly continuous maps from the metric space (X, d) into the metric space (Y, d') and the class of continous maps from the metric space (X, d) into the metric space (Y, d'). He also proved that a map $f : (X, d) \to (Y, d')$ is Cauchy sequentially regular whenever $f : (X, d) \to (Y, d')$ is continuous and (X, d) is complete. Finally Snipes in [12] investigated extension maps of Cauchy sequentially-regular maps into a complete metric space.

After Snipes paper [12], the concept of Cauchy sequential regularity got an interest in the community of mathematicians. For instance in [8], Jain and Kundu proved that each Cauchy sequentailly-regular map is uniformly continuous if and only if the completion of a

Email address: olivier.olelaotafudu@nwu.ac.za
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metric space (X, d) is a UC-space (space on which each continuous function is uniformly continuous).

Furthermore, in [6], Di Maio et al. proved that a metric space is complete if and only if each continuous function defined on it is a Cauchy sequentially-regular map.

Moreover, in [2,3], Beer, characterized Cauchy sequentially-regular maps in terms of a family on which they must be strongly uniformly continuous on totally bounded sets and uniformly continuous on totally bounded sets. Recently, Beer proved that on a metric space (X, d), a subset B of X is totally bounded if and only its image $f(B)$ is d'-bounded whenever $f : (X, d) \to (Y, d')$ is Cauchy sequentially-regular map and (Y, d') is a metric space. In addition, he showed that a subset B of a metric space (X, d) is totally bounded if and only if its image $f(B)$ is bounded subset of $\mathbb{R}$ whenever $f : (X, d) \to (\mathbb{R}, |.|)$ is uniformly locally Lipschitz.

It is no doubt that uniform continuity and continuity plays an important role in the study of Cauchy sequentially-regular maps on metric spaces. Observe that for any two quasi-metric spaces (X, q) and (Y, p). If a map $f : (X, q) \to (Y, p)$ is quasi-unformly continuous (or uniformly continuous), then $f : (X, q^s) \to (Y, p^s)$ is also uniformly continuous but the converse is not true, in general (see Example 3.3). It is well-know that on a quasi-pseudometric space, if a sequence is q^s-Cauchy, then it is left (right) K-Cauchy sequence but the converse is not true (see for instance [4,13]). As might be expected these have led to the conjecture that the maps from a quasi-pseudometric space into another quasi-pseudometric space that preserve left (right) K-Cauchy sequences that we call *left (right) K-Cauchy sequentially regular maps* (see Definition 4.8) need to be studied carefully.

The aim of this paper is a careful study of the above-mentioned conjecture. Moreover, for map $f : (X, q) \to (Y, p)$, where (X, q) and (Y, p) are quasi-pseudometric spaces, we study connections between left (right) K-Cauchy sequentially regular maps and q^s-Cauchy sequentially maps. We also show that a continuous map from a left (right) Smyth complete quasi-metric space into a quasi-pseudometric space is left (right) K-Cauchy sequentially-regular. Finally, we characterize totally bounded sets on a quasi-pseudometric space in terms of left and right K-Cauchy sequentially-regular maps and uniformly locally semi-Lipschitz maps which extend an important result due to Beer and Garrido ([1, Theorem 3.2]) in our settings.

2. Preliminaries

In this section we summarize some basic results on quasi-pseudometric spaces. For more details about quasi-pseudometric spaces we recommend the following articles [4,7,9,13].

Let X be a set and $q : X \times X \to [0, \infty)$ be a function. Then q is an *quasi-pseudometric* on X if

(a) $q(x, x) = 0$ for all $x \in X$,
(b) $q(x, y) \leq q(x, z) + q(z, y)$ for all $x, y, z \in X$.

If q is a quasi-pseudometric on X, then the pair (X, q) is called an *quasi-pseudometric space*.

If the function q satisfies the condition
(c) for any $x, y \in X, q(x, y) = 0 = q(y, x)$ implies $x = y$ instead of condition (a), then q is called a T_0-quasi-metric on X and the pair (X, q) is called T_0-*quasi-metric space* (see for instance [9]).

Furthermore, if q is a quasi-pseudometric on X, then the function $q^t : X \times X \to [0, \infty)$ defined by $q^t(x, y) = q(y, x)$, for all $x, y \in X$ is also a quasi-pseudometric on X and it is called the *conjugate quasi-pseudometric* of q.

Note that for any q quasi-pseudometric on X, the function q^s defined by $q^s(x,y) := \max\{q(x,y), q^t(x,y)\}$ for all $x, y \in X$ is a pseudometric on X, usually called the *symmetrised* quasi-pseudometric of q.

