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Research Article

The Brauer indecomposability of Scott modules with vertex $Q_{2^n} \times C_{2^m}$

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Abstract

We prove that the Scott module whose vertex is isomorphic to a direct product of a generalized quaternion 2-group and a cyclic 2-group is Brauer indecomposable. This result generalizes similar results which are obtained for abelian, dihedral, generalized quaternion, semidihedral and wreathed 2-group vertices.

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1. Introduction

The goal of this paper is to prove that the Scott module whose vertex is isomorphic to a direct product of a generalized quaternion 2-group and a cyclic 2-group is Brauer indecomposable. The Brauer indecomposability of Scott modules is an important notion because it serves a key ingredient for the Scott module to realize a splendid Morita equivalence between certain principal blocks with isomorphic defect groups (see [11–17, 22]).

Let G be a finite group and k an algebraically closed field of characteristic $p > 0$. For a finite-dimensional kG-module M, the Brauer construction $M(Q)$ with respect to Q has a natural $kN_G(Q)$-module structure on it (see Section 11 of [21]). As defined in [12], M is called *Brauer indecomposable* if $\mathrm{Res}^{N_G(Q)}_{Q\,C_G(Q)} M(Q)$ is indecomposable or zero as a $k(Q\,C_G(Q))$-module for any p-subgroup Q of G.

There is a relationship between saturation of the fusion system $\mathcal{F}_P(G)$ and Brauer indecomposability of p-permutation kG-modules with vertex P, where P is a p-subgroup of G. In fact, in Theorem 1.1 of [12] it is proved that, if M is a Brauer indecomposable p-permutation kG-module with vertex P, then $\mathcal{F}_P(G)$ is a saturated fusion system. The converse of this theorem is not true in general (see Remarks in page 99 of [12]). However, if M is the Scott kG-module with vertex P, there are results such that the converse is shown to hold under some extra conditions. This is shown when P is an abelian p-group ([12, Theorem 1.2]), P is a dihedral 2-group ([13, Theorem 1.3, Corollary 4.4]), P is a generalized quaternion 2-group ([14, Lemma 2.2]), P is a semidihedral 2-group ([16, Theorems 1.1 and 1.2]), P is a wreathed 2-group ([17, Theorem 1.1]). The 2-groups in the preceeding sentence have a common property that their 2-rank is at most 2, where the 2-rank of a finite group is defined to be the largest elementary abelian subgroup of

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a Sylow 2-subgroup of this group. In this paper, we focus on another family of 2-groups whose 2-rank is equal to 2. One of the main results of this paper is the following:

Theorem 1.1. *Let $P = Q_{2^n} \times C_{2^m}$ where $n \geq 3$ and $m \geq 1$. Assume that G is a finite group containing P such that the fusion system $\mathcal{F}_P(G)$ is saturated. Assume that $C_G(Q)$ is 2-nilpotent for every fully $\mathcal{F}_P(G)$-normalized non-trivial subgroup Q of P. Then the Scott module $\mathrm{Sc}(G, P)$ is Brauer indecomposable.*

In applications, when constructing stable equivalences of Morita type between principal blocks of two finite groups G and G', the Brauer indecomposability of $\mathrm{Sc}(G \times G', \Delta P)$ where $\Delta P := \{(u, u) \in P \times P\}$ and P is a common Sylow p-subgroup of G and G' gains importance. Hence, the following theorem serves a base step for obtaining these kind of equivalences. The second main theorem of this paper is the following result.

Theorem 1.2. *Let $P = Q_{2^n} \times C_{2^m}$ where $n \geq 3$ and $m \geq 1$. Assume that G and G' are two finite groups with a common Sylow 2-subgroup P. Assume that the fusion systems of G and G' on P are the same, namely $\mathcal{F}_P(G) = \mathcal{F}_P(G')$. Then the Scott module $\mathrm{Sc}(G \times G', \Delta P)$ is Brauer indecomposable.*

This result generalizes Lemma 2.2 of [14]. The paper is divided into four sections. In Section 2, we give some old and new results which will help us to accomplish our aim. Section 3 deals with the fusion inside $Q_{2^n} \times C_{2^m}$. We prove our main theorems in Section 4.

2. Preliminary results

In this section, we give some quoted and also some new results which will be helpful for showing Brauer indecomposability of Scott modules. Before stating these results, let us set some notation.

For a p-subgroup P of a finite group G, the *fusion system* $\mathcal{F}_P(G)$ is defined as the category whose objects are the subgroups of P and whose morphisms from Q to R are the group homomorphisms induced from conjugation by an element of G. If P is a Sylow p-subgroup of G, then $\mathcal{F}_P(G)$ is a saturated fusion system. For a detailed information on fusion systems, we refer the reader to [1, 3, 6, 18]. For a subgroup $H \leq G$, the *Scott kG-module with respect to H*, denoted by $\mathrm{Sc}(G, H)$, is defined as the unique indecomposable kG-module which is a direct summand of $\mathrm{Ind}_H^G(k)$ and which contains the trivial kG-module in its socle or in its top. If Q is a Sylow p-subgroup of H, then Q is a vertex of $\mathrm{Sc}(G, H)$ and it follows that $\mathrm{Sc}(G, H) = \mathrm{Sc}(G, Q)$ (see Corollary 4.8.5 of [19]). For more information on Scott modules see [4] and Chapter 4, Section 8 of [19].

The following results due to Ishioka and Kunugi constitute the framework of our strategy to deduce Brauer indecomposability of Scott modules.

Theorem 2.1 (Theorem 1.3 of [9]). *Assume that P is a p-subgroup of G and $\mathcal{F}_P(G)$ is saturated. Then the following assertions are equivalent:*

(a) *$\mathrm{Sc}(G, P)$ is Brauer indecomposable.*
(b) *$\mathrm{Res}^{N_G(Q)}_{Q C_G(Q)}\big(\mathrm{Sc}(N_G(Q), N_P(Q))\big)$ is indecomposable for each fully $\mathcal{F}_P(G)$-normalized subgroup Q of P.*

If these conditions are satisfied, then $(\mathrm{Sc}(G, P))(Q) \cong \mathrm{Sc}(N_G(Q), N_P(Q))$ for each fully $\mathcal{F}_P(G)$-normalized subgroup $Q \leq P$.

Theorem 2.2 (Theorem 1.4 of [9]). *Assume that P is a p-subgroup of G and $\mathcal{F}_P(G)$ is saturated. Let Q be a fully $\mathcal{F}_P(G)$-normalized subgroup of P. Assume further that there exists a subgroup H_Q of $N_G(Q)$ satisfying the following conditions:*

(a) *$N_P(Q)$ is a Sylow p-subgroup of H_Q and*
(b) *$|N_G(Q) : H_Q| = p^a$ for an integer $a \geq 0$.*

Then $\operatorname{Res}^{N_G(Q)}_{Q\,C_G(Q)}\big(\mathrm{Sc}(N_G(Q), N_P(Q))\big)$ *is indecomposable.*

In order to check that the conclusion of the previous theorem is satisfied, we will use the following result. The proof of this lemma comes directly from applying [10, Theorem 1.7].

Lemma 2.3. *Let G be a finite group with a p-subgroup P.*

(i) *For every subgroup $Q \le P$, $Q\,C_P(Q)$ is a maximal element of the set $N_P(Q) \cap_{N_G(Q)} Q\,C_G(Q) := \{{}^gN_P(Q) \cap Q\,C_G(Q) \mid g \in N_G(Q)\}$.*

(ii) *We have that*

$$\mathrm{Sc}(Q\,C_G(Q), Q\,C_P(Q)) \,\Big|\, \operatorname{Res}^{N_G(Q)}_{Q\,C_G(Q)}\big(\mathrm{Sc}(N_G(Q), N_P(Q))\big),$$

and

$$\mathrm{Sc}(C_G(Q), C_P(Q)) \,\Big|\, \operatorname{Res}^{N_G(Q)}_{C_G(Q)}\big(\mathrm{Sc}(N_G(Q), N_P(Q))\big).$$

The p-nilpotency of centralizers of p-subgroups plays an important role in deriving Brauer indecomposability when we are using Theorem 2.2. The following easy observation will be used frequently in the paper.

Lemma 2.4. *Let G be a finite group and suppose that $P \in \mathrm{Syl}_p(G)$. If $Q \le P$ is $\mathcal{F}_P(G)$-centric, then $C_G(Q) = Z(Q) \times O_{p'}(C_G(Q))$. In particular, $C_G(Q)$ is p-nilpotent.*

Proof. Follows from [6, Proposition 4.43]. □

The following result shows that there is a relationship between the indecomposabilities of Brauer quotient of a Scott module with respect to certain p-subgroups when restricted to their centralizers. This result can also be seen, in some way, as a generalization of Lemma 4.4 of [12].