If (X,q) is a quasi-pseudometric space. Then we associete the topology $\tau(q)$ on X with respect to q where its base is given by the family $\{D_q(x,\epsilon) : x \in X \text{ and } \epsilon > 0\}$ with $D_q(x,\epsilon) = \{y \in X : q(x,y) < \epsilon\}$.

If $A \subseteq X$ and $\epsilon > 0$, then the set $D_q(A,\epsilon)$ is defined by

$$D_q(A,\epsilon) := \bigcup_{a\in A} D_q(a,\epsilon).$$

We say that a subset A of X is *q-totally bounded* provided that A is q^s-totally bounded, i.e. for any $\epsilon > 0$, there exists $F \in \mathfrak{F}$ with $F \subseteq A \subseteq D_{q^s}(F,\epsilon)$, where $\mathfrak{F}$ is the collection of all finite subsets of X (see [14, Definition 5]).

In the sequel we denote by $\mathscr{TB}_q(X)$ the collection of all q-totally bounded subsets of X.

It is well-known that $\mathscr{TB}_q(X)$ forms a bornology on X, i.e.

(a) $\{x\} \in \mathscr{TB}_q(X)$ whenever $x \in X$,
(b) if $A \subseteq B$ with $B \in \mathscr{TB}_q(X))$, then $A \in \mathscr{TB}_q(X)$,
(c) if $A, B \in \mathscr{TB}_q(X)$, then $A \cup B \in \mathscr{TB}_q(X)$.

It is easy to see that $\mathscr{TB}_{q^s}(X) = \mathscr{TB}_{q^t}(X) = \mathscr{TB}_q(X)$.

Definition 2.1. Let (X,q) be a quasi-pseudometric space. A sequence (x_n) in X is called:

(a) *left K-Cauchy* if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$q(x_k, x_m) < \epsilon \text{ whenever } n, k \in \mathbb{N} \text{ with } N \leq k \leq n.$$

(b) *right K-Cauchy* if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$q^t(x_k, x_m) < \epsilon \text{ whenever } n, k \in \mathbb{N} \text{ with } N \leq k \leq n.$$

(c) q^s-Cauchy if it is a Cauchy sequence in the symmetrised quasi-pseudometric q^s.

Note that if we want to emphazise the quasi-pseudometric q on X, we shall say that a sequence is right q-K-Cauchy and left q-K-Cauchy.

For a quasi-pseudometric space (X,q). It well-known that these above three concepts are associated as follows:

$$q^s\text{-Cauchy} \implies \text{left } K\text{-Cauchy and } q^s\text{-Cauchy} \implies \text{right } K\text{-Cauchy}.$$

Remark 2.2. Let (X,q) be quasi-pseudometric space.

(a) A sequence (x_n) in X is left q-K-Cauchy if and only if (x_n) is right q^t-K- Cauchy.
(b) A sequence (x_n) in X is q^s-Cauchy if and only if the sequence (x_n) is both left q-K-Cauchy and right q-K-Cauchy.

The following definition can be found for instance on [4].

Definition 2.3. Let (X,q) be quasi-pseudometric space. We say that (X,q) is:

(a) *bicomplete* if its associated pseudometric space (X,q^s) is complete, that is, every q^s-Cauchy sequence is q^s-convergent;
(b) *sequentially left (right) K-complete* if every left (right) K-Cauchy sequence is q-convergent;
(c) *sequentially left (right) Smyth complete* if every left (right) K-Cauchy sequence is q^s-convergent;

3. Uniformly continuous and semi-lipschitz maps

An *asymmetric norm* on a real vector space X is a function $\|.| : X \to [0, \infty)$ satisfying the conditions

(1) $\|x| = \| - x| = 0$ then $x = 0$;
(2) $\|ax| = a\|x|$;
(3) $\|x + y| \leq \|x| + \|y|$,

for all $x, y \in X$ and $a \geq 0$. Then the pair $(X, \|.|)$ is called *an asymmetric normed space.*

The *conjugate asymmetric norm* $|.\|$ of $\|.|$ and the *sysmmetrisation norm* $\|.\|$ of $\|.|$ are defined respectively by

$$|x\| := \| - x| \quad \text{and} \quad \|x\| := \max\{|x\|, \|x|\} \text{ for any } x \in X.$$

An asymmetric norm $\|.|$ on X induces a quasi-metric $q_{\|.|}$ on X defined by

$$q_{\|.|}(x, y) = \|x - y| \text{ for any } x, y \in X.$$

If $(X, \|.\|)$ is normed lattice space, then the function $\|.|$ defined by $\|x| := \|x^+\|$, where $x^+ = \max\{x, 0\}$ is an asymmetric norm on X.

A basic but interesting example we point out the asymmetric norm u on $\mathbb{R}$ (considered as a real vector space) defined for any $y \in \mathbb{R}$ by $u(y) = y^+$, where $y^+ = \max\{y, 0\}$, it follows that $u^t(y) = \max\{-x, 0\} = y^-$ and $u^s(y) = \max\{y^+, y^-\} = |x|$. In addition, the asymmetric norm u induces the quasi-metric q_u on $\mathbb{R}$ defined by $q_u(x, y) = (x-y)^+ = \max\{x-y, 0\}$ whenever $x, y \in \mathbb{R}$.

The following is a well-known definition.

Definition 3.1. Let (X, q) and (Y, p) be quasi-pseudometric spaces. A map $f : (X, q) \to (Y, p)$ is called *quasi-uniformly continuous* (or *uniformly continuous*) if for any $\epsilon > 0$, there exists $\delta > 0$ such that $q(x, y) \leq \delta$, then $p(\varphi(x), \varphi(y)) < \epsilon$ for all $x, y \in X$.

Lemma 3.2. *Let (X, q) and (Y, p) be quasi-pseudometric spaces. If the map $f : (X, q) \to (Y, p)$ is uniformly continuous, then the function $f : (X, q^s) \to (Y, p^s)$ is uniformly continuous.*

Example 3.3. We equip $X = \mathbb{R}_+ = [0, \infty)$ with the quasi-metric q defined by $q(x, y) = (y - x)^+$ for any $x, y \in [0, \infty)$ and $Y = \mathbb{R}$ is equipped with the T_0-quasi-metric p defined by $p(x, y) = (y - x)^+$ for any $x, y \in \mathbb{R}$. Then

(i) the function $f(x) = -\sqrt{x}$ whenever $x \in \mathbb{R}_+$ is uniformly continuous from $(\mathbb{R}_+, |.|)$ into $(\mathbb{R}, |.|)$.
(ii) the function $f(x) = -\sqrt{x}$ whenever $x \in \mathbb{R}_+$ is not uniformly continuous from $(\mathbb{R}_+, q)$ into $(\mathbb{R}, p)$.

Let (X, q) be a quasi-metric space and $(Y, \|.|)$ be an asymmetric normed space. Then a map $f : (X, q) \to (Y, \|.|)$ is called *semi-Lipschitz* if there exists $k \geq 0$ such that

$$\|f(x) - f(y)| \leq kq(x, y) \quad \text{for all } x, y \in X. \tag{3.1}$$

The number k satisfying (3.1) is called *semi-Lipschitz constant* for f and the map f is called k-semi-Lipschitz. For more details about semi-Lipschitz maps we recommend the reader to see [5].

Definition 3.4. Let (X, q) be a quasi-metric space and $(Y, \|.|)$ be an asymmetric normed space. Then:

(a) A map $f : (X, q) \to (Y, \|.|)$ is called *locally semi-Lipschitz* provided that for all $x \in X$, then there exists $\delta(x) > 0$ such that $f|_{D_q(x,\delta(x))}$ is semi-Lipschtz.

(b) A function $f:(X,q)\to(Y,\|.|)$ is called *uniformly locally semi-Lipschitz* provided that for all $x\in X$, there exists $\delta>0$ (δ does not depend to x) such that $f|_{D_q(x,\delta)}$ is semi-Lipschtz.

Lemma 3.5. *Let (X,q) be a quasi-metric space and $(Y,\|.|)$ be an asymmetric normed space. If function $f:(X,q)\to(Y,\|.|)$ is locally semi-Lipschitz, then $f:(X,q^s)\to(Y,\|.\|)$ is locally semi-Lipschitz.*

Proof. Suppose that $f:(X,q)\to(Y,\|.|)$ is locally semi-Lipschitz. Let $x\in X$, there exists $\delta(x)>0$ and $k\geq 0$ such that for any

$$y,z\in D_{q^s}(x,\delta(x))\subseteq D_q(x,\delta(x))$$

we have

$$\|f(y)-f(z)|\leq kq(y,z)\leq kq^s(y,z) \tag{3.2}$$

and

$$\|f(z)-f(y)|\leq kq(z,y)\leq kq^s(y,z). \tag{3.3}$$

Combining (3.2) and (3.3) for some $k\geq 0$ we have

$$\|f(y)-f(z)\|\leq kq(y,z)\leq kq^s(y,z)$$

whenever $y,z\in D_{q^s}(x,\delta(x))$. Thus the function $f:(X,q^s)\to(Y,\|.\|)$ is locally semi-Lipschitz. □

Remark 3.6. Let (X,q) be a quasi-metric space and $(Y,\|.|)$ be an asymmetric normed space. If a function $fi:(X,q)\to(Y,\|.|)$ is locally semi-Lipschitz, then $\varphi|_{D_q(x,\delta_x)}$ is continuous whenever $x\in X$ and for some $\delta_x>0$.