Lemma 2.5. *Let P be an arbitrary finite p-group and G, G' be finite groups such that $P \in \mathrm{Syl}_p(G) \cap \mathrm{Syl}_p(G')$, $\mathcal{F} := \mathcal{F}_P(G) = \mathcal{F}_P(G')$ and $\mathcal{G} := G \times G'$. Assume that Q is a fully $\mathcal{F}$-normalized subgroup of P and that $C := Q\,C_P(Q)$ is normal in both $Q\,C_G(Q)$ and $Q\,C_{G'}(Q)$. Furthermore, suppose that $N_{\mathcal{G}}(\Delta Q) = C_{\mathcal{G}}(\Delta Q)\,N_{\Delta P}(\Delta Q)$. Set $M := \mathrm{Sc}(\mathcal{G}, \Delta P)$. Suppose that $\operatorname{Res}^{N_{\mathcal{G}}(\Delta C)}_{\Delta C\,C_{\mathcal{G}}(\Delta C)}(M(\Delta C))$ is indecomposable. If $M(\Delta Q)$ is indecomposable as an $N_{\mathcal{G}}(\Delta Q)$-module, then $\operatorname{Res}^{N_{\mathcal{G}}(\Delta Q)}_{\Delta Q\,C_{\mathcal{G}}(\Delta Q)}(M(\Delta Q))$ is indecomposable.*

Proof. By our assumption, we have that $\mathcal{C} := \Delta Q\,(C_P(Q) \times C_P(Q)) \trianglelefteq \Delta Q\,C_{\mathcal{G}}(\Delta Q)$. Moreover, by Lemma 2.3(ii),

$$\mathrm{Sc}(\Delta Q\,C_{\mathcal{G}}(\Delta Q), \Delta C) \,\Big|\, \operatorname{Res}^{N_{\mathcal{G}}(\Delta Q)}_{\Delta Q\,C_{\mathcal{G}}(\Delta Q)}\big(\mathrm{Sc}(N_{\mathcal{G}}(\Delta Q), N_{\Delta P}(\Delta Q))\big).$$

Since $M(\Delta Q)$ is indecomposable as an $N_{\mathcal{G}}(\Delta Q)$-module, the fourth line of the proof of Theorem 1.3 in [9] implies that $M(\Delta Q) = \mathrm{Sc}(N_{\mathcal{G}}(\Delta Q), N_{\Delta P}(\Delta Q))$, so that

$$\operatorname{Res}^{N_{\mathcal{G}}(\Delta Q)}_{\Delta Q\,C_{\mathcal{G}}(\Delta Q)}(M(\Delta Q)) = \operatorname{Res}^{N_{\mathcal{G}}(\Delta Q)}_{\Delta Q\,C_{\mathcal{G}}(\Delta Q)}\big(\mathrm{Sc}(N_{\mathcal{G}}(\Delta Q), N_{\Delta P}(\Delta Q))\big).$$

It follows that $\mathrm{Sc}(\Delta Q\,C_{\mathcal{G}}(\Delta Q), \Delta C) \mid \operatorname{Res}^{N_{\mathcal{G}}(\Delta Q)}_{\Delta Q\,C_{\mathcal{G}}(\Delta Q)}(M(\Delta Q))$, namely

$$\operatorname{Res}^{N_{\mathcal{G}}(\Delta Q)}_{\Delta Q\,C_{\mathcal{G}}(\Delta Q)}(M(\Delta Q)) = \mathrm{Sc}(\Delta Q\,C_{\mathcal{G}}(\Delta Q), \Delta C) \bigoplus X$$

where X is a $\Delta Q\,C_{\mathcal{G}}(\Delta Q)$-module. On the other hand,
$\mathrm{Sc}(N_{\mathcal{G}}(\Delta Q), N_{\Delta P}(\Delta Q)) \mid \operatorname{Ind}^{N_{\mathcal{G}}(\Delta Q)}_{N_{\Delta P}(\Delta Q)}(k)$ by definition, so gathering these information together, we get that

$$\operatorname{Res}^{N_{\mathcal{G}}(\Delta Q)}_{\Delta Q\,C_{\mathcal{G}}(\Delta Q)}(M(\Delta Q)) \,\Big|\, (\operatorname{Res}^{N_{\mathcal{G}}(\Delta Q)}_{\Delta Q\,C_{\mathcal{G}}(\Delta Q)} \circ \operatorname{Ind}^{N_{\mathcal{G}}(\Delta Q)}_{N_{\Delta P}(\Delta Q)})(k) = \operatorname{Ind}^{\Delta Q\,C_{\mathcal{G}}(\Delta Q)}_{\Delta C}(k)$$

by the Mackey formula and by our assumption that $N_{\mathcal{G}}(\Delta Q) = C_{\mathcal{G}}(\Delta Q)\, N_{\Delta P}(\Delta Q)$.

Consequently, we have that $X \mid \mathrm{Ind}_{\Delta C}^{\Delta Q\, C_{\mathcal{G}}(\Delta Q)}(k)$. Now, let us restrict X to $\mathcal{C}$. Then, we have that

$$\mathrm{Res}\,_{\mathcal{C}}^{\Delta Q\, C_{\mathcal{G}}(\Delta Q)}(X) \Bigm| (\mathrm{Res}\,_{\mathcal{C}}^{\Delta Q\, C_{\mathcal{G}}(\Delta Q)} \circ \mathrm{Ind}_{\Delta C}^{\Delta Q\, C_{\mathcal{G}}(\Delta Q)})(k).$$

Let us look more closely to the right hand side of the line above. The Mackey formula implies that

$$(\mathrm{Res}\,_{\mathcal{C}}^{\Delta Q\, C_{\mathcal{G}}(\Delta Q)} \circ \mathrm{Ind}_{\Delta C}^{\Delta Q\, C_{\mathcal{G}}(\Delta Q)})(k) = \bigoplus_g \mathrm{Ind}_{\mathcal{C}\cap {}^g(\Delta C)}^{\mathcal{C}}(k)$$

where g runs through the double cosets of $[\mathcal{C}\backslash \Delta Q\, C_{\mathcal{G}}(\Delta Q)/\Delta C]$. Since $\mathcal{C}$ is a normal subgroup of $\Delta Q\, C_{\mathcal{G}}(\Delta Q)$, we have that $\mathcal{C}\cap {}^g(\Delta C) = {}^g\mathcal{C}\cap {}^g(\Delta C) = {}^g(\mathcal{C}\cap \Delta C) = {}^g(\Delta C)$. Therefore,

$$(\mathrm{Res}\,_{\mathcal{C}}^{\Delta Q\, C_{\mathcal{G}}(\Delta Q)} \circ \mathrm{Ind}_{\Delta C}^{\Delta Q\, C_{\mathcal{G}}(\Delta Q)})(k) = \bigoplus_g \mathrm{Ind}_{{}^g(\Delta C)}^{\mathcal{C}}(k),$$

and consequently, we have that

$$(\mathrm{Res}\,_{\mathcal{C}}^{\Delta Q\, C_{\mathcal{G}}(\Delta Q)}(X) \Bigm| \bigoplus_g \mathrm{Ind}_{{}^g(\Delta C)}^{\mathcal{C}}(k).$$

Suppose that X is non-zero. Then by the above line and the fact that Green's indecomposability theorem implies that each $\mathrm{Ind}_{{}^g(\Delta C)}^{\mathcal{C}}(k)$ is indecomposable, we have that $(\mathrm{Res}\,_{\mathcal{C}}^{\Delta Q\, C_{\mathcal{G}}(\Delta Q)}(X)$ is a direct sum of some $\mathrm{Ind}_{{}^g(\Delta C)}^{\mathcal{C}}(k)$'s. Hence, we deduce that $X({}^g(\Delta C))$ is non-zero for some $g \in \Delta Q\, C_{\mathcal{G}}(\Delta Q)$ from [4, 1.4]. It follows that $0 \neq X({}^g(\Delta C)) = {}^gX({}^g(\Delta C)) = {}^g[X(\Delta C)]$, so that $X(\Delta C) \neq 0$.

Let us take Brauer quotients of both sides of the following identity with respect to ΔC:

$$\mathrm{Res}\,_{\Delta Q\, C_{\mathcal{G}}(\Delta Q)}^{N_{\mathcal{G}}(\Delta Q)}(M(\Delta Q)) = \mathrm{Sc}(\Delta Q\, C_{\mathcal{G}}(\Delta Q), \Delta C) \bigoplus X$$

then we get that

$$[\mathrm{Res}\,_{\Delta Q\, C_{\mathcal{G}}(\Delta Q)}^{N_{\mathcal{G}}(\Delta Q)}(M(\Delta Q))](\Delta C) = [\mathrm{Sc}(\Delta Q\, C_{\mathcal{G}}(\Delta Q), \Delta C)](\Delta C) \bigoplus X(\Delta C).$$

Note that ΔC acts trivially on $\mathrm{Sc}(\Delta Q\, C_{\mathcal{G}}(\Delta Q), \Delta C)$, so

$$[\mathrm{Sc}(\Delta Q\, C_{\mathcal{G}}(\Delta Q), \Delta C)](\Delta C) = \mathrm{Sc}(\Delta Q\, C_{\mathcal{G}}(\Delta Q), \Delta C).$$

So the above identity becomes

$$[\mathrm{Res}\,_{\Delta Q\, C_{\mathcal{G}}(\Delta Q)}^{N_{\mathcal{G}}(\Delta Q)}(M(\Delta Q))](\Delta C) = \mathrm{Sc}(\Delta Q\, C_{\mathcal{G}}(\Delta Q), \Delta C) \bigoplus X(\Delta C).$$

Since from the previous paragraph we have that $X(\Delta C) \neq 0$, the right hand side of the above identity is not indecomposable. Now, let us look at the left hand side of this identity. Since taking Brauer quotients and taking restriction commute, and since $\Delta C \leq \Delta Q\, C_{\mathcal{G}}(\Delta Q)$, we have that

$$[\mathrm{Res}\,_{\Delta Q\, C_{\mathcal{G}}(\Delta Q)}^{N_{\mathcal{G}}(\Delta Q)}(M(\Delta Q))](\Delta C) \cong \mathrm{Res}\,_{N_{\Delta Q\, C_{\mathcal{G}}(\Delta Q)}(\Delta C)}^{N_{N_{\mathcal{G}}(\Delta Q)}(\Delta C)}[(M(\Delta Q))(\Delta C)].$$