4. Left (right) K-Cauchy sequentially regular maps

Definition 4.1 (compare [10, Definition 3.1 and Definition 3.4])**.** Let (X,q) be a quasi-pseudometric space. Let (x_n) and (y_n) be sequences in X.

(a) We say that the sequences (x_n) and (y_n) are *parallel* with respect to q (noted by $(x_n)||_q(y_n)$) if for any $\epsilon>0$, there exists $n_\epsilon\in\mathbb{N}$ such that $q(x_n,y_n)<\epsilon$ whenever $n\geq n_\epsilon$.

(b) We say that the sequences (x_n) and (y_n) are *equivalent* with respect to q (noted by $(x_n)\equiv_q(y_n)$) if for any $\epsilon>0$, there exists $n_\epsilon\in\mathbb{N}$ such that $q^s(x_k,y_n)<\epsilon$ whenever $n,k\geq n_\epsilon$.

Note that the concept of parallel sequences in quasi-pseudometric spaces is not new. For instance, in [10], Moshoko introduced concepts of parallel sequences and equivalent sequences in order to study extensions of maps that preserve q^s-Cauchy sequences in a quasi-pseudometric space (X,q). But it is well-known that on a quasi-pseudometric space (X,q), any q^s-Cauchy sequence in X is left K-Cauchy (right K-Cauchy), still the converse is not true in general. Our definitions of parallel and equivalent sequences are motivated from metric point of view of parallel and equivalent sequences (see [12]) and by the fact that parallel sequences are preserved by uniformly continuous maps and equivalent sequences are preserved by Cauchy-sequentially-regular maps. However, we are studying maps that preserve left K-Cauchy (right K-Cauchy) sequences. This explains why our Definition 4.1(2) is more general than [10, Definition 3.4]. We point out that in [7], Doitchinov introduced the concept of cosequence sequences which is similar to the concept of parallel sequences with connections to Cauchy sequences in a quasi-pseudometric space. From cosequence sequences, he defined equivalent sequences for a quasi-metric space satisfying some properties (that he called balanced quasi-metric space).

Lemma 4.2. *Let (X,q) be a quasi-pseudometric space. Let (x_n) and (y_n) be sequences in X and $a\in X$. If (x_n) is q^s-convergent to a and (y_n) is q^s-convergent to a, then $(x_n)\equiv_q(y_n)$.*

Proof. Let $\epsilon > 0$. Suppose that (x_n) is q^s-convergent to a and (y_n) is q^s-convergent to a. We show that $(x_n)\equiv_{q^t}(y_n)$. Then there exists $n_\epsilon \in \mathbb{N}$ and $n_\epsilon^{'} \in \mathbb{N}$ such that

$$q^s(a, x_n) < \frac{\epsilon}{2} \quad \text{if} \quad n \geq n_\epsilon \tag{4.1}$$

and

$$q^s(y_n, a) < \frac{\epsilon}{2} \quad \text{if} \quad n \geq n_\epsilon^{'}. \tag{4.2}$$

Let $N = \max\{n_\epsilon, n_\epsilon^{'}\}$. If $N \leq k, n$, then

$$\begin{aligned} q^s(x_k, y_n) &= q^s(y_n, x_k) \leq q^s(y_n, a) + q^s(a, x_k) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence, $(x_n)\equiv_q(y_n)$. □

The following lemma is a consequence of the definition and Remark 2.2.

Lemma 4.3. *Let (X, q) be a quasi-pseudometric space. Let (x_n) and (y_n) be sequences in X. If $(x_n)\equiv_q(y_n)$, then the sequence (x_n) is left K-Cauchy and right K-Cauchy.*

We leave the proof of the following lemma.

Lemma 4.4. *Let (X, q) be a quasi-pseudometric space and (x_n) and (y_n) be any two sequences in X and $a \in X$. If (x_n) is q-convergent to a and $(y_n)||_q(x_n)$, then (y_n) is q-convergent to a.*

Lemma 4.5. *Let (X, q) be a quasi-pseudometric space and (x_n) and (y_n) be any two sequences in X. It is true that $(x_n)\equiv_q(y_n)$ if and only if the sequence (z_n) is left K-Cauchy and right K-Cauchy, where $z_n := (x_1, y_1, x_2, y_2, x_3, y_3, \cdots)$.*

Proof. ($\Rightarrow$) Let $\epsilon > 0$. Suppose that $(x_n)\equiv_q(y_n)$. Then there exists $n_\epsilon \in \mathbb{N}$ such that

$$q^s(x_k, y_m) < \epsilon \quad \text{whenever} \quad k, m \geq n_\epsilon.$$

It follows that the sequence $z_n = (x_1, y_1, x_2, y_2, x_3, y_3, \cdots)$ is q^s-Cauchy sequence. Hence the sequence z_n is left K-Cauchy and right K-Cauchy.

($\Leftarrow$) Suppose that the sequence $z_n = (x_1, y_1, x_2, y_2, x_3, y_3, \cdots)$ is left K-Cauchy and right K-Cauchy. Then the sequence $z_n = (x_1, y_1, x_2, y_2, x_3, y_3, \cdots)$ is q^s-Cauchy. Therefore, we have that $(x_n)\equiv_q(y_n)$ by [12, Theorem 1 (4)]. □

The following proposition is obvious. Therefore, we omit the proof.

Proposition 4.6 (compare [10, Theorem 3.2])**.** *Let (X, q) and (Y, p) be quasi-pseudometric spaces. Then the following statements are equivalent.*

(1) *The map $f : (X, q) \to (Y, p)$ is uniformly continuous.*
(2) *Whenever $(x_n)||_q(y_n)$ in X and $f : (X, q) \to (Y, p)$ is a map, then $(f(x_n))||_p(f(y_n))$ in Y.*

Remark 4.7. We point out that it is easy to find an example of two sequences which are parallel with respect to q but they are not parallel with respect to q^t.

Definition 4.8. Let (X, q) and (Y, p) be quasi-pseudometric spaces. A map $f : (X, q) \to (Y, p)$ is called:

(a) A *left K-Cauchy sequentially-regular* if for any left K-Cauchy sequence (x_n) in X, then the sequence $(f(x_n))$ is left K-Cauchy in Y.
(b) A *right K-Cauchy sequentially-regular* if for any right K-Cauchy sequence (x_n) in X, then the sequence $(f(x_n))$ is left K-Cauchy in Y.

Proposition 4.9. *Let (X,q) and (Y,p) be quasi-pseudometric spaces and $f:(X,q)\to(Y,p)$ be a map. Then we have that the map f is left K-Cauchy sequentially-regular in X if and only if whenever $(x_n)\equiv_q(y_n)$ in X, then $(f(x_n))\equiv_p(f(y_n))$ in Y.*

Proof. ($\Rightarrow$) Suppose that f is left K-Cauchy and right K-Cauchy sequentially-regular. If $(x_n)\equiv_q(y_n)$ in X, then it follows that the sequence $(x_1,y_1,x_2,y_2,\cdots)$ is left K-Cauchy and right K-Cauchy sequence in X by Lemma 4.5.

Thus the sequence $(f(x_1),f(y_1),f(x_2),f(y_2),\cdots)$ is left K-Cauchy and right K-Cauchy sequence in Y from the assumption on the map f. Hence $(f(x_n))\equiv_q(f(y_n))$ in Y by Lemma 4.5.

($\Leftarrow$) Assume that f preserves equivalent sequences. Let (x_n) be a left K-Cauchy sequence and right K-Cauchy in X. Since $(x_n)\equiv_q(x_n)$, then we have that $(f(x_n))\equiv_q(f(x_n))$ in Y. Therefore, the sequence $(f(x_n))$ is left K-Cauchy and right K-Cauchy sequence in Y. □

Theorem 4.10. *Let (X,q) and (Y,p) be quasi-pseudometric spaces and $f:(X,q)\to(Y,p)$ be a map. Then the following hold.*

(1) *If the map f is uniformly continuous, then f is left K-Cauchy (right K-Cauchy) sequentially-regular.*
(2) *If the map f is right K-Cauchy and right K-Cauchy sequentially-regular, then f is continuous with respect to $\tau(q^s)$ and $\tau(p^s)$.*

Proof. (1) Let $\epsilon>0$. Suppose that f is uniformly continuous. We only show that f is left K-Cauchy sequentially regular and for f right K-Cauchy will follow by symmetry. Let (x_n) be any left q-K-Cauchy sequence in X.

Then there exists $\delta>0$ because f is uniformly continuous such that

$$q(x_k,x_n)<\delta \text{ whenever } N\le k\le n$$

for some $N\in\mathbb{N}$ since (x_n) is left K-Cauchy sequence in X. It follows that

$$p(f(x_k),f(x_n))<\epsilon \text{ whenever } N\le k\le n$$

for some $N\in\mathbb{N}$. Hence $(f(x_n))$ is left K-Cauchy in Y.