Note also that, Proposition 1.5(3) of [5] implies that

$$(M(\Delta Q))(\Delta C) \cong \mathrm{Res}\,_{N_{\mathcal{G}}(\Delta C)\cap N_{\mathcal{G}}(\Delta Q)}^{N_{\mathcal{G}}(\Delta C)}\Big(M(\Delta C)\Big)$$

since $\Delta Q \trianglelefteq \Delta C$. Combining the last two lines of identity, we get that

$$[\mathrm{Res}\,_{\Delta Q\, C_{\mathcal{G}}(\Delta Q)}^{N_{\mathcal{G}}(\Delta Q)}(M(\Delta Q))](\Delta C) \cong \mathrm{Res}\,_{N_{\Delta Q\, C_{\mathcal{G}}(\Delta Q)}(\Delta C)}^{N_{\mathcal{G}}(\Delta C)}\Big(M(\Delta C)\Big).$$

The left hand side of the latest identity is indecomposable. Indeed, if it is not indecomposable, then since $\Delta C\, C_{\mathcal{G}}(\Delta C) \leq N_{\Delta Q\, C_{\mathcal{G}}(\Delta Q)}(\Delta C) = \Delta Q\, C_{\mathcal{G}}(\Delta Q) \cap N_{\mathcal{G}}(\Delta C)$, we would have that $\mathrm{Res}\,_{\Delta C\, C_{\mathcal{G}}(\Delta C)}^{N_{\mathcal{G}}(\Delta C)}(M(\Delta C))$ is not indecomposable. This contradicts with our assumption. Since we have deduced that the left hand side of the latest identity is indecomposable, this will imply in turn that $X(\Delta C) = 0$. So we should have that X is zero. Therefore, $\operatorname{Res}^{N_{\mathcal{G}}(\Delta Q)}_{\Delta Q\, C_{\mathcal{G}}(\Delta Q)}(M(\Delta Q)) = \operatorname{Sc}(\Delta Q\, C_{\mathcal{G}}(\Delta Q), \Delta C)$ as required. □

3. Fusion inside $Q_{2^n} \times C_{2^m}$

Let $P = \langle x, y, z \mid x^{2^{n-1}} = z^{2^m} = [x,z] = [y,z] = 1,\ y^2 = x^{2^{n-2}},\ yxy^{-1} = x^{-1}\rangle = \langle x, y\rangle \times \langle z\rangle \cong Q_{2^n} \times C_{2^m}$ where $n \geq 3$ and $m \geq 1$. From now on, the notation P, x, y and z will be fixed till the end of the paper, unless otherwise stated.

Let G be a finite group containing P. In this section, we analyze the G-fusion in P by making use of the classification of saturated (non-exotic) fusion systems defined on P given in Lemma 2.2 of [20]. This result classifies all saturated block fusion systems that can be defined on P. However when the block is the principal block of G, then the corresponding block fusion system is isomorphic to the fusion system of G over P by Brauer's third main theorem (see [1, Theorem IV.5.9]). So, we can rephrase this result as follows.

Lemma 3.1 (Lemma 2.2 of [20]). *Let G be a finite group and P a 2-subgroup of G. Assume that $\mathcal{F} := \mathcal{F}_P(G)$ is a saturated (non-exotic) fusion system. Let $Q_1 := \langle x^{2^{n-3}}, y, z\rangle \cong Q_8 \times C_{2^m}$ and $Q_2 := \langle x^{2^{n-3}}, xy, z\rangle \cong Q_8 \times C_{2^m}$. Then Q_1 and Q_2 are the only candidates for proper $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups up to conjugation. Moreover, one of the following cases occur:*

- (i) *Either $n = 3$, $Q_1 = Q_2 = P$ with $\operatorname{Out}_{\mathcal{F}}(P) \cong C_3$ or $n \geq 4$ and $\operatorname{Out}_{\mathcal{F}}(Q_1) \cong \operatorname{Out}_{\mathcal{F}}(Q_2) \cong S_3$,*
- (ii) *$n \geq 4$, $N_G(Q_1) = N_P(Q_1)\, C_G(Q_1)$ and $\operatorname{Out}_{\mathcal{F}}(Q_2) \cong S_3$,*
- (iii) *$n \geq 4$, $\operatorname{Out}_{\mathcal{F}}(Q_1) \cong S_3$ and $N_G(Q_2) = N_P(Q_2)\, C_G(Q_2)$,*
- (iv) *$N_G(Q_1) = N_P(Q_1)\, C_G(Q_1)$ and $N_G(Q_2) = N_P(Q_2)\, C_G(Q_2)$.*

The notation Q_1 and Q_2 will be fixed as in the preceeding lemma for the rest of the paper. As we will see, determination of the subgroups of P which are isomorphic to quaternion group of order 8 will be important to decide the subgroups of P which have an odd order automorphism. The following lemma will be helpful for this aim.

Lemma 3.2. *Assume that Q is a subgroup of P which is isomorphic to Q_8. Then, Q is P-conjugate to one of the following groups:*

- (i) *$\langle x^{2^{n-3}} z^i, yz^j\rangle$ where $i, j \in \{0, 2^{m-1}\}$,*
- (ii) *$\langle x^{2^{n-3}} z^{2^{m-2}}, yz^{2^{m-2}}\rangle$,*
- (iii) *$\langle x^{2^{n-3}} z^i, xyz^j\rangle$ where $i, j \in \{0, 2^{m-1}\}$,*
- (iv) *$\langle x^{2^{n-3}} z^{2^{m-2}}, xyz^{2^{m-2}}\rangle$.*

Moreover, for all cases, we have that $C_P(Q) = Z(P) = \langle y^2\rangle \times \langle z\rangle \cong C_2 \times C_{2^m}$.

Proof. There are three involutions in P: $z^{2^{m-1}}$, $x^{2^{n-2}} = y^2$, $x^{2^{n-2}} z^{2^{m-1}} = y^2 z^{2^{m-1}}$. Exactly one of these elements should lie in Q, since Q contains a unique involution. Let $x^i y^j z^k$ be an arbitrary element of P, then

$$(x^i y^j z^k)^2 = \begin{cases} y^{2j} z^{2k} & \text{if } j \text{ is odd},\\ x^{2i} z^{2k} & \text{if } j \text{ is even}.\end{cases}$$

Thus, if $x^i y^j z^k$ is an element of order 4, then one of the following cases can occur:

1. j is odd, that is $j \in \{1, 3\}$, and $k \in \{0, 2^{m-2}, 2^{m-1}\}$,
2. j is even, that is $j \in \{0, 2\}$, and $i \in \{0, 2^{n-3}\}$, and $k \in \{0, 2^{m-2}, 2^{m-1}\}$.

We claim that $z^{2^{m-1}}$ can not lie in Q. From the above computation, it can easily be seen that there are two square roots of $z^{2^{m-1}}$, namely $y^2 z^{2^{m-2}}$ and $z^{2^{m-2}}$. But if one of these elements lies in Q, then $C_4 \leq Z(Q)$ since both of these elements are central in P, but since $Z(Q) \cong C_2$ this is impossible. So our claim is established.

So among the three involutions, either y^2 or $y^2 z^{2^{m-1}}$ lie in Q. On the other hand, we observe that $x^j(x^i z^k)x^{-j} = x^i z^k$ and that $y(x^i z^k)y^{-1} = x^{-i}z^k$; also that $x^j(x^i y z^k)x^{-j} = x^{2j+i}yz^k$ and that $y(x^i y z^k)y^{-1} = x^{-i}yz^k$. It follows that yz^k and xyz^k belong to disjoint P-conjugacy classes of P. Hence, noting that $x^{2^{n-3}}y^2 = x^{-2^{n-3}}$ we deduce that the P-conjugacy classes of subgroups of order 4 are $\langle z^{2^{m-2}}\rangle, \langle x^{2^{n-3}}z^k\rangle, \langle yz^k\rangle, \langle xyz^k\rangle$ where $k \in \{0, 2^{m-2}, 2^{m-1}\}$. By the previous paragraph, $\langle z^{2^{m-2}}\rangle$ can not be a subgroup of Q. So, Q should be generated by two of the following subgroups: $\langle x^{2^{n-3}}z^k\rangle, \langle yz^k\rangle, \langle xyz^k\rangle$ where $k \in \{0, 2^{m-2}, 2^{m-1}\}$. If $n \geq 4$, the automorphism of P which is induced by conjugation with yz^k does not invert the element xyz^k and vice versa. But, the automorphism induced by conjugation with yz^k or xyz^k inverts the element $x^{2^{n-3}}z^k$. Hence, when $n \geq 4$, the subgroup $\langle x^{2^{n-3}}z^k\rangle$ should always be a subgroup of Q where $k \in \{0, 2^{m-2}, 2^{m-1}\}$. The possibilities for Q are divided according to which one of the other two P-conjugacy classes of subgroups of order 4 given above lies in Q. If $n = 3$, the automorphism of P induced by conjugation with yz^k inverts the element xyz^k and vice versa. Note also that when $n = 3$, if $x^{2^{n-3}}z^k$ and yz^k are in Q, then xyz^k should lie in Q, so that the group in (i) and (ii) is equal to the group in (iii) and (iv), respectively. □

The following lemma gives all subgroups of P with an odd order $\mathcal{F}$-automorphism.