(2) Suppose that f is right K-Cauchy and right-K-Cauchy sequentially-regular. If (x_n) be sequence in X such that (x_n) is q^s-convergent to $a\in X$. We show that the sequence $(f(x_n))$ is p^s-convergent to $f(a)$.
We consider the constant sequence (a) which is q^s-convergent to a. Then we have that $(x_n)\equiv_q(a)$ by Lemma 4.2. It follows that the sequence $(x_1,a,x_2,a,\cdots)$ is left K-Cauchy and right K-Cauchy (q^s-Cauchy) in X.

From our assumption we have $(f(x_1),f(a),f(x_2),f(a),\cdots)$ is left K-Cauchy and right K-Cauchy (p^s-Cauchy) in Y with a convergent subsequence $(f(a))$ which p^s-convergent to $f(a)$. Thus the sequence $(f(x_n))$ is p^s-convergent to $f(a)$. □

Example 4.11 (compare [15, Example 1])**.** Let $X=\{0\}\cup\{1/n:n\in\mathbb{N}\}\cup\{n:n\in\mathbb{N}\backslash\{1\}\}$. We equip X with the quasi-metric q defined by $q(x,x)=0$ for any $x\in X$, $q(0,1/n)=1/n$ for any $n\in\mathbb{N}$, $q(1/n,1/m)=1/n$ whenever $n<m$, $q(0,n)=2^{-n}$ whenever $n\in\mathbb{N}\setminus\{1\}$, $q(n,m)=|2^{-1}-2^{-m}|$ whenever $n,m\in\mathbb{N}\setminus\{1\}$ and $q(x,y)=1$ otherwise.

It is easy to see that sequences $(1/n)$ and (n) are left q-K-Cauchy in X and both are q-convergent to 0. If we consider the function $g:(X,q)\to(X,q)$ defined by

$$g(x)=\begin{cases}0 & \text{if} \quad x=0\\ 1/n & \text{if} \quad x=n\in\mathbb{N}\\ n & \text{if} \quad x=1/n \quad \text{and} \quad n\in\mathbb{N}\setminus\{1\}.\end{cases}$$

Then the function g preserves left q-K-Cauchy sequences since $g((1/n)) = (n)$ and $g((n)) = (1/n)$ and g is continuous.

Theorem 4.12. *Let (X,q) and (Y,p) be quasi-pseudometric spaces and $f:(X,q)\to(Y,p)$ be a uniformly continuous map. Then $f:(X,q)\to(Y,p)$ is left q-K-Cauchy sequentially-regular if and only if $f:(X,q^t)\to(Y,p^t)$ is right q^t-K-Cauchy sequentially-regular.*

Proof. We only prove the necessary condition and the sufficiant condition follows by similar arguments. It is obvious that $f:(X,q)\to(Y,p)$ is uniformly continuous if and only if $f:(X,q^t)\to(Y,p^t)$ is uniformly continuous.

Suppose that $f:(X,q)\to(Y,p)$ is left q-K-Cauchy sequentially-regular and let (x_n) be a right q^t-K-Cauchy sequence. Then the sequence (x_n) is left q-K-Cauchy in X by Remark 2.2 (1).

Moreover, the sequence $(f(x_n))$ is left p-K-Cauchy in Y from the assumption. But the sequence $(f(x_n))$ is right p^t-K-Cauchy in Y again by Remark 2.2 (1). Hence $f:(X,q^t)\to(Y,p^t)$ is right q^t-K-Cauchy sequentially-regular. □

Theorem 4.13. *Let (X,q) and (Y,p) be quasi-pseudometric spaces. If the uniformly continuous map $f:(X,q)\to(Y,p)$ is left K-Cauchy and right K-Cauchy sequentially-regular, then $f:(X,q^s)\to(Y,p^s)$ is q^s-Cauchy sequentially regular.*

Proof. Let (x_n) be a q^s-Cauchy sequence in X. Then (x_n) is both left and right q-K-Cauchy in X by Remark 2.2 (2).

Furthemore, $(f(x_n))$ is both left and right p-K-Cauchy in Y since $f:(X,q)\to(Y,p)$ is both left K-Cauchy and right K-Cauchy sequentially-regular.

Moreover, the sequence $(f(x_n))$ is p^s-Cauchy by Remark 2.2 (2). Therefore, the uniformly continuous map $f:(X,q^s)\to(Y,p^s)$ is q^s-Cauchy in X. □

Theorem 4.14. *Let (X,q) and (Y,p) be quasi-pseudometric spaces and $f:(X,q)\to(Y,p)$ be a map. Then whenever (X,q) is left Smyth complete and the map f is continuous, then f is left K-Cauchy sequentially-regular.*

Proof. Suppose that (X,q) is left Smyth complete and the map f is continuous. If the sequence (x_n) is left K-Cauchy , then there exists $x\in X$ such that (x_n) is q^s-convergent to x by the left Smyth completeness of (X,q).

Then, the sequence $(f(x_n))$ is p^s-convergent to $f(x)$ since the map $f:(X,q^s)\to(Y,p^s)$ is continous. Hence the sequence $(f(x_n))$ is q^s-Cauchy. Therefore, the sequence $(f(x_n))$ is left K-Cauchy by Remark 2.2(2). □

Corollary 4.15. *Let (X,q) and (Y,p) be quasi-pseudometric spaces and $f:(X,q)\to(Y,p)$ be a map. Then whenever (X,q) is right Smyth complete and the map f is continuous, then f is right K-Cauchy sequentially-regular.*

Remark 4.16. In Theorem 4.14, if we replace the left Smyth completeness by the sequentially left K-completeness, the theorem does not hold because for a sequence being left K-Cauchy does not guarantee the existence of the limit (see [13, Example 2]).

5. Total boundedness and left K-Cauchy sequential regularity

Let (X,q) be a quasi-pseudometric space. An arbitrary subset A of X is called *q-bounded* if and only if there exists $x\in X$, $r>0$ and $s>0$ such that $A\subseteq D_q(x,r)\cap D_{q^t}(x,s)$. Note that one can replace $D_q(x,r)\cap D_{q^t}(x,s)$ by $D_q[x,r]\cap D_{q^t}[x,s]$.

Note that the above definition is slightly different from [16]. In the sense of [16] a subset A of X can be q-bounded and not necessary q^t-bounded. Obviously in our context a subset A is q-bounded if and only if it is q^t-bounded. But q-boundedness (or q^t-boundedness) does not imply q^s-boundedness. Moreover, if q is an extended quasi-pseudometric on X (i.e. the distance between two point can be ∞), then a subset B of X can be included in $D_q(x,\epsilon)$ for some $x \in X$ but its diameter $\operatorname{diam}(B) = \{q(y,z) : y,z \in B\} = \infty$ (see [16, p. 2022]).

Let $\mathscr{B}_q(X)$ be the collection of all q-bounded subsets of X whenever (X,q) is quasi-pseudometric space. It is easy to see that

(a) whenever $x \in X$, then $\{x\} \in \mathscr{B}_q(X)$,
(b) whenever $A \subseteq B \subseteq X$ and $B \in \mathscr{B}_q(X)$, then $A \in \mathscr{B}_q(X)$,
(c) whenver $A, B \in \mathscr{B}_q(X)$, then $A \cup B \in \mathscr{B}_q(X)$.

It follows that $\mathscr{B}_q(X)$ forms a bornology on X and this bornology is called the *quasi-metric bornology* determined by q. Furthermore, We have the following observations instead of the one observed in [11]

$$\mathscr{B}_{q^s}(X) = \mathscr{B}_q(X) \tag{5.1}$$

and

$$\mathscr{B}_{q^s}(X) = \mathscr{B}_{q^t}(X). \tag{5.2}$$

Remark 5.1. Let (X,q) be a quasi-pseudometric space and $A \subseteq X$. It is easy to see that:

(i) If $A \in \mathscr{T}\mathscr{B}_q(X)$, then $A \in \mathscr{B}_q(X)$.
(ii) Whenever F is finite subset of X, $F \in \mathscr{T}\mathscr{B}_q(X)$.

Proposition 5.2. *Let (X,q) and (Y,p) be quasi-pseudometric spaces and $f : (X,q) \to (Y,p)$ be a map. Then $f|_T$ is uniformly continuous, whenever $T \in \mathscr{T}\mathscr{B}_q(X)$ if and only if f is Cauchy sequentially regular.*

Proof. ($\Rightarrow$) Assume that $f : (T,q) \to (Y,p)$ is uniformly continuous with T is q^s-totally bounded. Let (x_n) be a q^s-Cauchy sequence. Then $\{x_n : n \in \mathbb{N}\}$ is q^s-totally bounded and $f : (T,q^s) \to (Y,p^s)$ is uniformly continuous. It follows that f is Cauchy sequentially regular from [2, Proposition 5.7(2)].