Lemma 3.3. *Let G be a finite group with a 2-subgroup P. Suppose that $\mathcal{F} := \mathcal{F}_P(G)$ is saturated. Assume that $Q \leq P$ is a fully $\mathcal{F}$-normalized subgroup where $\mathrm{Out}_{\mathcal{F}}(Q)$ is not a 2-group. Then, we have that $Q = R \times Z$ where $R = \langle x^{2^{n-3}}z^i, yz^i\rangle \cong Q_8$ or $R = \langle x^{2^{n-3}}z^i, xyz^i\rangle \cong Q_8$ where $i = 0$ or 2^{m-2} and $Z = \langle z^{2^{m-j}}\rangle$ where $0 \leq j \leq m$, so that $Q \cong Q_8 \times C_{2^j}$. In particular, either $Q \leq Q_1$ or $Q \leq Q_2$ (see the notation in Lemma 3.1). Moreover, if $n \geq 4$, then $\mathrm{Out}_{\mathcal{F}}(Q) \cong S_3$ and if $n = 3$, then $\mathrm{Out}_{\mathcal{F}}(Q) \cong C_3$.*

Proof. P contains three involutions, so Q has either a unique involution or three involutions. If Q has a unique involution, then Q is either cyclic or Q_8. Since cyclic 2-groups don't have odd order automorphisms, Q_8 is the only possible group in this case. If Q has three involutions, we will use the classification of 2-groups with three involutions which has odd order automorphisms given in [7, Theorem 6.1], and determine which of the subgroups in this classification can lie as a subgroup of P.

In this paragraph, we show that there are some subgroups Q of P where $Q \cong Q_8 \times C_{2^j}$ for $0 \leq j \leq m$ with the property that Q contains an odd order $\mathcal{F}$-automorphism. By Lemma 3.2, if $R \cong Q_8$ is a subgroup of P, then $C_P(R) = Z(P) = \langle y^2\rangle \times \langle z\rangle$. So if a is an element lying in $Z(P)$ and a satisfies $\langle a\rangle \cap R = 1$, then $Q = R \times \langle a\rangle \cong Q_8 \times C_{2^j}$ where $0 \leq j \leq m$. Note that since $a \in Z(P)$, by Lemma 3.2, we have that $Q \leq Q_k$ for $k = 1$ or $k = 2$. We also have $C_P(Q) = C_P(R)$ so that $Q\,C_P(Q) = Q_k \cong Q_8 \times C_{2^m}$. Suppose that R is one of the subgroups in (i) or (iii) with $(i, j) = (0, 0)$ or (ii) or (iv) in Lemma 3.2. If $n \geq 4$, the normalizer of Q in P contains $x^{2^{n-4}}$, (but does not contain $x^{2^{n-5}}$ if $n \geq 5$) so that $N_P(Q) = \langle x^{2^{n-4}}, Q\,Z(P)\rangle$. Thus, $N_P(Q) \cong Q_{16} \times C_{2^m}$ and $N_P(Q)/Q\,C_P(Q) \cong C_2$. If $n = 3$, then $N_P(Q) = Q\,C_P(Q) = P$, so $N_P(Q)/Q\,C_P(Q)$ is trivial. Assume that Q_k is $\mathcal{F}$-radical so that $\mathrm{Out}_{\mathcal{F}}(Q_k) \cong S_3$. Note that from the outer automorphism of Q_k of order 3, we can construct an actual automorphism α of Q_k of order 3 since $\mathrm{Inn}(Q_k) \cong C_2 \times C_2$. Lemma 2.3 of [20] says that α fixes the elements of $Z(Q_k) = Z(P)$. It follows that α permutes the three non-trivial elements of $Q_k/Z(Q_k) \cong C_2 \times C_2$ and fixes the elements of $Z(P)$. Hence α restricts to an $\mathcal{F}$-automorphism of Q of order 3 which permutes the three non-trivial elements of $Q/Z(Q) \cong C_2 \times C_2$ and fixes the elements of $Z(Q)$ since $Z(Q) \leq Z(Q_k)$. Moreover, from [18, Proposition 2.5], we have that $\mathrm{Out}_P(Q) = N_P(Q)/Q\,C_P(Q)$ is a Sylow 2-subgroup of $\mathrm{Out}_{\mathcal{F}}(Q)$. We also have that $|\mathrm{Aut}(Q)| = 3 \cdot 2^r$ by Theorem 6.1(1) of [7]. As a result, when $n \geq 4$ we have that $\mathrm{Out}_{\mathcal{F}}(Q) \cong S_3$ and when $n = 3$, we have that $\mathrm{Out}_{\mathcal{F}}(Q) \cong C_3$. Now, let us consider the other possibilities for R. In this case, R is equal to one of the groups in (i) or (iii) with $(i,j) \in \{(0, 2^{m-1}), (2^{m-1}, 0), (2^{m-1}, 2^{m-1})\}$ in Lemma 3.2. Since all these cases have the same property, it is enough to consider only one of them. Let $i = 0$ and $j = 2^{m-1}$, then the cyclic subgroups of order 4 of R are $\langle x^{2^{n-3}}\rangle, \langle yz^{2^{m-1}}\rangle$ and $\langle x^{2^{n-3}}yz^{2^{m-1}}\rangle$. By Lemma 2.3 of [20], it follows that $z^{2^{m-1}}$ is an $\mathcal{F}$-central element in P, hence $\langle x^{2^{n-3}}\rangle$ is stabilized under any $\mathcal{F}$-automorphism of Q, so that the subgroup Q can not have an element of order 3. Thus Theorem 6.1(1) of [7] implies that $\mathrm{Out}_{\mathcal{F}}(Q)$ is a 2-group in this case.

By using Lemma 3.2, we deduce that the centralizer of any subgroup which is isomorphic to Q_8 has abelian centralizer in P. So P does not have a subgroup which is isomorphic $Q_8 \times Q_{2^k}$ for $k \geq 3$.

Now consider a subgroup Q of P, which is isomorphic to $C_{2^k} \times C_{2^k}$ where $k \geq 1$. Then Q contains all of the three involutions of P. By [20, Lemma 2.5], these three involutions are not $\mathcal{F}$-conjugate. But if Q has an odd order automorphism, this automorphism has order 3 ([7, Theorem 6.1 (3)]), and this automorphism permutes the three involutions. This gives a contradiction. As a result, P does not contain a subgroup isomorphic to $C_{2^k} \times C_{2^k}$ which has an odd order $\mathcal{F}$-automorphism.

Let Q be a subgroup of P (of order 2^k where $k \geq 6$) which is isomorphic to either X_k or Y_k (the construction of these groups are explained in the third paragraph of Section 6 in [7]), then Q is a non-split extension of $Q_8 \times C_{2^{k-4}}$ by C_2. But by the definition of P, we see that Q is isomorphic to either $Q_{16} \times C_{2^{k-4}}$ or $Q_8 \times C_{2^{k-3}}$. From the constructions of X_k and Y_k, now it is easy to see that we have a contradiction. Thus, such a Q can not occur as a subgroup of P.

Let Q be a subgroup of P which is isomorphic to a Sylow 2-subgroup of $\mathrm{U}_3(4) = \mathrm{PSU}_3(4)$ which is a Suzuki 2-group. Let us call it Suz. Then by [7, Theorem 4.2], Q has exponent 4, so that $Q \leq \Omega_2(P) = \langle x^{2^{n-3}}, y, z^{2^{m-2}}\rangle \cong Q_8 \times C_4$. But note that Suz has order 64, so this is impossible. That is, Q can not be a subgroup of P. □

Lemma 3.4. *Let $n \geq 4$ and let G be a finite group with a 2-subgroup P. Suppose that $\mathcal{F}_P(G)$ is saturated, Assume that Q_k (see the notation in Lemma 3.1) is $\mathcal{F}$-radical for $k = 1$ or $k = 2$. Let $Q \leq Q_k$ for $k = 1$ or $k = 2$ and let Q satisfy $\mathrm{Out}_{\mathcal{F}}(Q) \cong S_3$. Then we have that $N_G(Q) = N_G(Q_k)\,C_G(Q)$.*

Proof. Set $\mathcal{F} := \mathcal{F}_P(G)$. Since Q_k is an $\mathcal{F}$-radical subgroup, Lemma 3.1 implies that $\mathrm{Out}_{\mathcal{F}}(Q_k) \cong S_3$. To show that $N_G(Q) \leq N_G(Q_k)\,C_G(Q)$, we claim that every automorphism in $\mathrm{Aut}_{\mathcal{F}}(Q)$ extends to an automorphism in $\mathrm{Aut}_{\mathcal{F}}(Q_k)$. By Lemma 3.3, $Q = R \times \langle z^{2^{m-j}}\rangle$ for some j where $0 \leq j \leq m$ and where R is the specified copy of Q_8 as stated in this lemma. It follows from [20, Lemma 2.5] that each element of $Z(Q_k) = Z(P)$ is $\mathcal{F}$-central. In particular, since $Z(Q) \leq Z(Q_k)$, every element of $Z(Q)$ is $\mathcal{F}$-central, too, which implies that any $\mathcal{F}$-automorphism of Q fixes all elements of $Z(Q)$. So an $\mathcal{F}$-automorphism β of Q fixes all elements of $Z(Q)$ and it permutes some of the non-trivial elements of $Q/Z(Q)$. Now, we create an automorphism α of Q_k with the property that α fixes all elements of $Z(P)$ and α corresponds to the same permutation as β corresponds of the non-trivial elements of $Q_k/Z(Q_k) = R/Z(R) = Q/Z(Q)$. Then $\alpha \in \mathrm{Aut}_{\mathcal{F}}(Q_k)$ since $\mathrm{Out}_{\mathcal{F}}(Q_k) \cong S_3$ and it follows that $\alpha|_Q = \beta$. Note that if β corresponds to the trivial permutation, then it follows that β is an inner automorphism of Q, then accordingly α is an inner automorphism of Q_k. So our claim is established. Therefore, if $g \in N_G(Q)$ then $c_g \in \mathrm{Aut}_{\mathcal{F}}(Q)$, then there is an $h \in N_G(Q_k)$ such that $c_h|_Q = c_g$ which implies that $h^{-1}g \in C_G(Q)$. So we have that $g \in N_G(Q_k)\,C_G(Q)$. Hence, we get that $N_G(Q) \leq N_G(Q_k)\,C_G(Q)$. Conversely, let $g \in N_G(Q_k)$, then $c_g : Q_k \to Q_k$ is in $\mathrm{Aut}_{\mathcal{F}}(Q_k)$, then by [20, Lemma 2.3] that c_g fixes all elements in $Z(Q_k)$, so c_g corresponds to a permutation of non-trivial elements of $Q_k/Z(Q_k) = R/Z(R) = Q/Z(Q)$. Then, since

$Z(Q) \le Z(Q_k)$, it follows that c_g fixes all elements of $Z(Q)$, so it follows that c_g restricts to an $\mathcal{F}$-automorphism of Q. Therefore, $g \in N_G(Q)$ and the proof is finished. □

The following result will help us to prove Theorem 3.6.