($\Leftarrow$) Without loss of generality we suppose that $f : (T,q^s) \to (Y,p^s)$ is not uniformy continuous and T is q^s-totally bounded. Then for any $n \in \mathbb{N}$, there exists two sequences $(x_n),(t_n)$ in T such that

$$q^s(x_n,t_n) < \frac{1}{n} \text{ and } p^s(f(x_n),f(t_n)) \geq \epsilon \text{ for some } \epsilon > 0. \tag{5.3}$$

From the q^s-totally boundedness of T, suppose that the sequence (t_n) is q^s-Cauchy, then the sequence $(t_1,x_1,t_2,x_2,\cdots)$ is q^s-Cauchy but its image $(f(t_1),f(x_1),f(t_2),f(x_2),\cdots)$ under f is not p^s-Cauchy from (5.3). □

Theorem 5.3. *Let (X,q) be a quasi-pseudometric space and F be a nonempty subset of X. Then following conditions are equivalent:*

(1) *F is q-totally bounded;*
(2) *Whenever $(Y,\|.|)$ is an asymmetric normed space and the map $f : (X,q) \to (Y,q_{\|.|})$ is left and right K-Cauchy sequentially regular, then $f(F) \in \mathscr{B}_{q_{\|.|}}(Y)$;*
(3) *Whenever $(Y,\|.|)$ is an asymmetric normed space and the map $f : (X,q) \to (Y,q_{\|.|})$ is uniformly locally semi-Lipschitz, then $f(F) \in \mathscr{B}_{q_{\|.|}}(Y)$;*
(4) *Whenever the function $f : (X,q) \to (\mathbb{R},q_u)$ is uniformly locally semi-Lipschitz, then $f(F)$ is q_u-bounded set $\mathbb{R}$.*

Proof. (1) $\Longrightarrow$ (2) Suppose $f : (X, q) \to (Y, q_{\|.|})$ is left and right K-Cauchy sequentially regular and F is q-totally bounded. We have that $f : (X, q^s) \to (Y, q_{\|.\|})$ is q^s-Cauchy sequentially regular by Theorem 4.13. Since F is q^s-totally bounded as a q-totally bounded. Then $f(F) \in \mathscr{B}_{q_{\|.\|}}(Y)$ by [1, Theorem 3.2]. Thus from inclusion (5.1) we have $f(F) \in \mathscr{B}_{q_{\|.|}}(Y)$.

(2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (4) Follows from Lemma 3.5 and [1, Theorem 3.2].

(4) $\Longrightarrow$ (1) Suppose that F is not q-totally bounded. We show that there exists a semi-Lipschitz function $g : (D_q(x, \delta), q) \to (\mathbb{R}, q_u)$ for any $x \in X$ and some $\delta > 0$.

Since F is not q-totally bounded, then we have that

$$F \nsubseteq \bigcup_{k=1}^{n} D_{q^s}(f_k, \epsilon), \text{ where } f_k \in F \text{ whenever } k \in \{1, \cdots, n\}$$

for some $\epsilon > 0$. By induction, we construct a sequence (f_n) in F such that whenever $n \in \mathbb{N}$ we have $f_{n+1} \notin \bigcup_{k=1}^{n} D_{q^s}(f_k, \epsilon)$.

It follows that the family $\{D_{q^s}(f_n, \frac{\epsilon}{4}) : n \in \mathbb{N}\}$ is uniformly discrete. Furthermore, we have for any $x \in X$, there exists $n' \in \mathbb{N}$ such that

$$\emptyset \neq D_{q^s}(x, \epsilon/4) \cap D_{q^s}(f_{n'}, \epsilon/4) \subseteq D_q(x, \epsilon/4) \cap D_q(f_{n'}, \epsilon/4)$$

from [2, Proposition 3.8].

Let g be a function defined by

$$g(x) = \begin{cases} n - \frac{4n}{\epsilon} q(f_n, x) & \text{if} \quad x \in D_q(f_n, \epsilon/4) \\ 0 & \text{otherwise.} \end{cases}$$

It is obvious that the function g is unbounded with respect to u. We now show that g is a semi-Lipschitz. Consider $x, y \in D_q(f_n, \epsilon/4)$, then

$$\begin{aligned} q_u(g(x), g(y)) &= (g(x) - g(y))^+ = \left[\left(n - \frac{4n}{\epsilon} q(f_n, x)\right) - \left(n - \frac{4n}{\epsilon} q(f_n, y)\right)\right] \\ &= \frac{4n}{\epsilon}[q(f_n, y) - q(f_n, x)] \\ &\leq \frac{4n}{\epsilon} q(x, y). \end{aligned}$$

Therefore, we have that the function g is semi-Lipschitz with $k = \dfrac{4n}{\epsilon}$. □

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