Lemma 3.5 ([13, Lemma 4.2])**.** *Let Q be a normal 2-subgroup of G such that $G/Q \cong S_3$. Assume further that there is an involution $t \in G - Q$. Then G has a subgroup H such that $t \in H \cong S_3$.*

Recall that, we will use Theorem 2.2 to prove our main theorems. We will use the following result to check that the necessary conditions of this theorem are satisfied by the subgroups of P.

Theorem 3.6. *Let $n \ge 4$ and P a 2-subgroup of G and assume that $\mathcal{F} := \mathcal{F}_P(G)$ is a saturated fusion system. Let $Q \le P$. Assume that $Q \cong Q_8 \times C_{2^j}$ where $0 \le j \le m$ and Q is fully $\mathcal{F}_P(G)$-normalized. Assume, moreover, that $C_G(Q)$ is 2-nilpotent and $\mathrm{Out}_{\mathcal{F}}(Q) \cong N_G(Q)/Q\,C_G(Q)$ is not a 2-group. Then there exists a subgroup H_Q of $N_G(Q)$ such that $N_P(Q)$ is a Sylow 2-subgroup of H_Q and $|N_G(Q) : H_Q|$ is a power of 2 (possibly 1).*

Proof. Since $C_G(Q)$ is 2-nilpotent, the group $Q\,C_G(Q)$ is also 2-nilpotent. Let $K_Q := O_{2'}(Q\,C_G(Q))$ and let $S_Q \in \mathrm{Syl}_2(Q\,C_G(Q))$ containing $Q\,C_P(Q)$, so that $Q\,C_G(Q) = K_Q \rtimes S_Q$. Note that since $O_{2'}(Q\,C_G(Q)) = O_{2'}(C_G(Q))$ we have $[K_Q, Q] = 1$ and so $K_Q \rtimes Q = K_Q \times Q$. Note that $(K_Q \times Q) \trianglelefteq (K_Q \rtimes N_P(Q))$. Moreover, since K_Q is a characteristic subgroup of $Q\,C_G(Q)$ and $Q\,C_G(Q)$ is a normal subgroup of $N_G(Q)$, we have that $K_Q \trianglelefteq N_G(Q)$, so that $K_Q \times Q \trianglelefteq N_G(Q)$.

Set $L_Q := K_Q \times Q$ and use the notation $\overline{H}$ to denote the image of $H \le N_G(Q)$ under the natural epimorphism $\pi_{L_Q} : N_G(Q) \twoheadrightarrow N_G(Q)/L_Q$. Then $\overline{Q\,C_G(Q)} \cong S_Q/Q$ is a normal 2-subgroup of $\overline{N_G(Q)}$.

Let $Q = Q_k$ where $k = 1$ or $k = 2$. Then since $\mathrm{Out}_{\mathcal{F}}(Q_k)$ is not a 2-group, Q_k is $\mathcal{F}$-radical and using Lemma 3.1, we get that

$$\overline{N_G(Q_k)} \;/\; \overline{Q_k\,C_G(Q_k)} \cong N_G(Q_k)/Q_k\,C_G(Q_k) \cong S_3.$$

Since Q_k is $\mathcal{F}$-centric, we have that $Q_k\,C_P(Q_k) = Q_k$ and since $N_P(Q_k) = \langle x^{2^{n-4}}, Q_k\rangle$, it follows that

$$\overline{N_P(Q_k)} \;/\; \overline{Q_k\,C_P(Q_k)} = \overline{N_P(Q_k)} \;/\; \overline{Q_k} \cong N_P(Q_k)/Q_k \cong C_2,$$

in fact, $\overline{N_P(Q_k)} = \langle\overline{x^{2^{n-4}}}\rangle \cong C_2$ and $\overline{x^{2^{n-4}}} \notin \overline{Q_k\,C_G(Q_k)}$. Hence, by Lemma 3.5 there is a subgroup H of $\overline{N_G(Q_k)}$ such that $\overline{x^{2^{n-4}}} \in H \cong S_3$. Set H_{Q_k} as the preimage of H under $\pi_{L_{Q_k}}$. As a result, we deduce that there is a subgroup $H_{Q_k} \le N_G(Q_k)$ such that $N_P(Q_k) \in \mathrm{Syl}_2(H_{Q_k})$ and $|N_G(Q_k) : H_{Q_k}|$ is a power of 2.

Now let $Q \ne Q_k$. Since $\mathrm{Out}_{\mathcal{F}}(Q)$ is not a 2-group, from Lemma 3.3, we deduce that $Q = R \times Z$ where $R = \langle x^{2^{n-3}} z^i, yz^i\rangle$ and $Q < Q_1$ or $R = \langle x^{2^{n-3}} z^i, xyz^i\rangle$ and $Q < Q_2$ where $i = 0$ or 2^{m-2} and $Z = \langle z^{2^{m-j}}\rangle$ where $0 \le j \le m$. Moreover, the same lemma implies that $\mathrm{Out}_{\mathcal{F}}(Q) \cong S_3$. It follows that, if $Q < Q_1$, then the elements $x^{2^{n-3}}, y$ and $x^{2^{n-3}}y$ belong to the same $\mathcal{F}$-conjugacy class of P and if $Q < Q_2$ then the elements $x^{2^{n-3}}, xy$ and $x^{2^{n-3}+1}y$ belong to the same $\mathcal{F}$-conjugacy class of P since $Z(Q) \le Z(Q_k)$ and each element of $Z(Q_k)$ is $\mathcal{F}$-central by [20, Lemma 2.5]. Hence, Lemma 2.5 of [20] implies that one of the cases (i), (ii), (iii) in Lemma 3.1 holds. As a result, Q_k is $\mathcal{F}$-radical if $Q < Q_k$, that is $\mathrm{Out}_{\mathcal{F}}(Q_k) \cong S_3$.

As we observe in the proof of Lemma 3.3 that

$$N_P(Q) = \langle x^{2^{n-4}}, Q\,C_P(Q)\rangle = \langle x^{2^{n-4}}, Q_k\rangle = N_P(Q_k).$$

Now consider $H_Q := H_{Q_k} K_Q$. Then H_Q is a subgroup of $N_G(Q)$ by Lemma 3.4. We claim that H_Q is the subgroup of $N_G(Q)$ that we are looking for. To show this, we pause for a moment to show that

$$Q_k C_G(Q_k) = N_G(Q_k) \cap (Q C_G(Q)). \tag{3.1}$$

Note that since $Q < Q_k$ we have that $C_G(Q_k) \le C_G(Q)$ so that $Q_k C_G(Q_k) \le Q_k C_G(Q) = Q C_G(Q)$. Hence, it follows that $Q_k C_G(Q_k) \le N_G(Q_k) \cap (Q C_G(Q))$. Conversely, let $g \in N_G(Q_k) \cap (Q C_G(Q))$, then since $g \in N_G(Q_k)$ by using Lemma 2.3 of [20], we deduce that c_g fixes every element of $Z(Q_k)$. Note that since $Z(Q) \le Z(Q_k)$, it follows that c_g fixes every element of $Z(Q)$. Also, since $g \in Q C_G(Q)$, we deduce that $c_g \in \mathrm{Inn}(Q)$, that is c_g acts as identity on $Q/Z(Q) \cong C_2 \times C_2$. So as a result c_g acts as identity on $Q_k/Z(Q_k)$ since $Q_k/Z(Q_k) = R/Z(R) = Q/Z(Q) \cong C_2 \times C_2$. Since c_g fixes all elements of $Z(Q_k)$, it follows that $c_g \in \mathrm{Inn}(Q_k)$, so that $g \in Q_k C_G(Q_k)$. So we have that $N_G(Q_k) \cap (Q C_G(Q)) \le Q_k C_G(Q_k)$ which establishes (3.1).

Note that any element of K_{Q_k} has odd order, so any element of K_{Q_k} should lie in K_Q because K_Q contains any odd order element in $Q C_G(Q)$ since $Q C_G(Q)$ is 2-nilpotent. Hence, we deduce that $K_{Q_k} \le K_Q$. Also, we have that $|S_{Q_k}| \le |S_Q|$ and both of them are powers of 2.

Now we can compute the index of H_Q in $N_G(Q)$. Since Lemma 3.4 implies that $N_G(Q) = N_G(Q_k)(Q C_G(Q))$, by collecting all observations done above, the index of H_Q in $N_G(Q)$ is equal to

$$\frac{|N_G(Q_k)| \cdot |K_Q| \cdot |S_Q|}{|K_{Q_k}| \cdot |S_{Q_k}|} : \frac{|H_{Q_k}| \cdot |K_Q|}{|H_{Q_k} \cap K_Q|}.$$

Note also that $H_{Q_k} \cap K_Q \le N_G(Q_k) \cap K_Q \le N_G(Q_k) \cap Q C_G(Q) = Q_k C_G(Q_k)$, so that $H_{Q_k} \cap K_Q = K_{Q_k} \cap K_Q = K_{Q_k}$, where almost all of the (in)equalites follows from the above observations. So the index becomes,

$$\frac{|N_G(Q_k)|}{|H_{Q_k}|} \cdot \frac{|S_Q|}{|S_{Q_k}|}$$

which is a power of 2, as can be easily seen. Now, it remains to show that $N_P(Q)$ is a Sylow 2-subgroup of $N_G(Q)$. Recall that $N_P(Q_k) = N_P(Q)$ and $N_P(Q_k)$ is a Sylow 2-subgroup of H_{Q_k}. Since we have that $H_{Q_k} \le H_Q$ and the index of H_{Q_k} in H_Q is an odd number (which is a divisor of $|K_Q|$), the result follows. □

The following three consecutive results will help us to get rid of the 2-nilpotency condition on the centralizers of fully normalized subgroups of P in Theorem 1.1 and prove Theorem 1.2.

Lemma 3.7. *Suppose that $P \in \mathrm{Syl}_2(G)$ and $Q \le P$ such that Q is a non-abelian fully $\mathcal{F}_P(G)$-normalized in P. Then $C_G(Q)$ is 2-nilpotent.*

Proof. Set $\mathcal{F} := \mathcal{F}_P(G)$. Since $P \in \mathrm{Syl}_2(G)$, $\mathcal{F}$ is saturated by [3, Proposition 1.3]. Since Q is fully $\mathcal{F}$-normalized in P, by Lemma [18, Proposition 2.5] Q is fully $\mathcal{F}$-centralized, so that from Lemma 2.10(i) of [18], $C_P(Q) \in \mathrm{Syl}_2(C_G(Q))$.

Q is a non-abelian subgroup of P, so there exists a non-central element $a := x^i y^j z^k$ where $0 \le i \le 2^{n-1} - 1$, $0 \le j \le 3$ and $0 \le k \le 2^m - 1$. We compute the centralizer of a in P by considering three cases as follows:

Case 1: If j is even, then $i \ne 0$ and $i \ne 2^{n-2}$, let us call this element *a non-central element of type I*, then $C_P(a) = C_P(x^i) = \langle x \rangle \times \langle z \rangle$.

Case 2: If j is odd, and $i \in \{0, 2^{n-2}\}$, let us call this element *a non-central element of type II*. If a is of type II, then $C_P(a) = C_P(y^j) = \langle y \rangle \times \langle z \rangle$.

Case 3: If j is odd, and $i \notin \{0, 2^{n-2}\}$, let us call this element *a non-central element of type III*. If a is of type III, then for $1 \le r \le 2^{n-2} - 1$, x^r does not centralize a because $x^r(x^i y^j z^k)x^{-r} = x^{2r+i} y^j z^k$, so we have that $C_P(a) = \langle y^2 \rangle \times \langle z \rangle = Z(P)$.

If Q contains a non-central element a of some type, by looking at the cases above, we deduce that Q should contain a non-central element b of either of the remaining types. Thus, we have that $C_P(Q) \leq C_P(a) \cap C_P(b) = Z(P)$ and since $Z(P) \leq C_P(Q)$ is always satisfied, we deduce that $C_P(Q) = Z(P) \cong C_2 \times C_{2^m}$.

By the first paragraph of the proof, $C_P(Q) \in \mathrm{Syl}_2(C_G(Q))$. Further, if $m \geq 2$, [7, Corollary 2.3(4)] implies that $\mathrm{Aut}(C_P(Q)) \cong \mathrm{Aut}(C_2 \times C_{2^m})$ is a 2-group. So the normalizer of $C_P(Q)$ in $C_G(Q)$ is equal to the centralizer of $C_P(Q)$ in $C_G(Q)$ since $C_P(Q)$ is abelian. Therefore, Burnside normal p-complement theorem [8, Theorem 7.4.3] yields that $C_G(Q)$ is 2-nilpotent. If $m = 1$, [8, Theorem 7.7.1] implies that there exists two possibilities, either $C_G(Q)$ has three conjugacy class of involutions or a unique conjugacy class of involutions. From Lemma 2.5 of [20], we have that P has three distinct G-conjugacy class of involutions. This implies that $C_P(Q)$ also has at least three $C_G(Q)$-conjugacy classes of involutions. Therefore, $C_G(Q)$ has exactly three conjugacy classes of involutions, and so Theorem 7.7.1 (ii) of [8] implies that $C_G(Q)$ is 2-nilpotent. □

Lemma 3.8. *Suppose that $P \in \mathrm{Syl}_2(G)$ and Q is a non-central cyclic fully $\mathcal{F}_P(G)$-normalized subgroup of P. If $n \neq m+1$ and $m \neq 2$, then $C_G(Q)$ is 2-nilpotent.*

Proof. By using the same argument as in the first paragraph of the proof of Lemma 3.7, we have that $C_P(Q) \in \mathrm{Syl}_2(C_G(Q))$.

By [20, Lemma 2.5], if $Q := \langle a \rangle$ is a non-central, fully $\mathcal{F}_P(G)$-normalized subgroup of P, then up to $\mathcal{F}_P(G)$-conjugacy either $a = x^i z^j$ or $a = yz^j$ where $i = 1, \ldots, 2^{n-2} - 1$ and $j = 0, 1, \ldots, 2^m - 1$. Note that in the former case, $C_P(a) = C_P(x^i)$, then $C_P(Q) = C_P(a) = \langle x \rangle \times \langle z \rangle \cong C_{2^{n-1}} \times C_{2^m}$. In the latter case $C_P(Q) = C_P(a) = C_P(y) = \langle y \rangle \times \langle z \rangle \cong C_4 \times C_{2^m}$.

Since, $n \neq m+1$ and $m \neq 2$, by [7, Corollary 2.3(4)], $\mathrm{Aut}(C_P(Q))$ is a 2-group. So the normalizer of $C_P(Q)$ in $C_G(Q)$ is equal to the centralizer of $C_P(Q)$ in $C_G(Q)$ since $C_P(Q)$ is abelian. Therefore, Burnside normal p-complement theorem [8, Theorem 7.4.3] yields that $C_G(Q)$ is 2-nilpotent. □

Lemma 3.9. *Suppose that $P \in \mathrm{Syl}_2(G)$ and Q is a non-central abelian fully $\mathcal{F}_P(G)$-normalized subgroup of P. Assume additionally that Q is not cyclic. If $n \neq m+1$ and $m \geq 3$, then $C_G(Q)$ is 2-nilpotent.*

Proof. We use the same argument as in the first paragraph of the proof of Lemma 3.7 and deduce that $C_P(Q) \in \mathrm{Syl}_2(C_G(Q))$.

Suppose that $Q = \langle a \rangle \times \langle b \rangle$. Since Q is non-central we can assume without loss of generality that a is a non-central element of P. We will calculate $C_P(Q)$ by dividing cases according to the type of a (see the proof of Lemma 3.7).

Case 1: If a is a non-central element of type I, then $b \in C_P(a) = \langle x \rangle \times \langle z \rangle$. So $b = x^i z^j$ for some integers i and j. If b is non-central, $C_P(b) = \langle x \rangle \times \langle z \rangle$ and if b is central $C_P(b) = P$. Hence, $C_P(Q) = C_P(a) \cap C_P(b) = \langle x \rangle \times \langle z \rangle \cong C_{2^{n-1}} \times C_{2^m}$.

Case 2: If a is of type II, then $C_P(a) = \langle y \rangle \times \langle z \rangle$. So $b = y^i z^j$ for some i and j. If b is non-central $C_P(b) = \langle y \rangle \times \langle z \rangle$ and if b is central $C_P(b) = P$. Hence, $C_P(Q) = C_P(a) \cap C_P(b) = \langle y \rangle \times \langle z \rangle \cong C_4 \times C_{2^m}$.

Case 3: If a is a non-central element of type III, then $C_P(a) = \langle y^2 \rangle \times \langle z \rangle = Z(P)$. So $b \in Z(P)$ and $C_P(b) = P$ and hence $C_P(Q) = C_P(a) \cap C_P(b) = Z(P) \cong C_2 \times C_{2^m}$.

Since $n \neq m+1$ and $m \geq 3$, by [7, Corollary 2.3(4)], $\mathrm{Aut}(C_P(Q))$ is a 2-group under all cases. So the normalizer of $C_P(Q)$ in $C_G(Q)$ is equal to the centralizer of $C_P(Q)$ in $C_G(Q)$ since $C_P(Q)$ is abelian. Therefore, Burnside normal p-complement theorem [8, Theorem 7.4.3] yields that $C_G(Q)$ is 2-nilpotent. □

4. Scott modules with vertices isomorphic to $Q_{2^n} \times C_{2^m}$

Lemma 4.1. *Let P be a 2-subgroup of G with $\mathcal{F} := \mathcal{F}_P(G)$ is saturated. If Q is a fully $\mathcal{F}$-normalized subgroup of P where $C_G(Q)$ is 2-nilpotent, then $\mathrm{Res}^{N_G(Q)}_{Q\,C_G(Q)}\big(\mathrm{Sc}(N_G(Q), N_P(Q))\big)$ is indecomposable.*

Proof. Suppose that $Q \ncong Q_8 \times C_{2^j}$ where $0 \le j \le m$. Then by Lemma 3.3, we have that $N_G(Q)/Q\,C_G(Q)$ is a 2-group. Then $N_G(Q)$ is 2-nilpotent since $C_G(Q)$ is assumed to be 2-nilpotent. So we can write $N_G(Q) = K \rtimes S$ where $K := O_{2'}(N_G(Q))$ and $S \in \mathrm{Syl}_2(N_G(Q))$. Since $N_P(Q)$ is a 2-subgroup of $N_G(Q)$, without loss of generality we can assume that $N_P(Q) \le S$. Set $H_Q := K \rtimes N_P(Q)$, then $N_P(Q) \in \mathrm{Syl}_2(H_Q)$ and $|N_G(Q) : H_Q|$ is a power of 2.

Suppose that $Q \cong Q_8 \times C_{2^j}$ where $0 \le j \le m$. If $N_G(Q)/Q\,C_G(Q)$ is a 2-group, then we again have the desired subgroup H_Q as in the previous paragraph. So we can assume that $N_G(Q)/Q\,C_G(Q)$ is not a 2-group. If $n \ge 4$, Theorem 3.6 implies that the subgroup $H_Q \le N_G(Q)$ with the required properties exists. So, Theorem 2.2 implies that $\mathrm{Res}^{N_G(Q)}_{Q\,C_G(Q)}\big(\mathrm{Sc}(N_G(Q), N_P(Q))\big)$ is indecomposable.

If $n = 3$, then Lemma 3.1 implies that P is the only candidate for an $\mathcal{F}$-centric and $\mathcal{F}$-radical subgroup, so Alperin's fusion theorem implies that $\mathcal{F}_P(G) = \mathcal{F}_P(N_G(P))$. Then every non-trivial subgroup Q of P becomes fully $\mathcal{F}$-normalized and so for each non-trivial subgroup Q of P, the centralizer $C_G(Q)$ is 2-nilpotent by our assumption. Hence, Theorem 1.2 of [11] implies that $M := \mathrm{Sc}(G, P)$ is Brauer indecomposable. Then Lemma 2.2 (ii) of [11] gives us that $M(Q) = \mathrm{Sc}(N_G(Q), N_P(Q))$ for every subgroup Q of P. Then, by the definition of Brauer indecomposability, we reach the required conclusion. □

Proof of Theorem 1.1. If $n \ge 4$ the result follows from Lemma 4.1 together with Theorem 2.1. If $n = 3$, as discussed in the proof of Lemma 4.1, every non-trivial subgroup of P is fully $\mathcal{F}$-normalized and [11, Theorem 1.2] yields the result. □

Now, we give the proof of Theorem 1.2.

Proof of Theorem 1.2. Set $\mathcal{G} := G \times G'$ and $M := \mathrm{Sc}(\mathcal{G}, \Delta P)$. Since $P \in \mathrm{Syl}_2(G)$, $\mathcal{F} := \mathcal{F}_P(G)$ is a saturated fusion system (see Proposition 1.3 of [3]). Furthermore, $\mathcal{F}_{\Delta P}(\mathcal{G})$ is also saturated because $\mathcal{F}_{\Delta P}(\mathcal{G}) \cong \mathcal{F}_P(G)$ since $\mathcal{F}_P(G) = \mathcal{F}_P(G')$. Moreover note that $C_{\mathcal{G}}(\Delta Q) = C_G(Q) \times C_{G'}(Q)$ for any subgroup Q of P. Note also that since $\mathcal{F}$ is saturated and Q is fully $\mathcal{F}$-normalized in P, by Lemma [18, Proposition 2.5] Q is fully $\mathcal{F}$-centralized, so that from Lemma 2.10(i) of [18], $C_P(Q) \in \mathrm{Syl}_2(C_G(Q)) \cap \mathrm{Syl}_2(C_{G'}(Q))$. We shall prove that $\mathrm{Res}^{N_{\mathcal{G}}(\Delta Q)}_{\Delta Q\, C_{\mathcal{G}}(\Delta Q)}(M(\Delta Q))$ is indecomposable for any fully $\mathcal{F}$-normalized subgroup Q of P by using induction on $|P : Q|$.

If $|P : Q| = 1$, the assertion holds by [12, Lemma 4.3(ii)]. Now, assume that $Q \lneqq P$ and that $M(\Delta R)$ is indecomposable as a $(\Delta R \cdot C_{\mathcal{G}}(\Delta R))$-module for all fully $\mathcal{F}$-normalized subgroups R with $|P : R| < |P : Q|$. We first claim that $M(\Delta Q)$ is indecomposable as an $N_{\mathcal{G}}(\Delta Q)$-module.

Suppose that $M(\Delta Q) = M_1 \oplus \ldots \oplus M_r$ where each M_i is an indecomposable $N_{\mathcal{G}}(\Delta Q)$-module and $r \ge 1$. We can set $M_1 := \mathrm{Sc}(N_{\mathcal{G}}(\Delta Q), N_{\Delta P}(\Delta Q))$ by using Theorem 4.8.6(ii) of [19] . Since $M(\Delta Q) \mid \mathrm{Res}^{\mathcal{G}}_{N_{\mathcal{G}}(\Delta Q)}(M)$, we have that $M_i \mid \mathrm{Res}^{\mathcal{G}}_{N_{\mathcal{G}}(\Delta Q)}(M)$ for each i. If possible, let us fix some $j \ge 2$. Then since $M \mid \mathrm{Ind}^{\mathcal{G}}_{\Delta P}(k)$, by Mackey decomposition we have that

$$M_j \mid \bigoplus_g \mathrm{Ind}^{N_{\mathcal{G}}(\Delta Q)}_{N_{\mathcal{G}}(\Delta Q) \cap\, {}^g(\Delta P)}(k)$$

where g runs over representatives of the double cosets in $N_{\mathcal{G}}(\Delta Q)\backslash\mathcal{G}/\Delta P$ which satisfies $\Delta Q \le {}^g(\Delta P)$ by 1.4 of [4]. Hence a vertex ΔR of M_j lies in $N_{{}^g(\Delta P)}(\Delta Q) = N_{\mathcal{G}}(\Delta Q) \cap$

${}^g(\Delta P)$ for some $g \in \mathcal{G}$. It follows that

$$\Delta R \leq N_{^g(\Delta P)}(\Delta Q) \leq_{N_{\mathcal{G}}(\Delta Q)} N_{\Delta P}(\Delta Q)$$

by Lemma 3.2 of [9]. Since $N_{\Delta P}(\Delta Q)$ is a vertex of M_1, we have that $M_1(\Delta R) \neq 0$. On the other hand, since ΔQ is a proper subgroup of ΔP, by applying Burry-Carlson-Puig's Theorem for ΔQ (see Theorem 4.4.6(ii)] of [19]), we deduce that M_j does not have ΔQ as a vertex. Hence ΔQ is a proper normal subgroup of ΔR. Also, if ΔR is not fully $\mathcal{F}$-normalized, by using the same idea as in [9, page 445, lines 18–22], we can change it with a fully $\mathcal{F}$-normalized $\mathcal{F}$-conjugate of itself and it follows from our induction hypothesis that $M(\Delta R)$ is indecomposable as a $\Delta R\, C_{\mathcal{G}}(\Delta R)$-module. Furthermore, since ΔQ is a normal subgroup of ΔR, [5, Proposition 1.5(3)] implies that

$$M_1(\Delta R) \oplus M_j(\Delta R) \mid (M(\Delta Q))(\Delta R) \cong \ M(\Delta R)$$

as $N_{\mathcal{G}}(\Delta R) \cap N_{\mathcal{G}}(\Delta Q)$-modules, but $\Delta R\, C_{\mathcal{G}}(\Delta R) \leq N_{\mathcal{G}}(\Delta R) \cap N_{\mathcal{G}}(\Delta Q)$, so the isomorphism above restricts to as $\Delta R\, C_{\mathcal{G}}(\Delta R)$-modules. This gives us a contradiction. Therefore, $r = 1$ and $M(\Delta Q)$ is indecomposable as an $N_{\mathcal{G}}(\Delta Q)$-module as claimed. In other words, $M(\Delta Q) = \mathrm{Sc}(N_{\mathcal{G}}(\Delta Q), N_{\Delta P}(\Delta Q))$. Now, it remains us to show that

$$\mathrm{Res}\,^{N_{\mathcal{G}}(\Delta Q)}_{\Delta Q\, C_{\mathcal{G}}(\Delta Q)}\Big(M(\Delta Q)\Big) = \mathrm{Res}\,^{N_{\mathcal{G}}(\Delta Q)}_{\Delta Q\, C_{\mathcal{G}}(\Delta Q)}\Big(\mathrm{Sc}(N_{\mathcal{G}}(\Delta Q), N_{\Delta P}(\Delta Q))\Big)$$

is indecomposable.

Note that $O_{2'}(\Delta Q\, C_{\mathcal{G}}(\Delta Q))$ char $\Delta Q\, C_{\mathcal{G}}(\Delta Q) \trianglelefteq N_{\mathcal{G}}(\Delta Q)$. So $O_{2'}(\Delta Q\, C_{\mathcal{G}}(\Delta Q))$ is a normal subgroup of $N_{\mathcal{G}}(\Delta Q)$ and it follows that $O_{2'}(\Delta Q\, C_{\mathcal{G}}(\Delta Q)) \leq O_{2'}(N_{\mathcal{G}}(\Delta Q))$. Since the Scott module lies in the principal block by its definition, it follows that $O_{2'}(\Delta Q\, C_{\mathcal{G}}(\Delta Q))$ is included in the kernel of $\mathrm{Sc}(N_{\mathcal{G}}(\Delta Q), N_{\Delta P}(\Delta Q))$. Hence without loss of generality, we can assume that $O_{2'}(\Delta Q\, C_{\mathcal{G}}(\Delta Q))$ is trivial.

If Q is a central subgroup of P, by using Lemma 4.4 of [12], we get that

$$\mathrm{Res}\,^{N_{\mathcal{G}}(\Delta Q)}_{\Delta Q\, C_{\mathcal{G}}(\Delta Q)}\Big(M(\Delta Q)\Big) = \mathrm{Res}\,^{N_{\mathcal{G}}(\Delta Q)}_{\Delta Q\, C_{\mathcal{G}}(\Delta Q)}\Big(\mathrm{Sc}(N_{\mathcal{G}}(\Delta Q), N_{\Delta P}(\Delta Q))\Big)$$

is indecomposable. So assume from now on that Q is a non-central subgroup of P.

Case 1: $n \neq m + 1$ and $m \geq 3$.

By Lemma 3.7, Lemma 3.8 and Lemma 3.9, we have that both $C_G(Q)$ and $C_{G'}(Q)$ are 2-nilpotent, so is $C_{\mathcal{G}}(\Delta Q)$. Hence, by using Lemma 4.1, we deduce that

$$\mathrm{Res}\,^{N_{\mathcal{G}}(\Delta Q)}_{\Delta Q\, C_{\mathcal{G}}(\Delta Q)}\Big(\mathrm{Sc}(N_{\mathcal{G}}(\Delta Q), N_{\Delta P}(\Delta Q))\Big)$$

is indecomposable.

Case 2: $n = m + 1$ so that $m \geq 2$.

If Q is non-abelian, Lemma 3.7 and Lemma 4.1 imply that $M(\Delta Q)$ is indecomposable as a $\Delta Q\, C_{\mathcal{G}}(\Delta Q)$-module.

So let us assume that Q is an abelian fully $\mathcal{F}$-normalized subgroup of P. Since Q is non-central abelian, the proof of Lemma 3.8 and Lemma 3.9 imply that there exist subgroups Q for which $\Delta Q\, C_{\mathcal{G}}(\Delta Q)$ is 2-nilpotent, so Lemma 4.1 can be used to accomplish our claim in this case, similarly. Hence, using the computations in the proof of Lemma 3.8 and Lemma 3.9, we can assume that $C := Q\, C_P(Q) = C_P(Q) = C_{2^j} \times C_{2^j}$ for some $j \geq 2$. If $Q = C$, then it follows from [18, Proposition 4.3] that C is $\mathcal{F}$-centric. Thus Lemma 2.4 together with Lemma 4.1 implies that

$$\mathrm{Res}\,^{N_{\mathcal{G}}(\Delta C)}_{\Delta C\, C_{\mathcal{G}}(\Delta C)}\Big(M(\Delta C)\Big) = \mathrm{Res}\,^{N_{\mathcal{G}}(\Delta C)}_{\Delta C\, C_{\mathcal{G}}(\Delta C)}\Big(\mathrm{Sc}(N_{\mathcal{G}}(\Delta C), N_{\Delta P}(\Delta C))\Big)$$

is indecomposable. Now assume that $Q \neq C$, then Q is a proper subgroup of C. Since C is a Sylow 2-subgroup of both $Q\, C_G(Q)$ and $Q\, C_{G'}(Q)$, Theorem 1 of [2] implies that C is normal in $Q\,C_G(Q)$ and $Q\,C_{G'}(Q)$. Moreover, from Lemma 3.3, we have that $\mathrm{Out}_{\mathcal{F}}(\Delta Q)$ is a 2-group. Hence, Proposition 2.5 of [18] implies that

$$N_{\mathcal{G}}(\Delta Q)/\Delta Q\, C_{\mathcal{G}}(\Delta Q) \cong N_{\Delta P}(\Delta Q)/\Delta Q\, C_{\Delta P}(\Delta Q).$$

So it follows that $N_{\mathcal{G}}(\Delta Q) = N_{\Delta P}(\Delta Q)\, C_{\mathcal{G}}(\Delta Q)$. Therefore, we can apply Lemma 2.5, and deduce that $\mathrm{Res}^{N_{\mathcal{G}}(\Delta Q)}_{\Delta Q\, C_{\mathcal{G}}(\Delta Q)}\big(M(\Delta Q)\big) = \mathrm{Res}^{N_{\mathcal{G}}(\Delta Q)}_{\Delta Q\, C_{\mathcal{G}}(\Delta Q)}\big(\mathrm{Sc}(N_{\mathcal{G}}(\Delta Q), N_{\Delta P}(\Delta Q))\big)$ is indecomposable.

Case 3: $n \neq m+1$ and $m \leq 2$.

Assume that Q is non-abelian. Then Lemma 3.7 and Lemma 4.1 imply that $M(\Delta Q)$ is indecomposable as a $\Delta Q\, C_{\mathcal{G}}(\Delta Q)$-module.

Assume that Q is abelian. From the proof of Lemma 3.8 and Lemma 3.9, either Q is cyclic or non-cyclic, we observe that there are cases where $\Delta Q\, C_{\mathcal{G}}(\Delta Q)$ is 2-nilpotent. If Q is one of these, then by using Lemma 4.1, we can deduce that

$$\mathrm{Res}^{N_{\mathcal{G}}(\Delta Q)}_{\Delta Q\, C_{\mathcal{G}}(\Delta Q)}\big(\mathrm{Sc}(N_{\mathcal{G}}(\Delta Q), N_{\Delta P}(\Delta Q))\big)$$

is indecomposable. Otherwise, then $Q\,C_P(Q) = C_P(Q)$ is isomorphic to $C_2 \times C_2$ or $C_4 \times C_4$. If $C_P(Q) \cong C_2 \times C_2$, by using the same argument as in the last paragraph of the proof of Lemma 3.7, we deduce that $\Delta Q\, C_{\mathcal{G}}(\Delta Q)$ is 2-nilpotent. Then by using Lemma 4.1, it follows that

$$\mathrm{Res}^{N_{\mathcal{G}}(\Delta Q)}_{\Delta Q\, C_{\mathcal{G}}(\Delta Q)}\big(\mathrm{Sc}(N_{\mathcal{G}}(\Delta Q), N_{\Delta P}(\Delta Q))\big)$$

is indecomposable. If $C_P(Q) \cong C_4 \times C_4$, then we will repeat the similar argument that we use in Case 2. We add this argument here for completeness. Set $C := Q\,C_P(Q)$. If $Q = C$, then C itself is $\mathcal{F}$-centric. Thus Lemma 2.4 together with Lemma 4.1 implies that

$$\mathrm{Res}^{N_{\mathcal{G}}(\Delta C)}_{\Delta C\, C_{\mathcal{G}}(\Delta C)}\big(M(\Delta C)\big) = \mathrm{Res}^{N_{\mathcal{G}}(\Delta C)}_{\Delta C\, C_{\mathcal{G}}(\Delta C)}\big(\mathrm{Sc}(N_{\mathcal{G}}(\Delta C), N_{\Delta P}(\Delta C))\big)$$

is indecomposable. Now assume that $Q \neq C$, then Q is a proper subgroup of C, Since C is a Sylow 2-subgroup of both $Q\,C_G(Q)$ and $Q\,C_{G'}(Q)$, Theorem 1 of [2] implies that C is normal in $Q\,C_G(Q)$ and $Q\,C_{G'}(Q)$. Moreover, from Lemma 3.3, we have that $\mathrm{Out}_{\mathcal{F}}(\Delta Q)$ is a 2-group. Hence, Proposition 2.5 of [18] implies that

$$N_{\mathcal{G}}(\Delta Q)/\Delta Q\, C_{\mathcal{G}}(\Delta Q) \cong N_{\Delta P}(\Delta Q)/\Delta Q\, C_{\Delta P}(\Delta Q).$$

So it follows that $N_{\mathcal{G}}(\Delta Q) = N_{\Delta P}(\Delta Q) C_{\mathcal{G}}(\Delta Q)$. We can apply Lemma 2.5, and deduce that $\mathrm{Res}^{N_{\mathcal{G}}(\Delta Q)}_{\Delta Q\, C_{\mathcal{G}}(\Delta Q)}\big(M(\Delta Q)\big) = \mathrm{Res}^{N_{\mathcal{G}}(\Delta Q)}_{\Delta Q\, C_{\mathcal{G}}(\Delta Q)}\big(\mathrm{Sc}(N_{\mathcal{G}}(\Delta Q), N_{\Delta P}(\Delta Q))\big)$ is indecomposable.

Hence, we show that $\mathrm{Res}^{N_{\mathcal{G}}(\Delta Q)}_{\Delta Q\, C_{\mathcal{G}}(\Delta Q)}\big(\mathrm{Sc}(N_{\mathcal{G}}(\Delta Q), N_{\Delta P}(\Delta Q))\big)$ is indecomposable for any fully $\mathcal{F}$-normalized subgroup Q of P. Therefore Theorem 2.1 implies that $\mathrm{Sc}(\mathcal{G}, \Delta P)$ is Brauer indecomposable. □

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